The Variational/Complementarity Approach to Nash Equilibria, part I

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Contents of Presentation

- General Nash equilibrium
- Affine games and Lemke's method
- Equivalent formulations
- Existence results
- Multi-leader-follower games
- More extensions

Basic Components and Concepts

a deterministic, static, one-stage, non-cooperative game

- a finite set of selfish players, who compete non-cooperatively for optimal individual well-being

- a set of strategies for each player, that is generally dependent of rivals' strategies

- an objective for each player, dependent on rivals' strategies

- an optimal response set given rivals' plays

- a guiding principle of an equilibrium, i.e., a solution, of the game

- there is no leading player; but system welfare is of concern.

The Mathematical Setting

N	number of players
$x \equiv \left(x^i \right)_{i=1}^N$	a vector tuple of strategies, x^i for player i
$x^{-i} \equiv \left(x^j \right)_{j \neq i}$	a vector tuple of all players' strategies, except player i
$\theta_i(x)$	player i's objective, a function all players' strategies
$X^i(x^{-i}) \subseteq \Re^{n_i}$	player i's strategy set dependent on rivals' strategy x^{-i}

Anticipating rivals' strategies x^{-i}, player i solves

$$\boxed{\begin{array}{ll} \underset{x^i}{\text{minimize}} & \theta_i(x^i, x^{-i}) \\ \text{subject to} & x^i \in X^i(x^{-i}) \end{array}}$$

Player i's optimal response set: $\mathcal{R}^i(x^{-i}) \equiv \underset{x^i \in X^i(x^{-i})}{\text{argmin}} \; \theta_i(x^i, x^{-i})$.

Definition of a Nash equilibrium

A tuple $\widehat{x} = \left(\widehat{x}^i\right)_{i=1}^N$ is a Nash equilibrium if,

for all $i = 1, \cdots, N$, $\widehat{x}^i \in \mathcal{R}^i(\widehat{x}^{-i})$, i.e., $\widehat{x}^i \in X^i(\widehat{x}^{-i})$ and

$$\theta_i(\widehat{x}^i, \widehat{x}^{-i}) \leq \theta_i(x^i, \widehat{x}^{-i}), \ \forall\, x^i \in X^i(\widehat{x}^{-i}).$$

In words, a Nash equilibrium is a tuple of strategies, one for each player, such that no player has an incentive to unilaterally deviate from her designated strategy if the rivals play theirs.

Some immediate questions:

- Existence, multiciplicity, characterization, computation, and sensitivity?
- Can players be better off if they collude, i.e., form bargaining groups?
- Can players be given incentives to optimize system well-being while behaving selfishly?

Affine Games

- each $\theta_i(x)$ is quadratic:

$$\theta(x^i, x^{-i}) = \frac{1}{2}(x^i)^T A^{ii} x^i + (x^i)^T \left[\sum_{j \neq i} A^{ij} x^j + a^i \right]$$

with A^{ii} symmetric; and $A^{ij} \neq A^{ji}$ for $i \neq j$;

- each $X^i(x^{-i})$ is polyhedral given by

$$X^i(x^{-i}) \equiv \left\{ x^i \in \Re^{n_i}_+ : \sum_{j=1}^{n} B^{ij} x^j + b^i \geq 0 \right\};$$

note the dependence of B^{ij} on (i,j);

- extending a bimatrix (i.e., 2-person matrix) game, wherein $N = 2$, $(A^{11}, a^1) = 0$, $(A^{22}, a^2) = 0$, and $X^1(x^2)$ and $X^2(x^1)$ are both unit simplices (i.e., strategies are probability vectors).

A Linear Complementarity Formulation

By linear programming duality, $x^i \in \mathcal{R}^i(x^{-i})$ if and only if λ^i exists such that (the $\perp$ notation denotes complementarity slackness),

$$0 \leq x^i \;\perp\; a^i + \sum_{j=1}^{N} A^{ij} x^j - (\, B^{ii}\,)^T \lambda^i \geq 0$$
$$0 \leq \lambda^i \;\perp\; b^i + \sum_{j=1}^{N} B^{ij} x^j \geq 0;$$

concatenation yields an LCP in the variables $\left(x^i, \lambda^i\right)_{i=1}^{N}$.

Note that for each i, only B^{ii} appears in the first complementarity condition, whereas B^{ij} for all j appear in the second.

An Illustration for $N = 2$

$$
\begin{pmatrix} 0 \\ 0 \\ - \\ 0 \\ 0 \end{pmatrix} \leq \begin{pmatrix} x^1 \\ x^2 \\ - \\ \lambda^1 \\ \lambda^2 \end{pmatrix} \perp \begin{pmatrix} a^1 \\ a^2 \\ - \\ b^1 \\ b^2 \end{pmatrix} + \left[\begin{array}{cccc} A^{11} & A^{12} & | & -(\,B^{11}\,)^T & 0 \\ A^{21} & A^{22} & | & 0 & -(\,B^{22}\,)^T \\ -- & -- & | & --- & --- \\ B^{11} & B^{12} & | & 0 & 0 \\ B^{21} & B^{22} & | & 0 & 0 \end{array} \right] \begin{pmatrix} x^1 \\ x^2 \\ - \\ \lambda^1 \\ \lambda^2 \end{pmatrix} \geq 0.
$$

- There is no connection to a single quadratic program, let alone a convex one.

- There is presently no algorithm that is capable of processing this LCP in finite time.

- A major difficulty is due to the two off-diagonal blocks.

Common Coupled Constraints

$[B^{11} \ B^{12}] = [B^{21} \ B^{22}]$ and $b^1 = b^2 = b$

Is the game equivalent to the condensed LCP:

$$\begin{pmatrix} 0 \\ 0 \\ - \\ 0 \end{pmatrix} \leq \begin{pmatrix} x^1 \\ x^2 \\ - \\ \lambda \end{pmatrix} \perp \begin{pmatrix} a^1 \\ a^2 \\ - \\ b \end{pmatrix} + \begin{bmatrix} A^{11} & A^{12} & | & -(B^{11})^T \\ A^{21} & A^{22} & | & -(B^{22})^T \\ -- & -- & | & ----- \\ B^{11} & B^{12} & | & 0 \end{bmatrix} \begin{pmatrix} x^1 \\ x^2 \\ - \\ \lambda \end{pmatrix} \geq 0.$$

In general, every solution to the condensed LCP is a Nash equilibrium, but the converse is not necessarily true.

