Ambiguity, Variability, and Robustness

and Their Role in Decision Making

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Newsboy, Uncertainty, and Sensitivity

Let us consider the standard newsboy problem.

Wholesale price from the publisher: \$2 per piece

Retailer price on street: \$5 per piece

Unsold newspaper return to the wholesaler: \$1 per piece

Two scenarios:

- Good day: one can sell 100 pieces;
- Bad day: one can sell 50 pieces.

What is the optimal strategy of the newsboy?

Standard Stochastic Programming Formulation

The Stochastic Program Formulation:

$$
\begin{array}{llll}
(NB) & \text{minimize} & 2x & + \quad \mathsf{E}\left[-5y_\omega - z_\omega\right] \\
 & \text{subject to} & x \geq 0 & \\
 & & & y_\omega \leq \omega \\
 & & & y_\omega + z_\omega = x \\
 & & & y_\omega \geq 0,\ z_\omega \geq 0.
\end{array}
$$

Linear Programming Resolution

Let

p_1 be the probability of Good Day

p_2 be the probability of Bad Day

$$
\begin{aligned}
\min \quad & 2x \quad + p_1(-5y_1 - z_1) + p_2(-5y_2 - z_2) \\
\text{s.t.} \quad & x \geq 0 \\
& y_1 \leq 100 \\
& y_1 + z_1 = x \\
& y_1 \geq 0,\ z_1 \geq 0 \\
& y_2 \leq 50 \\
& y_2 + z_2 = x \\
& y_2 \geq 0,\ z_2 \geq 0.
\end{aligned}
$$

What to Do According to the Model?

If $(p_1, p_2) = (0.3, 0.7)$ then $x^* = 100;$

If $(p_1, p_2) = (0.2, 0.8)$ then $x^* = 50;$

If $(p_1, p_2) = (0.25, 0.75)$ then $x^* = 68.2776.$

Airline Revenue Management: Another Case Study

Single-leg flight: Static and Deterministic Model

– Flight capacity: C

– Fare classes: $i = 1, ..., m$ each with price r_i

– Demand for fare class i: d_i

The model:

$$
\begin{array}{rll}
v_1(C) := & \max & \sum_{i=1}^m r_i \min\{x_i, d_i\} \\
& \text{s.t.} & \sum_{i=1}^m x_i \le C, \\
& & x \in \mathcal{Z}_+^m,
\end{array}
$$

Single-leg flight: Static and Stochastic Model

$$
\begin{array}{rll}
v_2(C) := & \max & \sum_{i=1}^{m} r_i \mathsf{E}\left(\min\{x_i, D_i\}\right) \\
 & \text{s.t.} & \sum_{i=1}^{m} x_i \le C, \\
 & & x \in \mathcal{Z}_+^m .
\end{array}
$$

The computation can be done using the recursive formula

$$
R_p(y) = \max_{0 \le x_p \le y} \left\{ R_{p+1}(y - x_p) + r_p \mathsf{E}(\min\{x_p, D_p\}) \right\}.
$$

where

$$
R_p(y) = \max \left\{ \sum_{i=p}^{m} r_i \mathsf{E}(\min\{x_i, D_i\}) \,\middle|\, \sum_{i=p}^{m} x_i \le y, x_i \in \mathcal{Z}, i = p, ..., m \right\}.
$$

A Robust Model

Assume that random variable D_i, representing the total demand for fare class i, is concentrated on $\{0, \cdots, K\}$, and this demand has an estimated probability vector $\hat{p}_i = (\hat{p}_{i0}, \cdots, \hat{p}_{iK})$.

The true probability vectors p_i is in the ambiguity set P_i:

$$P_i = \{p_i \in \Re^{K+1} \mid p_i \in \hat{p}_i + \Delta_i,\ p_i^T e = 1\},$$

where

$$\Delta_i = \left\{ d_i = (d_{i0}, \cdots, d_{iK})^T \in \Re^{K+1} \,\middle|\, \sum_{k=0}^{K} \left(\frac{d_{ik}}{\hat{p}_{ik}} \right)^2 \leq \delta_i^2 \right\}$$

with $\delta_i \in [0, 1]$.

