The Variational/Complementarity Approach to Nash Equilibria

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Extended Abstract

Part I: Fundamentals. In the first part of this two-part lecture, we present the basic theory of Nash equilibria based on the variational/complementarity approach. We begin with a formal mathematical definition of a non-cooperative game and the central solution concept of a Nash equilibrium. Distinction is made between the standard version of the game where only the objective function of each player, and not her feasible set, is affected by the rival players' strategies, and the generalized version of the game where both the objective and the constraints of each player are dependent on all players' strategies. The remaining presentation focuses on 3 main topics: equivalent formulations, existence results, and extended games.

(a) After discussing two special games with affine structure and their solution by Lemke's complementary pivot algorithm, namely, the 2-person nonzero-sum matrix game and its n-person polymatrix game extension, we present equivalent formulations of the (generalized) Nash game as (i) a (quasi-)variational inequality, (ii) a mixed nonlinear complementarity problem, and (iii) a constrained optimization problem. We also introduce the penalization idea to convert a generalized Nash game into a sequence of standard games whereby the coupled constraints moved to the players' objectives.

(b) Each of the equivalent formulations is important in its own right. For the standard game, the variational inequality formulation allows a straightforward application of Kakutani's fixed-point theorem to yield the existence of a Nash equilibrium under appropriate convexity and compactness assumptions. The complementarity formulation enables the application of a degree-theoretic result that is the basis for proving the existence of an equilibrium solution for many realistic generalized games without the boundedness assumption; this formulation is also the basis for the application of the powerful PATH solver for solving generalized Nash games that is publicly available on the NEOS website. The constrained optimization formulation dates back to a 1955 paper by the economists Nikaido and Isoda and has been used by many contemporary authors for various purposes.

(c) We introduce several extensions of the basic Nash games: the Stackelberg game and its multi-leader extension, the open-loop differential Nash game, and a two-stage game with stochastic recourse. The Stackelberg game leads to the class of Mathematical Programs with Equilibrium Problems for which there have been extensive research and much algorithmic advances in recent years. The multi-leader-follower game is largely un-explored; there is even a lack of a fundamental existence theory. The treatment of the differential Nash game requires the methodology of *differential variational inequality*, which is a novel class of differential-algebraic problems defined by ordinary differential equations coupled with finite-dimensional variational inequalities. The investigation of Nash equilibria in the presence of uncertainty is very much in an infantile stage and promises a vast opportunity for future research.

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Part II: Applications and Distributed Algorithms. The second part of our presentation will begin with a formulation of the celebrated classic abstract economy model of Arrow and Debreu as a pioneering example of a generalized Nash equilibrium problem. We will then discuss some contemporary applications of this problem in three areas: electricity markets, power allocation in communication network, and supply-chain assembly systems, within each of which a representative model is presented and the existence of an equilibrium solution is established. Time permitting, we will also discuss the design and convergence analysis of distributed algorithms, which are inspired by the treatment of communication networks wherein the users have access to only local information and there is lack of central coordination.

(a) A recent practical issue of debate in deregulated electricity markets is the design of carbon dioxide trading systems for the allocation of emissions allowances, which includes auction, giving away fixed amounts, or by allocating based on output, fuel, or other decisions. The latter allocation system can bias investment, operations, and pricing decisions, and increase costs relative to other systems. A nonlinear complementarity model is used to investigate long-run equilibria that would result under alternative systems for power markets characterized by time varying demand and multiple generation technologies.

(b) We present a generalized Nash equilibrium model for the minimization of transmit power in Gaussian parallel interference channels, subject to a rate constraint for each user. The latter constraint adds much complication to the analysis of the model. Via a degree-theoretic approach, we establish the existence of a Nash equilibrium based on a nonlinear complementarity formulation. A sufficient condition for the uniqueness of such an equilibrium turns out to be also sufficient for the convergence of a distributed algorithm for computing the equilibrium.

(c) We consider a simple stochastic, decentralized supply-assembly system consisting of two suppliers of two complementary components for assembly into a final product by an assembler. The suppliers and the assembler constitute the three players in a non-cooperative game, who make optimal decisions regarding investments on capacities and inventories subject to a common product throughput constraint. Queuing approximations for the relevant performance measures are integrated into the players' optimization problems. The resulting Nash equilibrium model is of the generalized type wherein all the players' decision variables appear in the throughput formula. Again, existence of an equilibrium is established by the degree-theoretic result.

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