Example. Consider a 2-person game with a common coupled constraint:

$$\begin{array}{lll|lll} \underset{x_1}{\text{minimize}} & \theta_1(x_1, x_2) \equiv \frac{1}{2}(x_1 + x_2 - 1)^2 & & \underset{x_2}{\text{minimize}} & \theta_1(x_1, x_2) \equiv \frac{1}{2}(x_1 + x_2 - 2)^2 \\ \text{subject to} & x_1 + x_2 \leq 1 & & \text{subject to} & x_1 + x_2 \leq 1 \end{array}$$

The equivalent LCP:

$$
\begin{aligned}
0 &= -1 + x_1 + x_2 + \lambda_1 \\
0 &= -2 + x_1 + x_2 + \lambda_2 \\
0 &\leq 1 - x_1 - x_2 \qquad\qquad \perp \quad \lambda_1 \geq 0 \\
0 &\leq 1 - x_1 - x_2 \qquad\qquad \perp \quad \lambda_2 \geq 0
\end{aligned}
$$

has solutions $(x_1, x_2, \lambda_1, \lambda_2) = (\alpha, 1 - \alpha, 0, 1)$ all of which are Nash equilibria; whereas the condensed LCP:

$$
\begin{aligned}
0 &= -1 + x_1 + x_2 + \lambda \\
0 &= -2 + x_1 + x_2 + \lambda \\
0 &\leq 1 - x_1 - x_2 \qquad\qquad \perp \quad \lambda \geq 0
\end{aligned}
$$

obviously has no solution.

Thus, Nash equilibria exist, but no common multipliers to the common coupled constraint exist!

Solution of the condensed LCP by Lemke's complementary pivot algorithm has been studied by Eaves (1973).

Equivalent Formulations

- fixed-point: $\boxed{\widehat{x}^i \in \mathcal{R}^i(\widehat{x}^{-i})}$ for all $i = 1, \cdots, N$

– fully equivalent in general

- generalized quasi-variational inequality (GQVI):

$$\boxed{\begin{array}{l} \widehat{x} = \left(\widehat{x}^i\right)_{i=1}^N \in X(\widehat{x}) \textbf{ and for some } a^i \in \partial_{x^i}\theta_i(\widehat{x}), \\ \qquad \displaystyle\sum_{i=1}^N (\, x^i - \widehat{x}^i\,)^T a^i \geq 0, \ \forall\, x = \left(\, x^i\,\right)_{i=1}^N \in X(\widehat{x}) \end{array}}$$

where $X(\widehat{x}) \equiv \displaystyle\prod_{i=1}^N X^i(\widehat{x}^{-i})$ is a moving set and

$\partial_{x^i}\theta_i(\widehat{x}) \equiv \left\{ a^i \in \Re^{n_i} : \theta_i(x^i, \widehat{x}^{-i}) - \theta_i(\widehat{x}) \geq (x^i - \widehat{x}^i)^T a^i \ \ \forall\, x^i \in \Re^{n_i} \right\}$ is the sub-differential of $\theta_i(\bullet, \widehat{x}^{-i})$ with respect to x^i at $\widehat{x}^i$

- $\widehat{x}$ is a Nash equilibrium if and only if $\widehat{x}$ is a solution to the GQVI, provided that $\theta_i(\bullet, \widehat{x}^{-i})$ and $X^i(\widehat{x}^{-i})$ are both convex for all i.

A Standard VI under "Joint Convexity"

Let $\boxed{\mathcal{X} \equiv \{x : x \in X(x)\}}$ be the set of fixed-points of the set-valued map X.

A substitution assumption. Suppose that for every $\widetilde{x} \in \mathcal{X}$ and every $i \neq j$,

$$x^i \in X^i(\widetilde{x}^{-i}) \Rightarrow \widetilde{x}^j \in X^j(z^{-j}),$$

where z^{-j} is the vector whose k-component is $\widetilde{x}^k$ for $k \neq i$ and equals to x^i for $k = i$.

- Under the substitution assumption, every solution to the generalized VI:

$$\boxed{\begin{array}{l} \widehat{x} = \left(\widehat{x}^i\right)_{i=1}^N \in \mathcal{X} \textbf{ and for some } a^i \in \partial_{x^i}\theta_i(\widehat{x}), \\ \qquad \displaystyle\sum_{i=1}^N (x^i - \widehat{x}^i)^T a^i \geq 0, \ \forall x = \left(x^i\right)_{i=1}^N \in \mathcal{X} \end{array}}$$

is a solution to the GQVI; but not conversely; counterexample is provided by the previous 2-person generalized game with a common coupled constraint.

• Analysis of VIs typically requires the convexity of the defining set, which amounts to the "joint convexity" of the players' strategies.

• Facchinei-Kanzow (2007) coined the term "variational equilibrium" to mean a solution of the GVI.

• The difference between the GQVI and the GVI is the moving set $X(\widehat{x})$ in the former versus the stationary set $\mathcal{X}$ in the latter.

Yet Another Equivalent Formulations (cont.)

- Karush-Kuhn-Tucker conditions: assume

$$X^i(x^{-i}) \equiv \{ x^i \in \Re^{n_i} : g^i(x^i, x^{-i}) \leq 0 \},$$

where $g^i : \Re^n \to \Re^{m_i}$, where $n \equiv \sum_{j=1}^{N} n_j$.

The KKT conditions of player i's optimization problem:

$$0 = \nabla_{x^i} \theta_i(x) + \sum_{k=1}^{m_i} \lambda_k^i \nabla_{x^i} g_k^i(x)$$

$$0 \leq -g^i(x) \perp \lambda^i \geq 0;$$

concatenation yields a mixed nonlinear complementarity problem in the variables $\left(x^i, \lambda^i\right)_{i=1}^N$.