The robust model is

$$\begin{array}{rll} v_3(C) := & \max & \sum_{i=1}^m r_i \min_{p_i \in P_i} \{\mathsf{E}\,(\min\{x_i, D_i(p_i)\})\} \\ & \text{s.t.} & \sum_{i=1}^m x_i \le C, \\ & & x \in \mathcal{Z}_+^m. \end{array}$$

Let

$$G_i(x_i) = \min_{p_i \in P_i} \{\mathsf{E}\,(\min\{x_i, D_i(p_i)\})\}$$

and one can calculate that

$$G_i(x_i) = c(x_i)^T \hat{p}_i - \delta_i \sqrt{\sum_{k=1}^K \hat{p}_{ik}^2 c_k^2(x_i) - \frac{(\sum_{k=1}^K \hat{p}_{ik}^2 c_k(x_i))^2}{\sum_{k=0}^K \hat{p}_{ik}^2}}.$$

where

$$c(x_i)^T := (0, 1, \cdots, x_i - 1, x_i, x_i, \cdots, x_i).$$

A Dynamic Model

Let ξ_t denote the random demand in period t.

Assume that ξ_t may take $m+1$ different values $r_0, r_1, ..., r_m$ and its discrete density is given by

$$\text{Prob}\,\{\xi_t = r_i\} = p_{it}$$

with $i = 0, 1, ..., m$ and $t = 1, ..., T$.

Let $R_t(z)$ be the revenue generated from period t to T, before a request shows up in period t, while the number of available seats at the beginning of period t is z.

Let $J_t(z) := \mathsf{E}(R_t(z))$.

The Lautenbacher and Stidham Model

The dynamic programming formula is

$$J_t(z) = \mathsf{E}\left(\max\{\xi_t + J_{t+1}(z-1), J_{t+1}(z)\}\right),$$

with

$$J_T(z) = \begin{cases} \mathsf{E}(\xi_T), & \text{if } z > 0 \\ 0, & \text{if } z = 0. \end{cases}$$

Let

$$\Delta_{t+1}(z) := J_{t+1}(z) - J_{t+1}(z-1)$$

which can be shown to be nonnegative and non-increasing in z.

We then have

$$J_t(z) = J_{t+1}(z) + \mathsf{E}\left(\max\{\xi_t - \Delta_{t+1}(z), 0\}\right),$$

or specifically,

$$\boxed{J_t(z) = J_{t+1}(z) + \sum_{i=1}^{m} p_{it}(r_i - \Delta_{t+1}(z))_+}$$

Robust Dynamic Model

Under the same ambiguity assumption on the probabilities:

$$J_t(z) = J_{t+1}(z) + \sum_{i=1}^{m} \widehat{p}_{it}(r_i - (J_{t+1}(z) - J_{t+1}(z-1))_+ + H_t(z)$$

with

$$H_t(z) = -\delta_t \sqrt{\sum_{i=1}^{m} \hat{p}_{it}^2 c_{it}^2 - \frac{\left(\sum_{i=1}^{m} \hat{p}_{it}^2 c_{it}\right)^2}{\sum_{i=1}^{m} \hat{p}_{it}^2}},$$

where

$$c_{it} := (r_i - (J_{t+1}(z) - J_{t+1}(z-1)))_+, \quad i = 1, ..., m.$$

Simulation Results

The characteristic of a solution for the robust model is not being conservative: *it immunizes the variability from ambiguous data!*

The parameters in simulation (the static part):

Parameters	**Values**
$[M, N, K, C, m]$	[25, 250, 100, 100, 4]
(r_1, r_2, r_3, r_4)	(2, 3, 4, 6)
$(\kappa_1, \kappa_2, \kappa_3, \kappa_4)$	(40, 20, 10, 1)
$(\mu_1, \mu_2, \mu_3, \mu_4)$	(70, 40, 30, 10)

We use the truncated (by K) Poisson distributions to model the demands, with the rate λ_i being uniform in $[\kappa_i, \mu_i]$, $i = 1, 2, 3, 4$.

