# Optimized Randomness!

# Why and How?

Shuzhong Zhang
Department of Systems Engineering and Engineering Management
The Chinese University of Hong Kong

`Based on joint works with my collaborators:`
`S. He, Z. Luo, J. Nie, N. Sidiropoulos, P. Tseng, J. Xie`

32nd Conference on the Mathematics of Operations Research
'De Werelt', Lunteren, The Netherlands
January 17, 2007

## An Example for Randomization

Zhi-Quan Luo, *An Isotropic Universal Decentralized Estimation Scheme for a Bandwidth Constrained Ad Hoc Sensor Network.*

IEEE Journal on Selected Areas in Communications, 23 (4), 735 – 744, 2005.

## Data Transmission in Communication

- Ad hoc sensor network with $K$ sensors.
- Each sensor observes a real data in $[-U, U]$ independently.
- Each sensor sends back the data to the base-station.
- The base-station operates a least square estimation.

## Matters of Facts

- Sensors have weak batteries.
- The above scheme is an unbiased estimation.
- The statistical error is

$$\frac{U^2}{K}$$

# A Randomized Transmission Scheme!

- Each sensor observes the data in binary digits:

$$a_1 a_2 a_3 \cdots .$$

- Each sensor, say sensor $k$, independently tosses a coin to decide which single binary digit to transmit: $\xi = j$ with probability $1/2^j$, $j = 1, 2, ...$
- Then, sends this one bit data $a_\xi$ back to the station.
- The base-station simply adds up all the received digits.
- This is an unbiased estimation, with statistical error

$$\frac{4U^2}{K}$$

# Another Example: Transmit Beamforming

A transmitter utilizes an array of $n$ transmitting antennas to broadcast information within its service area to $m$ radio receivers.

The constraints model the requirement that the total received signal power at receiver $i$ must be above a given threshold (normalized to 1); or, equivalently, a signal-to-noise ratio (SNR) condition for receiver $i$, as commonly used in data communication.

The objective is to minimize the total transmit power subject to individual SNR requirements (one at each receiver).

## Measuring Quality of Decisions

In a *minimization* problem, the quality measure of a solution $x$ is a guaranteed bound $\theta$ such that

$$v(x) \leq \theta \times v^*$$

In this context, $\theta \geq 1$, e.g., $\theta = 150\%$.

In a *maximization* problem, the quality measure of a solution $x$ is a guaranteed bound $\theta$ such that

$$v(x) \geq \theta \times v^*$$

In this context, $\theta \leq 1$, e.g., $\theta = 85\%$.

The value $\theta$ is called *approximation ratio* of a method.

# Transmit Beaforming: A Quadratic Model

The problem of transmit beamforming as stated before can be precisely modelled by homogeneous complex quadratic minimization:

$$
\begin{array}{lll}
(QPc)_{\min} & \min & z^H C z \\
& \text{s.t.} & z^H Q_i z \geq 1, \quad i = 1, ..., m, \\
& & z \in \mathbf{C}^n.
\end{array}
$$

## Homogeneous Quadratic Minimization

In general, let us consider:

$$
\begin{array}{lll}
(QPr)_{\min} & \min & x^T C x \\
 & \text{s.t.} & x^T Q_i x \geq 1, \quad i = 1, ..., m, \\
 & & x \in \Re^n.
\end{array}
$$

All data matrices are assumed to be positive semidefinite.

This problem is clearly NP-hard.

Also, $(QPc)_{\min}$ is NP-hard.

# The SDP Relaxation

Consider the Semidefinite Programming relaxation for $(QPr)_{\min}$

$$(SDPr)_{\min} \quad \begin{array}{ll} \min & C \bullet X \\ \text{s.t.} & Q_i \bullet X \geq 1, \quad i = 1, ..., m, \\ & X \succeq 0, \end{array}$$

and similarly for $(QPc)_{\min}$:

$$(SDPc)_{\min} \quad \begin{array}{ll} \min & C \bullet Z \\ \text{s.t.} & Q_i \bullet Z \geq 1, \quad i = 1, ..., m, \\ & Z \succeq 0. \end{array}$$

# A Randomized Approach to $(QPr)_{\min}$

*But what to do with the solution of a relaxed problem?*

Let $X^*$ be the optimal solution of the SDP relaxation.