Note: differentiability is needed of all functions.

KKT Formulation and Nash Equilibrium

• Every MNCP solution is a Nash equilibrium, provided that each $\theta_i(\bullet, \widehat{x}^{-i})$ is convex and so is $g_k^i(\bullet, \widehat{x}^{-i})$ for all $k = 1, \cdots, m_i$.

• Conversely, a Nash equilibrium is an MNCP solution under standard constraint qualifications in nonlinear programming, such as that of Mangasarian-Fromovitz.

• The classical case treated by Rosen (1965) assumed $g^i = g$ for all i, where each component function g_k is convex, and certain proportionality condition on the players' multipliers λ_k^i for the (common) constraints.

The regularized Nikaido-Isoda function

For an arbitrary scalar $c > 0$, define the bivariate function: for $x = \left(x^i\right)_{i=1}^N$ and $y = \left(y^i\right)_{i=1}^N$ both in $\mathcal{X}$,

$$\phi_c(x,y) \equiv \sum_{i=1}^N \left[\theta_i(y^i, x^{-i}) - \theta_i(x^i, x^{-i}) + \frac{c}{2} (\, y^i - x^i \,)^T (\, y^i - x^i \,) \right].$$

The regularized Nikaido-Isoda function is the value function:

$$\boxed{\chi_c(x) \equiv \min_{y \in X(x)} \phi_c(x,y)}, \quad \forall\, x \in \mathcal{X}.$$

Clearly,

$$\chi_c(x) = \sum_{i=1}^N \left\{ \min_{y^i \in X^i(x^{-i})} \left[\theta_i(y^i, x^{-i}) + \frac{c}{2} (\, y^i - x^i \,)^T (\, y^i - x^i \,) \right] - \theta_i(x) \right\}.$$

Optimization Formulation

Proposition. Assume that $X^i(x^{-i})$ is closed convex and $\theta_i(\bullet, x^{-i})$ is convex for every i and every x^{-i}. For any scalar $c > 0$,

(a) $\chi_c(x)$ is a well-defined nonpositive function on the set $\mathcal{X}$;

(b) $\widehat{x}$ is a Nash equilibrium if and only if $\boxed{\widehat{x} \in \operatorname*{\mathbf{argmax}}_{x \in \mathcal{X}} \chi_c(x) \ \textbf{and} \ \chi_c(\widehat{x}) = 0}$;

(c) for every $x \in \mathcal{X}$, a unique $y(x) \equiv (y^i(x))_{i=1}^N \in X(x)$ exists such that $\chi_c(x) = \phi_c(x, y(x))$;

In particular, a vector $x \in \mathcal{X}$ satisfying $|\chi_c(x)| \leq \varepsilon$ for some $\varepsilon > 0$ can be considered an inexact Nash equilibrium.

- In general, $\chi_c(x)$ is not a friendly function to be maximized.

- Furthermore, the maximization problem is non-concave; yet Krawczyk-Uryasev have developed a "relaxation algorithm" and shown convergence under a certain "uniform positive definiteness" condition.

Existence of a Nash equilibrium

The Abstract Case A Nash equilibrium exists if
- each set-valued map X^i is continuous,
- compact convex sets K^i exist such that for every $x^{-i} \in K^{-i}$, $X^i(x^{-i})$ is a nonempty closed convex subset of K^i and $\theta_i(\bullet, x^{-i})$ is convex.

The Jointly Convex Case A Nash equilibrium exists if
- the set $\mathcal{X} = \{ x : x^i \in X^i(x^{-i}) \ \forall i \}$ is compact convex,
- the substitution assumption holds
- each $\theta_i(\bullet, x^{-i})$ is convex and continuously differentiable.

Both results extend the classical Nash existence theorem where each $X^i(x^{-i})$ is a **constant** convex compact set.

A Degree-Theoretic Existence Result

Consider the cone complementarity problem (CCP):

$$C \ni x \perp F(x) \in C^*,$$

where C is a closed convex cone in $\Re^n$ and

$$C^* \equiv \{ y \in \Re^n : y^T x \geq 0, \ \forall x \in C \}$$

is the dual cone of C.

Theorem (Facchinei-Pang) If F is continuous and

$$\sup_{\tau > 0} \sup \{ \| x \| : x \text{ satisfies } C \ni x \perp F(x) + \tau x \in C^* \} < \infty,$$

then the CCP has a solution.

By far the most widely applicable in the absence of boundedness.

Multi-Leader Follower Games

Stackelberg-Nash game (1952)

There are N Nash players whose strategy sets and objective functions are parameterized by a leader's variable z. The leader chooses z to optimize a performance measure, leading to a

Mathematical Program with Equilibrium Constraints

$$\boxed{\begin{array}{ll} \underset{x,z}{\text{minimize}} & \psi(x,z) \\ \text{subject to} & (\,x,z\,) \in \mathcal{Z} \\ \text{and} & x \in \text{NE}(z) \end{array}}$$

Solution existence depends on the closedness of the Nash equilibrium map.

More on the Stackelberg-Nash game

Nonlinear program with complementarity constraints

$$
\begin{array}{ll}
\underset{x,z,\lambda}{\text{minimize}} & \psi(x,z) \\
\text{subject to} & (\,x,z\,) \in \mathcal{Z} \\
\text{and} & \text{for all } i = 1,\cdots,N \\
& 0 = \nabla_{x^i}\,\theta_i(x,z) + \displaystyle\sum_{k=1}^{m_i} \lambda_k^i\,\nabla_{x^i}\,g_k^i(x,z) \\
& 0 \leq -g^i(x,z) \perp \lambda^i \geq 0
\end{array}
$$

- Disjunctive, nonconvex, non-standard first-order optimality conditions
- Albeit no certificate of optimality, NEOS solvers handle complementarity constraints effectively
- Recent study of global solution in the all-affine case.

Multi-Leader Follower Games (cont.)

There are M leaders competing at an upper level, whose strategies $z \equiv (z^\nu)_{\nu=1}^M$ induce a set-valued response NE(z) from N lower-level Nash players.