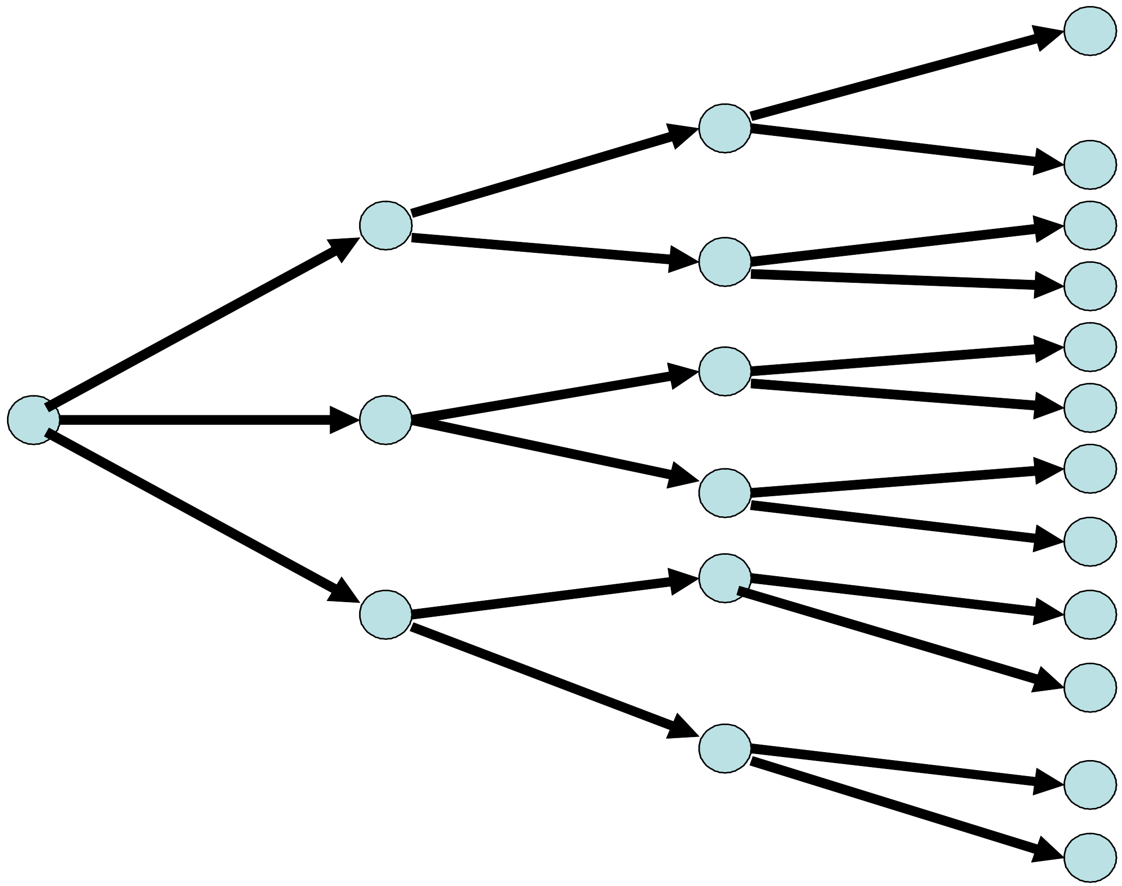
	Mean			**Standard Deviation**		
Run	$\mathbf{R}^{(a)}$	$\mathbf{Non\text{-}R}^{(b)}$	$(b-a)/b$	$\mathbf{R}^{(c)}$	$\mathbf{Non\text{-}R}^{(d)}$	$(d-c)/d$
1	259.9800	260.0200	0.0154%	18.0090	18.7750	4.0777%
2	275.6000	276.5600	0.3500%	12.8040	14.9030	14.0810%
3	277.1200	277.7400	0.2218%	11.9320	14.0740	15.2160%
4	287.7000	288.2400	0.1887%	13.1010	15.2410	14.0350%
5	283.8500	284.5100	0.2334%	13.1380	15.3610	14.4730%
6	299.5600	299.7600	0.0681%	17.5140	17.6740	0.9024%
7	304.3500	305.3700	0.3340%	16.9290	19.6080	13.6630%
8	285.9600	286.3300	0.1313%	13.2190	15.7330	15.9770%
9	289.0400	289.6900	0.2237%	15.5990	18.5220	15.7780%
10	268.0600	268.1600	0.0403%	15.2080	15.5560	2.2364%
11	291.6600	292.1000	0.1506%	14.8070	17.3390	14.6020%
12	261.2900	261.4400	0.0581%	15.1350	15.4880	2.2761%