1. Generate a random vector $\xi \in \Re^n$ from the real-valued normal distribution $\mathcal{N}(0, X^*)$.
2. Let
$$x^*(\xi) = \frac{\xi}{\min_{1\le i\le m} \sqrt{\xi^T Q_i \xi}}.$$

# Approximation Ratio

**Theorem.** (Luo, Sidiropoulos, Tseng, and Z.; 2005)

For $m \geq 2$, we have

$$v(QPr_{\min}) \leq \frac{27m^2}{\pi} v(SDPr_{\min}).$$

Moreover, there is an instance such that

$$v(QPr_{\min}) \geq \frac{2m^2}{\pi^2} v(SDPr_{\min}).$$

## The Complex Case: $(QPc)_{\min}$

1. Generate a random vector $\xi \in \mathbf{C}^n$ from the complex-valued normal distribution $\mathcal{N}_c(0, Z^*)$.
2. Let
$$x^*(\xi) = \frac{\xi}{\min_{1 \le i \le m} \sqrt{\xi^H Q_i \xi}}.$$

# Approximation Ratio

Theorem. (Luo, Sidiropoulos, Tseng, and Z.; 2005)

For $m \geq 2$, we have

$$v(QPc_{\min}) \leq 8m \cdot v(SDPc_{\min}).$$

Moreover, there is an instance such that

$$v(QPc_{\min}) \geq \frac{m}{\pi^2(2+\pi/2)^2} \, v(SDPc_{\min}).$$

## A Homogeneous Quadratic Maximization Model

The following model is considered by Nemirvoski, Roos, and Terlaky (1999):

$$
\begin{array}{lll}
(QPr)_{\max} & \max & x^T C x \\
 & \text{s.t.} & x^T Q_i x \le 1, \quad i = 1, ..., m, \\
 & & x \in \Re^n,
\end{array}
$$

where $Q_i \succeq 0$, $i = 1, ..., m$.

# A Homogeneous Quadratic Maximization Model

The corresponding SDP relaxation is

$$(SDPr)_{\max} \quad \max \quad C \bullet X$$
$$\text{s.t.} \quad Q_i \bullet X \le 1, \quad i = 1, ..., m,$$
$$X \succeq 0.$$

**Theorem.** (Nemirovski, Roos, Terlaky; 1999)
It holds that

$$v((QPr)_{\max}) \ge \frac{1}{2\ln(2m\mu)} v((SDPr)_{\max}),$$

where $\mu = \min\{m, \max_i \text{Rank}(Q_i)\}$.

# Complex Quadratic Maximization Problem

Consider

$$\begin{array}{lll} (QPc)_{\max} & \max & z^H C z \\ & \text{s.t.} & z^H Q_i z \le 1, \quad i = 1, ..., m, \\ & & z \in \mathbf{C}^n. \end{array}$$

The SDP relaxation is

$$\begin{array}{lll} (SDPc)_{\max} & \max & C \bullet Z \\ & \text{s.t.} & Q_i \bullet Z \le 1, \quad i = 1, ..., m, \\ & & Z \succeq 0. \end{array}$$

## A Randomization Method for $(QPc_{\max})$

Similar as before, we propose to solve the problem as follows

1. Generate a random vector $\xi \in \mathbf{C}^n$ from the complex-valued normal distribution $\mathcal{N}_c(0, Z^*)$.
2. Let
$$x^*(\xi) = \frac{\xi}{\max_{1\le i\le m} \sqrt{\xi^H Q_i \xi}}.$$

# Approximation Ratio

**Theorem.** (Luo, Sidiropoulos, Tseng, and Z.; 2005)

For $m \geq 2$, we have

$$v(QPc_{\max}) \geq \frac{1}{4\ln(100\mu)} v(SDPc_{\max}),$$

where $\mu = \sum_{i=1}^{m} \min\{\text{rank}(Q_i), \sqrt{m}\}$.

# Indefinite Constraints

*How about when some of the constraints are indefinite?*

There is no finite approximation ratio if more than one $Q_i$'s are indefinite!

**Theorem.** (He, Luo, Nie, and Z.; 2007)
If exactly one of $Q_i$'s is indefinite, then

$$v(QPr_{\min}) \le \frac{10^6 m^2}{\pi} v(SDPr_{\min}).$$

**Theorem.** (He, Luo, Nie, and Z.; 2007)
If exactly one of $Q_i$'s is indefinite, then

$$v(QPc_{\min}) \le 2400m \cdot v(SDPc_{\min}).$$

# Indefinite Quadratic Maximization

*The approximation ratio can be arbitrarily large, depending on the data matrices, if more than two $Q_i$'s are indefinite.*

**Theorem.** (Ben-Tal, Nemirovski, Roos; 2002)
If one of the $Q_i$'s is indefinite and $C$ indefinite, then

$$v(QPr_{\max}) \geq \frac{1}{2\log(16n^2\, m\mu)}\, v(SDPr_{\max}),$$

where $\mu = \sum_{i=1}^m \min\{\text{rank}(Q_i), \sqrt{m}\}$.