The overall model is to determine a Nash equilibrium $\hat{z}$ for the leaders, each of whose optimization problem is an MPEC resulting from a Stackelberg game parameterized by the rival leaders' strategies.

> **Open challenge:** Introduce a **sensible** notion of an equilibrium solution and establish its existence under a non-trivial set of **realistic** conditions.

More extensions

• Nash games under uncertainty: each player solves a stochastic program with recourse (Gürkan-Özge-Robinson 1999; Gürkan-Pang 2007)

• Differential Nash games: each player solves an optimal control problem with a differential state equation and constraints on control:

$$\begin{array}{ll} \underset{x^i, u^i}{\text{minimize}} & \theta_i(\boldsymbol{x}, \boldsymbol{u}) \\ \text{subject to} & \text{for almost all } t \in [0, T], \\ & \left\{ \begin{array}{l} \dot{x}^i(t) = g^i(t, x^i(t), u^i(t)) \\ u^i(t) \in U^i \subseteq \Re^{\ell_i}, \end{array} \right\} \\ \text{and} & x^i(0) = x^{0,i} \end{array}$$

Leading to a differential variational inequality in the differential variables $(\boldsymbol{x}, \boldsymbol{\lambda})$ and algebraic variable $\boldsymbol{u}$, where $\boldsymbol{\lambda}$ is the adjoint variable of the players' ODEs.

Concluding Remarks

We have

- introduced the Nash equilibrium
- presented several equivalent formulations
- given some existence theorems, and
- briefly mentioned several extensions.
- Many topics are omitted.

The Variational/Complementarity Approach to Nash Equilibria, part II

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Contents of Presentation

- Applications

— The classical Arrow-Debreu abstract economy

— Spectrum allocation in multiuser communication networks

— Emission permit allocations in electricity markets (among many)

— Integrating queueing delays in supply-chain assembly systems (omitted due to insufficient time)

- Iterative Algorithms

— Distributed optimization

— An illustration

— Sequential penalization

- Concluding remarks

The classical Arrow-Debreu abstract economy

• There are ℓ commodities, m producers, and n consumers.

• Producers maximize their profits equal to revenues less costs subject to production constraints described by the production set $Y^j \subseteq \Re^{m_j}$.

• Consumers maximize their utilities subject to budget and consumption constraints described by the consumption set $X^i \subseteq \Re^{n_i}$.

• A market clearing mechanism ensures market efficiency; i.e., price of a commodity is positive only if production is equal to consumption.

Model variables

p_k : $k = 1, \cdots, \ell$, commodity prices
y_k^j : $k = 1, \cdots, \ell$, $j = 1, \cdots, m$, production quantities
x_k^i : $k = 1, \cdots, \ell$, $i = 1, \cdots, n$, consumption quantities

Model constants

a_k^i : $k = 1, \cdots, \ell$, $i = 1, \cdots, n$, consumers' initial endowments of commodities

α_{ij} : $j = 1, \cdots, m$, $i = 1, \cdots, n$, consumers' shares of producers' revenues

Producer's problem: taking the price p as exogenously given,

$$\boxed{\underset{y^j \in Y^j}{\text{maximize}} \quad p^T y^j - c_j(y^j)}$$

Consumer: taking the price p and consumptions y^j as exogenously given,

$$\boxed{\begin{array}{ll} \underset{x^i \in X^i}{\text{maximize}} & u_i(x^i) \\ \text{subject to} & p^T x^i \leq p^T a^i + \displaystyle\sum_{j=1}^{m} \alpha_{ij}\, p^T y^j \end{array}}$$

Market clearing: with x^i and y^j taken as exogenously given,

$$\boxed{\begin{array}{c} \underset{p \geq 0}{\text{maximize}} \quad p^T \left(\displaystyle\sum_{i=1}^{n} (\, x^i - a^i \,) - \sum_{j=1}^{m} y^j \right) \\ \text{(prices can be normalized: } \displaystyle\sum_{k=1}^{\ell} p_k = 1\text{, if there are no production costs)} \end{array}}$$

Variations of the basic model abound; for example,

• consumers, instead of selfishly optimizing their individual utilities, may determine their consumptions by jointly optimizing their total utilities by solving

$$
\begin{array}{ll}
\underset{\boldsymbol{x}}{\text{maximize}} & \displaystyle\sum_{i=1}^{n} u_i(x^i) \\
\text{subject to} & \text{for all } i = 1, \cdots, n \\
& \left\{ \begin{array}{l} x^i \in X^i \\ p^T x^i \leq p^T a^i + \displaystyle\sum_{j=1}^{m} \alpha_{ij}\, p^T y^j \end{array} \right\}
\end{array}
$$

• alternatively, the market clearing mechanism may determine the price by maximizing a social welfare function, subject to the selfish behavior of the producers and consumers;

• yet a third variation is that the price could be determined by an econometric or market model and is a function of consumer consumptions.

Multiuser communication systems: characteristics

- m users connected to a service provider via frequency-selective channels
- user lines are bundled, causing interferences
- total bandwidth divided into n frequency tones, shared by all users
- each user allocates transmission power to all tones subject to: power budget (min) and achievable information rate (max)
- major channel impediment: crosstalk interference at each tone.

Notation

$p_k^i \geq 0$	user i's power spectrum allocated to tone k
$\sigma_k^i > 0$	background noise of user i's loop at tone k
$\alpha_k^{ij} \geq 0$	crosstalk of frequency tone k between users i and j with $\alpha_k^{ii} > 0$
$P_{\max}^i > 0$	user i's total power
$L_i > 0$	user i's target achievable rate

Information rate: logarithm of signal to noise ratios, summed over tones

$$R_i(p^1, \cdots, p^m) \equiv \sum_{k=1}^{n} \log \left(1 + \frac{\alpha_k^{ii}\, p_k^i}{\sigma_k^i + \sum_{j \neq i} \alpha_k^{ij}\, p_k^j} \right), \quad \text{for user } i.$$

Important consideration: user i can only estimate the rivals' interferences, i.e., the sum $\sum_{j \neq i} \alpha_k^{ij}\, p_k^j$, and has no knowledge of the individual summands.