	Mean			**Standard Deviation**		
Run	$\mathbf{R}^{(a)}$	**Non-R**$^{(b)}$	$(b-a)/b$	$\mathbf{R}^{(c)}$	**Non-R**$^{(d)}$	$(d-c)/d$
1	432.6600	437.3400	1.0692%	13.0110	13.7500	5.3766%
2	438.1000	443.0200	1.1088%	11.8790	15.3450	22.5850%
3	425.0600	427.3000	0.5252%	12.8420	14.9320	13.9940%
4	437.3300	444.0200	1.5071%	11.8860	13.7100	13.3050%
5	430.9800	435.9200	1.1314%	12.3960	14.5080	14.5550%
6	427.4600	432.5900	1.1854%	11.5500	14.9910	22.9550%
7	425.1600	430.3700	1.2092%	12.7460	15.4330	17.4100%
8	429.7400	436.3800	1.5198%	12.0690	14.7410	18.1240%
9	424.4900	428.8000	1.0047%	12.2520	13.7000	10.5710%
10	436.9900	441.6200	1.0480%	12.4890	15.5960	19.9190%
11	432.2000	438.5200	1.4412%	13.1890	14.9990	12.0700%
12	439.3000	445.0900	1.3013%	12.3520	15.3690	19.6310%

Run	**Perfect**$^{(a)}$	**Dynamic**$^{(b)}$	**Static**$^{(c)}$	$\%(a-b)/a$	$\%(a-c)/a$
1	429.0100	427.0300	410.7100	0.4622	4.2666
2	434.2700	432.5000	416.0400	0.4068	4.1983
3	432.2200	430.4100	413.6400	0.4179	4.2990
4	436.5800	434.9000	417.8100	0.3852	4.3001
5	438.1600	436.1400	419.4700	0.4612	4.2660
6	443.5300	441.5500	424.5700	0.4484	4.2762
7	431.6700	430.5000	413.7800	0.2701	4.1437
8	435.7300	434.6000	417.3700	0.2607	4.2145
9	433.0000	431.0500	414.3100	0.4495	4.3152
10	439.1600	437.5400	420.3800	0.3689	4.2776
11	439.1100	437.3000	420.3500	0.4122	4.2723
12	433.9600	432.8600	416.0800	0.2528	4.1208

Cost of Perfect Information: Static and Dynamic Models

Robust Multistage Scenario Trees: The Third Case Study



A Simple Investment Model Based on a Utility Function

$$
\begin{array}{lll}
(P_1) & \max & \sum_{i=1}^{m} \pi_i u(\phi^T r^i) \\
 & \text{s.t.} & \phi^T e = 1 \\
 & & \phi \in \Delta,
\end{array}
$$

where

n:	the number of stocks
m:	the number of sequent scenarios at each node
$\phi \in \Re^n$:	the holding of stocks
$r^i \in \Re^n$:	the return of n stocks if scenario i happens
π_i:	the probability that scenario i will occur
$e \in \Re^n$:	the vector of all 1's
Δ:	the set of admissible portfolios, which is assumed to be convex.

A Two-Stage Extension

$$
\begin{aligned}
(P_2) \quad & \max_{\phi} \sum_{i=1}^{m} \pi_i \left[\max_{\phi^i} \sum_{j=1}^{m} \pi_j^i u({\phi^i}^T r^{ij}) \right] \\
& \qquad\qquad\qquad\qquad \text{s.t.} \quad {\phi^i}^T e = \phi^T r^i, \forall i = 1, 2, \cdots, m \\
& \qquad\qquad\qquad\qquad\qquad\quad \phi^i \in \Delta^i \\
& \text{s.t.} \quad \phi^T e = 1 \\
& \qquad\quad \phi \in \Delta.
\end{aligned}
$$

Ambiguity and Robustness

Assume

- The topology of the scenario tree is reliable;
- The values on the scenario tree are ambiguous;
- The probability estimation is only an estimation.

