A Semidefinite Relaxation Scheme for Multivariate Quartic Polynomial Optimization With Quadratic Constraints

Zhi-Quan Luo* and Shuzhong Zhang†

In this work we consider the optimization of a multivariate fourth order (quartic) polynomial under quadratic constraints. The problem can take either the maximization form

$$\boxed{\begin{array}{ll} \text{maximize} & f(x) = \displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}x_ix_jx_kx_\ell \\ \text{subject to} & x^{\mathrm{T}}A_ix \le 1,\; i=1,...,m, \end{array}} \tag{1}$$

or the minimization form

$$\boxed{\begin{array}{ll} \text{minimize} & f(x) = \displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}x_ix_jx_kx_\ell \\ \text{subject to} & x^{\mathrm{T}}A_ix \ge 1,\; i=1,...,m, \end{array}} \tag{2}$$

where A_i's are positive semidefinite matrices in $\mathbb{R}^{n\times n}$, $i = 1,...,m$. Let $f_{\max}$ and $f_{\min}$ denote the optimal values of (1) and (2) respectively. Throughout, we assume $f_{\min} \ge 0$.

Quartic optimization problems arise in various engineering applications such as independent component analysis [2], blind channel equalization in digital communication [4] and sensor localization [9]. From the complexity standpoint, the nonconvex quartic polynomial optimization problems (1)–(2) are NP-hard. This motivates us to consider polynomial time relaxation procedures that can deliver provably high quality approximate solutions.

As a special case of the general polynomial optimization problem, the quartic optimization problems (1)–(2) can be relaxed using the standard SOS technique of semidefinite programming relaxation. Specifically, by representing each nonnegative polynomial as a sum of squares of some

*Department of Electrical and Computer Engineering, University of Minnesota, 200 Union Street SE, Minneapolis, MN 55455. Email: luozq@ece.umn.edu. The research of this author is supported in part by the National Science Foundation, Grant No. DMS-0312416.

†Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, Hong Kong. Email: zhang@se.cuhk.edu.hk. Research supported by Hong Kong RGC Earmarked Grants CUHK418505 and CUHK418406.

other polynomials (SOS) [3] of a given degree, it is possible to relax each polynomial inequality as a convex linear matrix inequality (LMI). In this way, as the polynomial degree in SOS representation increases, the nonconvex quartic optimization problems (1)–(2) can be approximated by a hierarchy of semidefinite programs (SDP) with increasing size. While this SOS relaxation scheme can achieve, at least theoretically, asymptotic global optimality, the size of the resulting SDPs in the hierarchy grows exponentially fast. This presents great computational challenges in practice, so much so that it severely limits the application scope of the SOS relaxation approach. Indeed, the most effective use of SDP relaxation so far has been for the quadratic optimization problems whereby only the first level relaxation in the SOS hierarchy is used. Even though such SDP relaxation does not always provide a tight approximation in general, it does lead to provably high quality approximate solution for certain type of quadratic optimization problems. The latter includes various graph problems such as the Max-Cut problem [1] as well as some homogeneous nonconvex quadratic optimization problems [5–8, 10].

In this work we present a general semidefinite relaxation scheme for n-variate quartic polynomial optimization under homogeneous quadratic constraints. Unlike the existing sum-of-squares (SOS) approach which relaxes the quartic optimization problems to a sequence of (typically large) linear semidefinite programs (SDP) over $\mathbb{R}^{n^2 \times n^2}$, our relaxation scheme leads to a (possibly nonconvex) quadratic optimization problem with linear constraints over the semidefinite matrix cone in $\mathbb{R}^{n \times n}$. It is shown that each α-approximate solution of the relaxed quadratic SDP can be used to generate in randomized polynomial time an $O(\alpha / \ln^2(mn))$-approximate solution for the original quartic optimization problem, where m is the number of constraints in the problem. In the case where only one quadratic constraint is present, we provide a polynomial time $\Omega((n \ln^2 n)^{-1})$-approximation algorithm for the quartic maximization problem and a polynomial time $O(n^2)$-approximation algorithm for the quartic minimization problem.

References

[1] Goemans, M. and Williamson, D., "Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming," *Journal of the ACM,* Vol. 42, No. 6, pp. 1115–1145, 1995.

[2] Cardoso, J.-F., "Blind Signal Separation: Statistical Principles," *Proceedings of the IEEE,* Vol. 90, No. 8, pp. 2009–2026, 1998.

[3] Lasserre, J., "Global Optimization with Polynomials and the Problem of Moments," *SIAM Journal on Optimization*, Vol. 11, No. 3, pp. 796–817, 2001.

[4] Maricic, B., Luo, Z.-Q. and Davidson, T.N., "Blind Constant Modulus Equalization via Convex Optimization," *IEEE Transactions on Signal Processing*, Vol. 51, pp. 805–818, March 2003.

[5] Luo, Z.-Q., Sidiropoulos, N.D., Tseng, P., and Zhang, S., "Approximation Bounds for Quadratic Optimization with Homogeneous Quadratic Constraints," *SIAM Journal on Optimization* 18, 1 – 28, 2007.

[6] He, S., Luo, Z.-Q., Nie, J.W., and Zhang, S., "Semidefinite Relaxation Bounds for Indefinite Homogeneous Quadratic Optimization," Accepted for publication in *SIAM Journal on Optimization*, December 2007.

[7] Nemirovski, A., Roos, C. and Terlaky, T., "On Maximization of Quadratic Form Over Intersection of Ellipsoids with Common Center," *Mathematical Programming,* Vol. 86, pp. 463–473, 1999.

[8] Nesterov, Y., "Semidefinite Relaxation and Non-convex Quadratic Optimization," *Optimization Methods and Software*, Vol. 12, pp. 1–20, 1997.

[9] Biswas, P., Liang, T.-C., Wang, T.-C. and Ye, Y., "Semidefinite Programming Based Algorithms for Sensor Network Localization," *ACM Transactions on Sensor Networks*, Vol. 2, pp. 188–220, 2006.

[10] So, A.M-C., Ye, Y. and Zhang, J., "A Unified Theorem on SDP Rank Reduction," accepted for publication in *Mathematics of Operations Research*, 2008.