Therefore, interested in a **distributed algorithm** that requires minimal user coordination, although a benchmark algorithm would be useful too.

Model I: rate maximization with budget constraint

Yu, Ginis, and Cioffi (2002)

Anticipating noises and interferences, user i, selfishly maximizes information rate subject to power budget; i.e., given $p^{-i} \equiv (p^j)_{j \neq i}$,

$$\boxed{\begin{array}{ll} \underset{p^i}{\text{maximize}} & R_i(p^i, p^{-i}) \\ \text{subject to} & p^i_k \geq 0, \quad k = 1, \cdots, n \\ \text{and} & \displaystyle\sum_{k=1}^{n} p^i_k = P^i_{\text{max}} \end{array}}$$

A Nash equilibrium is a tuple $\widehat{p} \equiv (\widehat{p}^{\,i})_{i=1}^m$ such that

$$\widehat{p}^{\,i} \in \text{ argmax of user } i\text{'s problem given } \widehat{p}^{\,j} \text{ for } j \neq i$$

for each $i = 1, \cdots, m$.

Model II: power minimization with rate constraint

ensuring quality of service

Anticipating noises and interferences, user i, selfishly minimizes power budget to ensure achievable rate; i.e., given $p^{-i} \equiv (p^j)_{j \neq i}$,

$$\boxed{\begin{array}{ll} \underset{p^i}{\text{minimize}} & \displaystyle\sum_{k=1}^{n} p_k^i \\ \text{subject to} & p_k^i \geq 0, \quad k = 1, \cdots, n \\ \text{and} & R_i(p^i, p^{-i}) \geq L_i. \end{array}}$$

A Nash equilibrium is similarly defined.

Model I: partitioned constraints, given P^i_{max}

Model II: joint constraints, given L_i.

The Karush-Kuhn-Tucker conditions

Model I: as a linear complementarity problem

$$0 \leq p_k^i \;\perp\; \sigma_k^i + \sum_{j=1}^n \alpha_k^{ij} \, p_k^j - \alpha_k^{ii} \, v_i \geq 0$$

$$0 \leq v_i \qquad \sum_{k=1}^n p_k^i = P_{\max}^i$$

Model II: as a **non**linear complementarity problem

$$0 \leq p_k^i \;\perp\; \sigma_k^i + \sum_{j=1}^n \alpha_k^{ij} \, p_k^j - \alpha_k^{ii} \, \lambda_i \geq 0$$

$$0 \leq \lambda_i \qquad \sum_{k=1}^n \log\left(1 + \frac{\alpha_k^{ii} \, p_k^i}{\sigma_k^i + \sum_{j \neq i} \alpha_k^{ij} \, p_k^j} \right) = L_i.$$

Existence of solutions: Model I

(Luo-Pang 2006): Computable by the finite Lemke algorithm, an equilibrium exists for all $\alpha_k^{ij} \geq 0$ with $\alpha_k^{ii} > 0$ and all $\sigma_k^i > 0$, albeit not necessarily unique. $\square$

The tone matrices: $M_k \equiv \left[\alpha_k^{ij}\right]_{i,j=1}^n$, $k = 1, \cdots, n$.

The normalized max-interference matrix

$$B \equiv \begin{bmatrix} 1 & \beta_{\max}^{12} & \cdots & \beta_{\max}^{1m} \\ \beta_{\max}^{21} & 1 & \cdots & \beta_{\max}^{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{\max}^{m1} & \beta_{\max}^{m2} & \cdots & 1 \end{bmatrix},$$

where $\beta_{\max}^{ij} \equiv \max\limits_{1 \leq k \leq n} \alpha_k^{ij} / \alpha_k^{ii} \quad i \neq j.$

Solution uniqueness: Model I

(Luo-Pang 2006) Nash equilibrium is unique if either

- each tone matrix M_k is positive definite, or
- B is an H-matrix;
- most generally, if $\max_{1 \le i \le m} \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha_k^{ij} \, p_k^i \, p_k^j > 0$, for all $\left(p^i \right)_{i=1}^{m} \neq 0$.

Many equivalent descriptions of an H-matrix, e.g.,

- $\text{Diag}(B)^{-1}\text{off-Diag}(B)$ has spectral radius less than 1, or
- $\text{Diag}(B) - \text{off-Diag}(B)$ has positive principal minors, or
- B is strictly (quasi-)diagonally dominant; e.g., $\max_{1 \le i \le m} \sum_{j \ne i} \beta_{\max}^{ij} < 1$.

Existence of solutions: Model II

Definition. A power tuple $p \equiv (p^i)_{i=1}^m$ is a noiseless equilibrium if $v_i \geq 0$ exists such that

$$\mathcal{NE}_0 : \quad 0 \leq p_k^i \perp \sum_{j=1}^n \alpha_k^{ij} p_k^j - \alpha_k^{ii} v_i \geq 0.$$

The noiseless asymptotical cone:

$$\widehat{\mathcal{NE}}_0(\mathbf{L}) \equiv \left\{ \mathbf{q} \in \mathcal{NE}_0 \setminus \{ 0 \} : \right.$$
$$\left. \sum_{k=1}^n \log \left(1 + \frac{\alpha_k^{ii} q_k^i}{\sum_{j \neq i} \alpha_k^{ij} q_k^j} \right) \leq L_i, \; i = 1, \cdots, m \right\}.$$

Main result. An equilibrium exists for all $\sigma_k^i > 0$ if $\widehat{\mathcal{NE}}_0(\mathbf{L}) = \emptyset$.

— Proof by a degree-theoretic argument.