The Robust Version of the Two-Stage Model

$$(RP_2) \quad \max_{\phi} \min_{r^i \in V^i, y \in U} \sum_{i=1}^{m} (\tilde{\pi}_i + y_i) \max_{\phi^i} \min_{r^{ij} \in V^{ij}, y^i \in U^i} \sum_{j=1}^{m} (\tilde{\pi}^i_j + y^i_j) u({\phi^i}^T r^{ij})$$

$$\text{s.t.} \quad {\phi^i}^T e = \phi^T r^i, \forall i = 1, 2, \cdots, m$$

$$\phi^i \in \Delta^i$$

$$\text{s.t.} \quad \phi^T e = 1$$

$$\phi \in \Delta.$$

Rule of the Game

Can we represent the above optimization model in finite terms so as to enable efficient solution methods?

Some Preparations

- A pointed convex cone $\mathcal{K}$ is a set satisfying
  - If $a \in \mathcal{K}$ and $-a \in \mathcal{K}$ then $a = 0$.
  - If $x \in \mathcal{K}$ then $tx \in \mathcal{K}$ for all $t > 0$.
  - If $a \in \mathcal{K}$ and $b \in \mathcal{K}$ then $a + b \in \mathcal{K}$.
- If K is convex cone then its dual cone is

$$\mathcal{K}^* = \{s \mid x^T s \geq 0 \, \forall x \in \mathcal{K}\}.$$

- If U is a convex set, then the corresponding homogenized cone is

$$H(U) = \mathrm{cl}\left\{ \begin{pmatrix} t \\ x \end{pmatrix} \middle| \, t > 0, \frac{x}{t} \in U \right\}.$$

An Important Cone

An $(n+1)$-dimensional Second Order Cone:

$$\mathrm{SOC}(n+1) = \left\{ \begin{pmatrix} t \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \;\middle|\; t \geq \sqrt{\sum_{j=1}^{n} x_j^2} \right\}.$$

Conic Optimization

$$
\begin{array}{ll}
\text{minimize} & c^T x \\
\text{subject to} & Ax = b \\
 & x \in \mathcal{K}
\end{array}
$$

- Known as ***Semidefinite Programming*** (SDP) if $\mathcal{K}$ is essentially the cone of positive semidefinite matrices.
  Available solvers: *SeDuMi, SDPA, SDPT3, ...*
- Known as ***Second Order Cone Programming*** (SOCP) if $\mathcal{K}$ is essentially the second order cone.
  Available solvers: *SeDuMi, MOSEK, ...*

A Specific Two-Stage Model

- There is no short selling: $\Delta = \Delta^i = \Re^n_+$.
- We use a semi-variance disutility function

$$d(w) = (R - w)_+^2.$$

- $V^i, V^{ij} \subseteq \Re^n_+$.
- The ambiguity sets are ellipsoidal:

$$\Pi = \{\pi \in \Re^m \mid \pi^T e = 1, \|\pi - \tilde{\pi}\| \leq \theta\}$$
$$V^i = \{r^i \in \Re^n \mid (r^i - \tilde{r}^i)^T Q^i (r^i - \tilde{r}^i) \leq \rho_i^2\}$$
$$\Pi^i = \{\pi \in \Re^m \mid {\pi^i}^T e = 1, \|\pi^i - \tilde{\pi}^i\| \leq \theta_i\}$$
$$V^{ij} = \{r^{ij} \in \Re^n \mid (r^{ij} - \tilde{r}^{ij})^T Q^{ij} (r^{ij} - \tilde{r}^{ij}) \leq \rho_{ij}^2\}.$$

Also,

$$U = \{y \in \Re^m \mid y^T e = 0, \|y\| \le \theta\}$$
$$U^i = \{y^i \mid {y^i}^T e = 0, \|y^i\| \le \theta^i\}, i = 1, 2, \cdots, m.$$

