Parameterized integer programming, part I and II

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Abstract

Central to this talk is the following *parametric integer linear programming* problem:

> Given a rational matrix $A \in \mathbb{Q}^{m \times n}$ and a rational polyhedron Q in the Euclidean space $\mathbb{R}^{m+p}$, decide if for all $b \in \mathbb{R}^m$, for which there exists an integral $z \in \mathbb{Z}^p$ such that $(b, z) \in Q$, the system of linear inequalities $Ax \leq b$ has an integral solution.

In other words, we need to check that for all vectors b in the set

$$Q/\mathbb{Z}^p \colon = \left\{ b \in \mathbb{Q}^m : (b, z) \in Q \text{ for some } z \in \mathbb{Z}^p \right\}$$

the corresponding *integer linear programming* problem $Ax \leq b$, $x \in \mathbb{Z}^n$ has a feasible solution. The set $Q/\mathbb{Z}^p$ is called the *integer projection* of Q. Using this notation, we can reformulate PILP as the problem of testing the following sentence

$$\forall b \in Q/\mathbb{Z}^p \quad \exists x \in \mathbb{Z}^n : \quad Ax \leq b. \tag{1}$$

We restrict ourselves to the case when n and p in (1) are fixed. In the talk we will see a polynomial algorithm to solve PILP in this case.

We begin with a review of the first algorithm for integer linear programming in fixed dimension of Lenstra [1]. Then we move to the study of parametric integer linear programming, which was pioneered by Kannan [2, 3]. In its general form, PILP belongs to the second level of the polynomial hierarchy and is Π_2^p-complete. Kannan presented a polynomial algorithm to decide the sentence (1) in the case when n, p and the affine dimension of Q are fixed. This result was applied to deduce a polynomial algorithm that solves the Frobenius problem when the number of input integers is fixed. We present an extension of Kannan's algorithm which runs in polynomial time under the assumption that only n and p are fixed. The presented results are joint work with Gennady Shmonin.

References

[1] Lenstra, Jr., H.W.: *Integer programming with a fixed number of variables*, Mathematics of Operations Research 8 (1983) 538–548.

[2] Kannan, R.: *Test sets for integer programs, ∀ ∃ sentences*, in: W.J. Cook and P.D. Seymour (eds.), *Polyhedral Combinatorics*, Volume 1 of DIMACS Series in Discrete Mathematics and Theoretical Computer Science pp. 39–47. American Mathematical Society, Providence, RI. Proceedings of a workshop on polyhedral combinatorics, held in the Headquarters Plaza Hotel, Morristown, NJ, June 12–16, 1989.

[3] Kannan, R.: *Lattice translates of a polytope and the Frobenius prolem*, Combinatorica 12 (1992) 161–177.