A matrix criterion

Define the nonnegative matrix $\mathbf{Z}_{\text{max}}$:

$$\begin{bmatrix} 1 & (e^{L_1}-1)\,\beta^{12}_{\text{max}} & \cdots & (e^{L_1}-1)\beta^{1m}_{\text{max}} \\ (e^{L_2}-1)\,\beta^{21}_{\text{max}} & 1 & \cdots & (e^{L_2}-1)\,\beta^{2m}_{\text{max}} \\ \vdots & \vdots & \ddots & \vdots \\ (e^{L_m}-1)\,\beta^{m1}_{\text{max}} & (e^{L_m}-1)\,\beta^{m2}_{\text{max}} & \cdots & 1 \end{bmatrix}.$$

Corollary. An equilibrium exists for all $\sigma_k^i > 0$ if $\mathbf{Z}_{\text{max}}$ is an H-matrix. In particular, this holds if

$$\chi \equiv 1 - \max_{1\le i\le m}\left[(e^{L_i}-1)\sum_{j\neq i}\beta^{ij}_{\text{max}}\right] > 0,$$

ensuring strict diagonal dominance of $\mathbf{Z}_{\text{max}}$.

Uniqueness can be established under a more restrictive condition.

Comparisons of results

Model I	Model II
• power budget restriction	• quality of service
• partitioned constraints	• joint constraints
• essentially a linear problem	• a nonlinear problem
• existence independent of crosstalk coefficients and power budgets	• existence dependent on crosstalk coefficients and achievable rates
• solvable by a finite algorithm	• no finite algorithm is known
• uniqueness independent of noises	• uniqueness dependent on ratios of noises
• admits a single optimization formulation with symmetric crosstalk	• no such formulation is known

Emissions Permit Allocation in Electric Markets

• Pollutant emission cap-and-trade systems existed since the 1990 Clean Act Amendments for SO_2 in the US, and later for NO_x and mercury.

• Recent emissions trading systems for greenhouse gas CO_2 by the European Union, which are expected to have much larger economic impacts with the potential of distorting market efficiency.

• Study the long-run effect of CO_2 permit allocation schemes on market efficiency, including generator investment and operation decisions and consumer prices, using complementarity modeling.

• Alternative emissions allocation rules: mixtures of

— grandfathering: initial allocation based on historical benchmark

— contingent allocation: depending on future input and output decisions.

Characteristics of model

- Allowing minimum output constraints
- Capacity markets in addition to energy markets
- Arbitrary temporal price-sensitive demand distribution
- Price-taking and/or price-participating firms
- Endogenous allocation allowances
- Some refinements are straightforward.

Notations

Parameters: all nonnegative

$\mathcal{F}$	Set of firms
$\mathcal{T}$	Set of time periods $\equiv \{1, \cdots, T\}$
CAP_f	Minimal amount of energy that firm f has to generate (MW)
MC_f	Marginal cost for firm f, excluding cost of emission allowances (EURO/MWh)
E_f	Emission rate for firm f (tons/MWh)
F_f	Annualized investment cost of firm f's capacity (EURO/MWyr)
R_f	Fraction of emission allowance for firm $f \neq 1$, normalized with respect to firm 1
$\widehat{R}_f$	Proportion of sales-based emission allowance for firm f
$\underline{\text{CAP}}$	Total capacity requirement (MW)
H_t	Time converter (hr/yr)
$\overline{E}$	Total emission allowances supply (tons/yr): $\overline{E} > E_{\text{GF}}$
E_{GF}	Amount of emission allowances grandfathered (tons/yr)
K	Unit converter = 1 MW^2 yr/EURO

Functions:

$d_t(\cdot)$	Demand function, strictly monotonically decreasing (MW)
$\phi_t(\cdot)$	The inverse of $d_t(\cdot)$; (EURO/MWh)
$e_{\mathrm{N}P}(\cdot)$	Nonpower emission, nonincreasing (tons/yr)

Variables:

p_t	Energy price during period t (EURO/MWh): function of total sales $\sum_{g \in \mathcal{F}} s_{gt}$
pe	Emission allowance price (EURO/ton)
pcap	Capacity price(EURO/MWyr)
α_f	Emission allowance for firm f (tons/MWyr)
s_{ft}	Energy sold by firm f in period t (MW)
$\overline{s}_{ft}$	$= s_{ft} - \mathrm{CAP}_f$ (MW)
$\mathrm{cap_f}$	Capacity for firm f (MW)
μ_{ft}	Dual variable associated with firm f's capacity constraint in period t (EURO/MWyr)

Firm f's profit maximization problem

Anticipating prices p_t^* and pe^* and rival firms' cap_g for all $g \neq f$,

$$\boxed{\begin{array}{ll} \underset{\mathsf{cap}_f,\,(s_{ft})_{t\in\mathcal{T}}}{\text{maximize}} & \displaystyle\sum_{t\in\mathcal{T}} H_t\,(\,p_t^* - MC_{ft} - pe^* E_f\,)s_{ft} + (\,pe^*\alpha_f^* - F_f\,)\,\mathsf{cap}_f \\ \text{subject to} & \mathsf{CAP}_f \leq s_{ft} \leq \mathsf{cap}_f, \quad \forall\, t \in \mathcal{T} \\ \text{and} & \mathsf{cap}_f + \displaystyle\sum_{\substack{g\in\mathcal{F} \\ g\neq f}} \mathsf{cap}_g \geq \underline{\mathsf{CAP}} \text{ a common joint constraint} \end{array}}$$

When firms exert market power, firm's revenue from energy sales becomes

$$\sum_{t\in\mathcal{T}} H_t\, s_{ft}\, p_t\left(\sum_{g\in\mathcal{F}} s_{gt}\right).$$

Emission and capacity markets

Allowance price is positive only when demands for allowances equal supplies:

$$\boxed{0 \leq pe \perp e_{\mathsf{N}P}(pe) - \left(\overline{E} - \sum_{g \in \mathcal{F}} \sum_{t \in \mathcal{T}} H_t \, E_g \, s_{gt}\right) \geq 0} \, .$$

Capacity price is positive only when demands for capacity equal available capacity:

$$\boxed{0 \leq \text{pcap} \perp \sum_{f \in \mathcal{F}} \text{cap}_f - \underline{\text{CAP}} \geq 0} \, ,$$

implying common multipliers for joint capacity constraint.