By linear transformations, one can assume

$$U = \{y \in \Re^m \mid y^T e = 0, \|y\| \le \theta\},$$
$$V^i = \{r^i \in \Re^n \mid \|(r^i - \tilde{r}^i)\| \le \rho_i\},$$
$$U^i = \{y^i \in \Re^m \mid {y^i}^T e = 0, \|y^i\| \le \theta_i\},$$
$$V^{ij} = \{r^{ij} \in \Re^n \mid \|(r^{ij} - \tilde{r}^{ij})\| \le \rho_{ij}\}.$$

A Finite Robust Formulation

$$
\begin{array}{lll}
\min\limits_{\phi, \phi^i, d_{ij}, t_i, t_0} & t_0 & \\
\text{s.t.} & \begin{pmatrix} t_0 - \tilde{\pi}^T \mathbf{t} \\ \theta[e \cdot \frac{\mathbf{t}^T e}{m} - \mathbf{t}] \end{pmatrix} \in \mathrm{SOC}(m+1) & \\
& \begin{pmatrix} t_i - \tilde{\pi}^{i^T} \mathbf{d_i} \\ \theta_i[e \cdot \frac{\mathbf{d_i}^T e}{m} - \mathbf{d_i}] \end{pmatrix} \in \mathrm{SOC}(m+1), \forall i = 1, 2, \cdots, m & \\
& \begin{pmatrix} \phi^T r^i - \phi^{i^T} e \\ \rho_i \phi \end{pmatrix} \in \mathrm{SOC}(n+1), \forall i = 1, 2, \cdots, m & \\
& \begin{pmatrix} d_{ij} + 1 \\ d_{ij} - 1 \\ 2\tau_{ij} \end{pmatrix} \in \mathrm{SOC}(3) & \\
& \begin{pmatrix} \tau_{ij} - R^{ij} + \phi^{i^T} \tilde{r}^{ij} \\ \rho_{ij} \phi^i \end{pmatrix} \in \mathrm{SOC}(n+1) & \\
& \tau_{ij} \geq 0 & \\
& \phi^i \geq 0 & \\
& \phi^T e = 1 & \\
& \phi \geq 0. &
\end{array}
$$

A Finite Robust Formulation for the General Model

$$
\begin{array}{ll}
\max\limits_{\phi,\phi^i,u_{ij},w_{ij},w_i,t_0} & t_0 \\
\text{s.t.} & \begin{pmatrix} \tilde{\pi}^T t - t_0 \\ t \end{pmatrix} \in \mathsf{H}(U)^* \\
& \begin{pmatrix} \tilde{\pi}^{i^T} \mathbf{u_i} - t_i \\ \mathbf{u_i} \end{pmatrix} \in \mathsf{H}(U^i)^* \\
& u_{ij} \le u(w_{ij}) \\
& \begin{pmatrix} -w_{ij} \\ \phi^i \end{pmatrix} \in \mathsf{H}(V^{ij})^* \\
& \begin{pmatrix} -\phi^{i^T} e \\ \phi \end{pmatrix} \in \mathsf{H}(V^i)^* \\
& \phi^i \in \Delta \\
& \phi^T e = 1 \\
& \phi \in \Delta^i .
\end{array}
$$

Numerical Results

We wish to choose a portfolio among four indices: Heng Seng Index, Dow Jones index, London index and Nikkei. The decision horizon is divided into two periods, and the length of each period is one month. The target return is assumed to be 0.3% for these two months in total, i.e. R=1.003. We use the monthly price from Jan. 2001 to Dec. 2004 (source: www.yahoo.com) as historical data to get a least square estimate for the VAR model:

$$h_t = c + \Omega h_{t-1} + \epsilon_t, \qquad \epsilon_t \sim N(0, \Sigma), \qquad\qquad t = 1, \cdots, T.$$

For each setting, we generate 30 scenarios trees. For each scenario tree, we simulate 500 random runs.

