On the Impact of Combinatorial Structure on Congestion Games*

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Congestion games are a natural and generally accepted approach to model resource allocation among selfish or myopic players. In a congestion game we have a set of resources. A strategy of a player corresponds to the selection of a subset of these resources. The strategy space is thus a set of sets of resources. The delay (cost, payoff) for each player from selecting a particular resource depends on the number of players choosing that resource, and her total delay is the sum of the delays associated with her selected resources. Almost needless to say, congestion games are fundamental to routing, network design and other kinds of resource sharing problems in distributed systems.

Rosenthal [3] shows with a potential function argument that every congestion game possesses at least one pure Nash equilibrium. This argument does not only prove the existence of pure Nash equilibria but it also shows that such an equilibrium is reached in a natural way when players iteratively play best responses. A recent result of Fabrikant et al. [2] shows, however, that these best response sequences may require an exponential number of iterations. Their analysis relates congestion games to local search problems. They show that it is PLS-complete to compute a Nash equilibrium for general congestion games. Their completeness proof is based on a *tight* PLS-reduction preserving lower bounds on the lengths of improvement sequences. This way, it follows from previous results about local search problems that there exist congestion games with initial configurations such that any best response sequence starting from these configurations needs an exponential number of iterations to reach a Nash equilibrium. Fabrikant et al. [2] are able to extend their negative results from general congestion games towards *network congestion games* in which each player aims at allocating a path in a network connecting a given source with a given destination node, provided that different players can have different source/destination pairs. The complexity changes if one assumes that all players have the same source/destination pair: For symmetric network congestion games, they present a polynomial time algorithm that computes a Nash equilibrium by solving a min-cost flow problem. This positive result leaves open, however, the question about the convergence time for best responses in symmetric network congestion games. As one of our results, we will see that, in contrast to the PLS-hardness results,

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the negative results for the convergence time of asymmetric network congestion games directly transfer to the symmetric case.

In this talk, we study the impact of combinatorial structure in congestion games on the complexity of computing pure Nash equilibria and the convergence time of best response sequences. In particular, we investigate which properties of the strategy spaces of individual players ensure a polynomial convergence time. We show, if the strategy space of each player consists of the bases of a matroid over the set of resources, then the lengths of all best response sequences are polynomially bounded in the number of players and resources. We can also prove that this result is tight, that is, the matroid property is a necessary and sufficient condition on the players' strategy spaces for guaranteeing polynomial time convergence to a Nash equilibrium. In addition, we present an approach that enables us to devise hardness proofs for various kinds of combinatorial games, including first results about the hardness of market sharing games and congestion games for overlay network design. Our approach also yields a short proof for the PLS-completeness of network congestion games. In particular, we can show that network congestion games are PLS-complete for directed and undirected networks even in case of linear latency functions.

References

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