Market clearing conditions and emission rules

- Supplies balancing demands: $\sum_{f \in \mathcal{F}} s_{ft} = d_t(p_t^*)$, for all $t \in \mathcal{T}$
- Balance of emissions allowances: $\sum_{f \in \mathcal{F}} \alpha_f^* \text{cap}_f = \overline{E} - E_{\text{G}F}$.

Input-based rule: $\dfrac{\alpha_f^*}{\alpha_1^*} = R_f > 0$, for all $f \in \mathcal{F}$;

An output-based rule: $\alpha_f \text{cap}_f = \dfrac{\sum_{t \in \mathcal{T}} H_t \, E_f \, s_{ft}}{\sum_{(g,t) \in \mathcal{F} \times \mathcal{T}} H_g \, E_g \, s_{gt}} (\overline{E} - E_{\text{G}F})$;

A general output-based rule: $\alpha_f \, \text{cap}_f = \sigma \, \widehat{R}_f \sum_{t \in \mathcal{T}} H_t \, E_f \, s_{ft}$.

"Fairness" of allocation rules

Input-based: $\alpha_1 = \dfrac{\overline{E} - E_{\mathsf{G}F}}{\sum_{g \in \mathcal{F}} R_g \mathsf{cap}_g}$ if denominator is positive; yielding

$$\boxed{\alpha_f \mathsf{cap}_f = \frac{R_f \mathsf{cap}_f}{\sum_{g \in \mathcal{F}} R_g \mathsf{cap}_g} (\overline{E} - E_{\mathsf{G}F})}$$ capacity-based allocation.

Output-based: $$\boxed{\alpha_f \mathsf{cap}_f = \frac{\sum_{t \in \mathcal{T}} H_t \, E_f \, s_{ft}}{\sum_{(g,t) \in \mathcal{F} \times \mathcal{T}} H_g \, E_g \, s_{gt}} (\overline{E} - E_{\mathsf{G}F})}$$ sales-based.

Generalized sales-based: $$\boxed{\frac{\overline{E} - E_{\mathsf{G}F}}{\sum_{(g,t) \in \mathcal{F} \times \mathcal{T}} H_g \, E_g \, s_{gt}} \longrightarrow \sigma \, \widehat{R}_f}$$

The Nonlinear Complementarity Formulation

Capacity-based allocation rule:

$$0 \leq \bar{s}_{ft} \quad \perp \quad H_t \left[-\phi_t \left(\sum_{g \in \mathcal{F}} (\bar{s}_{gt} + \mathsf{CAP}_g) \right) + \mathsf{MC}_f + pe\, E_f \right] + \mu_{ft} \geq 0,$$

$$\forall\, (f, t) \in \mathcal{F} \times \mathcal{T}$$

$$0 \leq \mu_{ft} \quad \perp \quad \mathsf{cap}_f - \bar{s}_{ft} - \mathsf{CAP}_f \geq 0, \;\; \forall\, f \in \mathcal{F};\; t \in \mathcal{T}$$

$$0 \leq \mathsf{cap}_f \quad \perp \quad -\mathsf{pcap} - R_f\, \sigma + F_f - \sum_{t \in \mathcal{T}} \mu_{ft} \geq 0, \;\; \forall\, f \in \mathcal{F}$$

$$0 \leq pe \quad \perp \quad \bar{E} - \sum_{g \in \mathcal{F}} \sum_{t \in \mathcal{T}} H_t\, E_g\, (\bar{s}_{gt} + \mathsf{CAP}_g) - e_{\mathsf{N}P}(pe) \geq 0$$

$$0 \leq \mathsf{pcap} \quad \perp \quad \sum_{g \in \mathcal{F}} \mathsf{cap}_g - \underline{\mathsf{CAP}} \geq 0$$

$$0 \leq \sigma \quad \perp \quad \sigma \sum_{g \in \mathcal{F}} R_g\, \mathsf{cap}_g - (\overline{E} - E_{\mathsf{G}F})\, pe \geq 0.$$

The equivalent variational inequality

Find $\bar{\mathbf{x}} \in K$ such that $(\mathbf{x} - \bar{\mathbf{x}})^T \Phi(\bar{\mathbf{x}}) \geq 0$ for all $\mathbf{x} \in K$, where Φ is non-monotone and K is unbounded:

$$\Phi(\bar{\mathbf{s}}, \mathbf{cap}, pe) \equiv$$

$$\left(\begin{array}{c} \left(H_t \left[-\phi_t \left(\sum_{g \in \mathcal{F}} (\, \bar{s}_{gt} - \mathsf{CAP}_g \,) \right) + \mathsf{MC}_f + pe\, E_f \right] \right)_{(f,t) \in \mathcal{F} \times \mathcal{T}} \\ \mathbf{F} - \dfrac{(\, \overline{E} - E_{\mathsf{G}F} \,)\, pe}{\displaystyle\sum_{g \in \mathcal{F}} R_g\, \mathsf{cap}_g} \mathbf{R} \\ \overline{E} - \displaystyle\sum_{(g,t) \in \mathcal{F} \times \mathcal{T}} H_t\, E_g\, (\, \bar{s}_{gt} - \mathsf{CAP}_g \,) - e_{\mathsf{N}P}(pe) \end{array} \right)$$

$$\text{and } K \equiv \left\{ \; (\, \bar{\mathbf{s}}, \mathbf{cap} \,) \geq 0 \; : \; \sum_{g \in \mathcal{F}} \mathsf{cap}_g - \underline{CAP} \geq 0 \right.$$

$$\left. \mathsf{cap}_f - \bar{s}_{ft} \geq 0, \;\; \forall\, (\, f, t \,) \in \mathcal{F} \times \mathcal{T} \right\} \times \mathcal{R}_+.$$