The comparison of the mean of 500 simulated disutility function values for (SP_2) and (RSP_2) under different parameter settings:

Parameter settings	mean(mean(ϕ_{SP}))	mean(mean(ϕ_{RSP}))
$m = 10, \theta = \theta_i = \rho_i = \rho_{ij} = 0.01$	0.016594	0.016472
$m = 10, \theta = \theta_i = \rho_i = \rho_{ij} = 0.1$	0.019674	0.019079
$m = 20, \theta = \theta_i = \rho_i = \rho_{ij} = 0.01$	0.038082	0.038143
$m = 20, \theta = \theta_i = \rho_i = \rho_{ij} = 0.1$	0.040925	0.043790
Parameter settings	std(mean(ϕ_{SP}))	std(mean(ϕ_{RSP}))
$m = 10, \theta = \theta_i = \rho_i = \rho_{ij} = 0.01$	0.000896	0.000838
$m = 10, \theta = \theta_i = \rho_i = \rho_{ij} = 0.1$	0.008564	0.003961
$m = 20, \theta = \theta_i = \rho_i = \rho_{ij} = 0.01$	0.001217	0.001168
$m = 20, \theta = \theta_i = \rho_i = \rho_{ij} = 0.1$	0.008844	0.005006
Parameter settings	min(mean(ϕ_{SP}))	min(mean(ϕ_{RSP}))
$m = 10, \theta = \theta_i = \rho_i = \rho_{ij} = 0.01$	0.015113	0.015049
$m = 10, \theta = \theta_i = \rho_i = \rho_{ij} = 0.1$	0.007236	0.011178
$m = 20, \theta = \theta_i = \rho_i = \rho_{ij} = 0.01$	0.034983	0.035368
$m = 20, \theta = \theta_i = \rho_i = \rho_{ij} = 0.1$	0.024528	0.031694
Parameter settings	max(mean(ϕ_{SP}))	max(mean(ϕ_{RSP}))
$m = 10, \theta = \theta_i = \rho_i = \rho_{ij} = 0.01$	0.018534	0.018310
$m = 10, \theta = \theta_i = \rho_i = \rho_{ij} = 0.1$	0.044821	0.029962
$m = 20, \theta = \theta_i = \rho_i = \rho_{ij} = 0.01$	0.040037	0.040258
$m = 20, \theta = \theta_i = \rho_i = \rho_{ij} = 0.1$	0.063347	0.053967

	Non-Robust			Robust		
Para.	**min**	**max**	**average**	**min**	**max**	**average**
(10, 0.01)	0.015113	0.018534	0.016594	0.015049	0.018310	0.016472
(10, 0.1)	0.007236	0.044821	0.019674	0.011178	0.029962	0.019079
(20, 0.01)	0.034983	0.040037	0.038082	0.035368	0.040258	0.038143
(20, 0.1)	0.024528	0.063347	0.040925	0.031694	0.053967	0.043790

Expected Utility: Robust vs. Non-Robust Models

	Non-Robust		Robust		% change	
Para.	**average**^{a}	**std**^{c}	**average**^{b}	**std**^{d}	$(a-b)/a$	$(c-d)/c$
(10, 0.01)	0.016594	0.000896	0.016472	0.000838	0.735%	6.473%
(10, 0.1)	0.019674	0.008564	0.019079	0.003961	3.024%	53.748%
(20, 0.01)	0.038082	0.001217	0.038143	0.001168	-0.160%	4.026%
(20, 0.1)	0.040925	0.008844	0.043790	0.005006	-7.000%	43.396%

`Variability: Robust vs. Non-Robust Models`

Conclusions

- *Uncertainty* and *ambiguity* issues in decision models;
- *Uncertainty* is captured by stochastic models;
- *Ambiguity* is handled by robust optimization models;
- *Robust* solutions are usually immunized against data ambiguities, in the form of reduced *variability*.

URL of the reports

`http://www.se.cuhk.edu.hk/~zhang/#workingpaper`

- *An Integrated Approach to Single-Leg Airline Revenue Management: The Role of Robust Optimization*, Technical Report SEEM2006-04, Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong, 2006 (with Ilker Birbil, Hans Frenk, and Joaquim Gromicho).

- *Robust Portfolio Selection Based on a Multi-stage Scenario Tree*, Technical Report SEEM2006-02, Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong, 2006 (with R.J. Shen).