The sales-based allocation formulation

$$0 \le \bar{s}_{ft} \quad \perp \quad H_t \left[-\phi_t \left(\sum_{g \in \mathcal{F}} (\, \bar{s}_{gt} + \mathsf{CAP}_g \,) \right) + \mathsf{MC}_f + pe\, E_f \right] + \mu_{ft} \ge 0, \quad \forall\, (\, f, t\,) \in \mathcal{F} \times \mathcal{T}$$

$$0 \le \mu_{ft} \quad \perp \quad \mathsf{cap}_f - \bar{s}_{ft} - \mathsf{CAP}_f \ge 0, \;\; \forall\, f \in \mathcal{F};\; t \in \mathcal{T}$$

$$0 \le \mathsf{cap}_f \quad \perp \quad -\mathsf{pcap} - \alpha_f\, pe + F_f - \sum_{t \in \mathcal{T}} \mu_{ft} \ge 0, \;\; \forall\, f \in \mathcal{F}$$

$$0 \le \alpha_f \quad \perp \quad \alpha_f\, \mathsf{cap}_f - \sigma\, \widehat{R}_f \sum_{t \in \mathcal{T}} H_t\, E_f\, \bar{s}_{ft} \ge 0, \;\; \forall f \in \mathcal{F}$$

$$0 \le pe \quad \perp \quad \overline{E} - \sum_{g \in \mathcal{F}} \sum_{t \in \mathcal{T}} H_t\, E_g\, (\, \bar{s}_{gt} + \mathsf{CAP}_g \,) - e_{\mathsf{N}P}(pe) \ge 0$$

$$0 \le \mathsf{pcap} \quad \perp \quad \sum_{g \in \mathcal{F}} \mathsf{cap}_g - \underline{\mathsf{CAP}} \ge 0$$

$$0 \le \sigma \quad \perp \quad \sum_{f \in \mathcal{F}} \alpha_f\, \mathsf{cap}_f - (\, \overline{E} - E_{\mathsf{G}F} \,) \ge 0$$

Iterative Algorithms

Approach I : Distributed optimization

Player i's optimization problem

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & \theta_i(x^i, x^{-i}) \\ \text{subject to} & x^i \in X^i(x^{-i}) \end{array}$$

A Jacobi iterative scheme.

At iteration ν, given $\boldsymbol{x}^\nu \equiv \left(x^{\nu,i}\right)_{i=1}^N$, compute $\boldsymbol{x}^{\nu+1} \equiv \left(x^{\nu+1,i}\right)_{i=1}^N$ by solving, for $i = 1, \cdots, N$,

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & \theta_i(x^i, x^{\nu,-i}) \\ \text{subject to} & x^i \in X^i(x^{\nu,-i}) \end{array}$$

Convergence has not been fully investigated in general.

Approach I: Illustration

$$\begin{array}{ll} \underset{p^i \geq 0}{\text{minimize}} & \sum_{k=1}^{n} p_k^i \\ \text{subject to} & \sum_{k=1}^{n} \log\left(1 + \alpha_k^{ii}\, p_k^i / \tau_k^i\right) \geq L_i \\ \text{where} & \tau_k^i \equiv \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij}\, p_k^j \end{array}$$

At iteration ν, given are, for all $i = 1, \cdots, m$ and $k = 1, \cdots, n$,

$$\tau_k^{\nu,i} \equiv \sigma_k^i + \sum_{j \neq i} \alpha_k^{ij}\, p_k^{\nu,j},$$

user i computes $p^{\nu+1,i} = \left(p_k^{\nu+1,i}\right)_{k=1}^{n}$ to satisfy

$$0 \leq p_k^{\nu+1,i} \perp \tau_k^{\nu,i} + \alpha_k^{ii}\, p_k^{\nu+1,i} - \alpha_k^{ii}\, \lambda_i^{\nu+1} \geq 0$$

$$\sum_{k=1}^{n} \log\left(1 + p_k^{\nu+1,i} / \tau_k^{\nu,i}\right) = L_i.$$

• Solve, via sorting, the univariate piecewise smooth equation for $\lambda_i^{\nu+1}$:

$$\boxed{\sum_{k=1}^{n} \underline{\log} \max\left(\lambda_i^{\nu+1},\ \tau_k^{\nu,i}/\alpha_k^{ii}\right) = L_i - \sum_{k=1}^{n} \log(\tau_k^{\nu,i}/\alpha_k^{ii})}$$

• set $\boxed{p_k^{\nu+1,i} \equiv \max(0,\ \lambda_i^{\nu+1} - \tau_k^{\nu,i}/\alpha_k^{ii})}$

• Sufficient convergence can be established under the same conditions for solution uniqueness.

• In practice, convergence is very fast.

Iterative Algorithms

Approach II : Sequential (Cartesian) Nash via penalization

Player i's optimization problem

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & \theta_i(x^i, x^{-i}) \\ \text{subject to} & g^i(x^i, x^{-i}) \leq 0, \;\; h^i(x^i) \leq 0 \end{array}$$

Let $\{\rho_\nu\}$ be a sequence of positive scalars satisfying $\rho_\nu < \rho_{\nu+1}$ and tending to ∞. Let $\{u^\nu\}$ be a given sequence of vectors.

At iteration ν, given $\boldsymbol{x}^\nu \equiv \left(x^{\nu,i}\right)_{i=1}^N$, compute $\boldsymbol{x}^{\nu+1} \equiv \left(x^{\nu+1,i}\right)_{i=1}^N$ as an equilibrium solution to a Nash subproblem, wherein player i's problem is

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & \theta_i(x^i, x^{-i}) + \dfrac{1}{2\,\rho_\nu} \displaystyle\sum_{k=1}^{m_i} \max(0, u_k^{\nu,i} + \rho_i g_k^i(x^i, x^{-i}))^2 \\ \text{subject to} & h^i(x^i) \leq 0. \end{array}$$

Alternatively,

$$\begin{array}{ll} \text{minimize} & \theta_i(x^i, x^{-i}) + \dfrac{1}{\rho_\nu} \displaystyle\sum_{k=1}^{m_i} u_k^{\nu,i} \exp\left(\, \rho_\nu g_i^i(x^i, x^{-i}) \,\right) \\ \text{subject to} & h^i(x^i) \leq 0, \end{array}$$

Concluding Remarks

• Applications of Nash equilibria abound in communication networks, electricity markets, supply chain systems, and other contexts.

• Most of these are complex and of large-scale.

• Variational and complementarity formulations offer a mathematically viable framework for the rigorous analysis and computational solution of these games.