# New Bound for Indefinite Quadratic Maximization

**Theorem.** (He, Luo, Nie, and Z.; 2007)

If one of the $Q_i$'s is indefinite and $C$ indefinite, then

$$v(QPr_{\max}) \geq \frac{1}{2\log(174\, m\mu)}\, v(SDPr_{\max}),$$

where $\mu = \sum_{i=1}^{m} \min\{\text{rank}(Q_i), \sqrt{m}\}$.

# A Key Ingredient

Ben-Tal, Nemirovski, Roos conjectured that

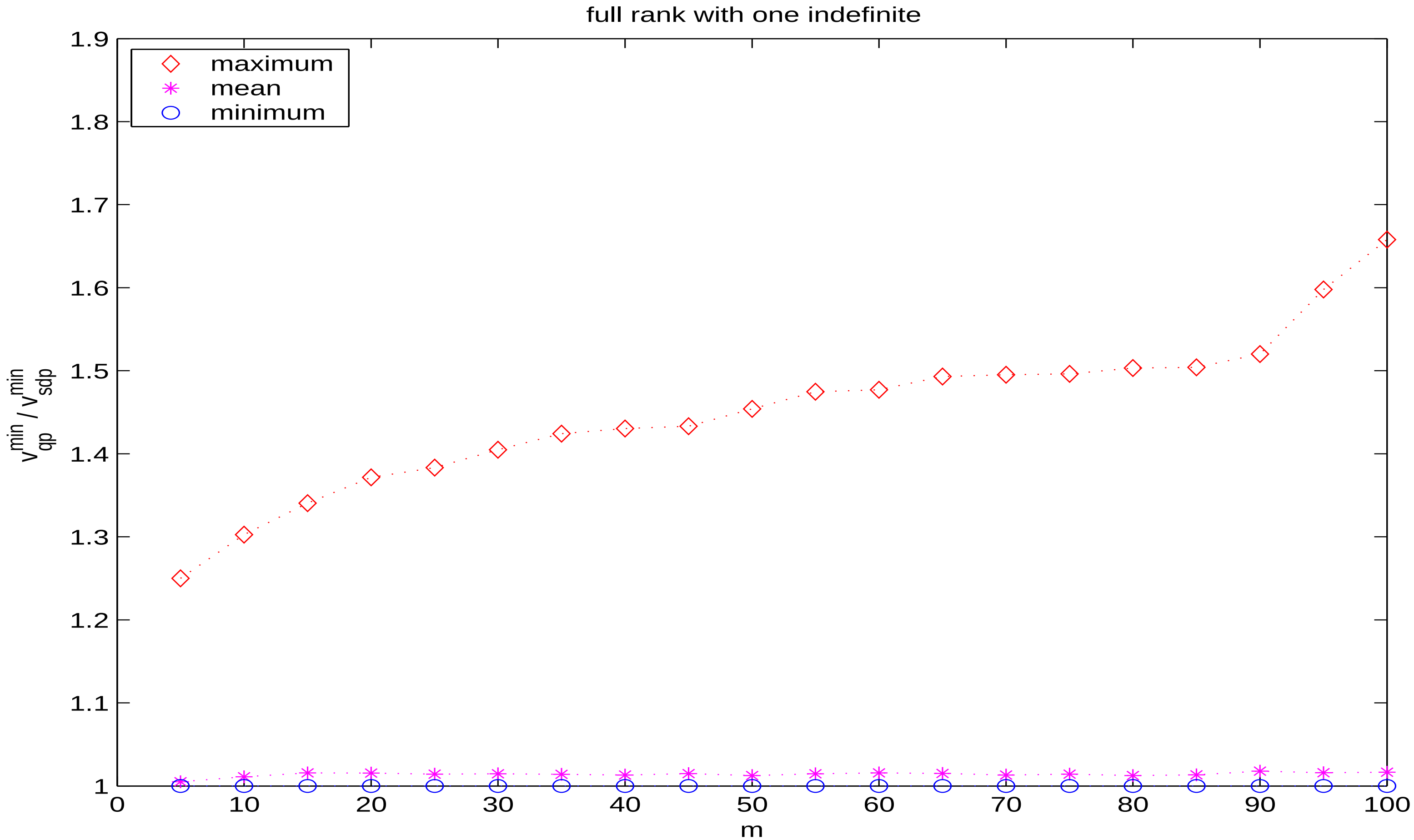
$$\text{Prob}\left\{\xi^T A\xi \le \mathsf{E}\left(\xi^T A\xi\right)\right\} \ge \frac{1}{4}, \qquad \forall\ A \text{ symmetric matrix},$$

for i.i.d. $\xi_i$'s, with $\text{Prob}\{\xi_i = +1\} = \text{Prob}\{\xi_i = -1\} = \frac{1}{2}$.

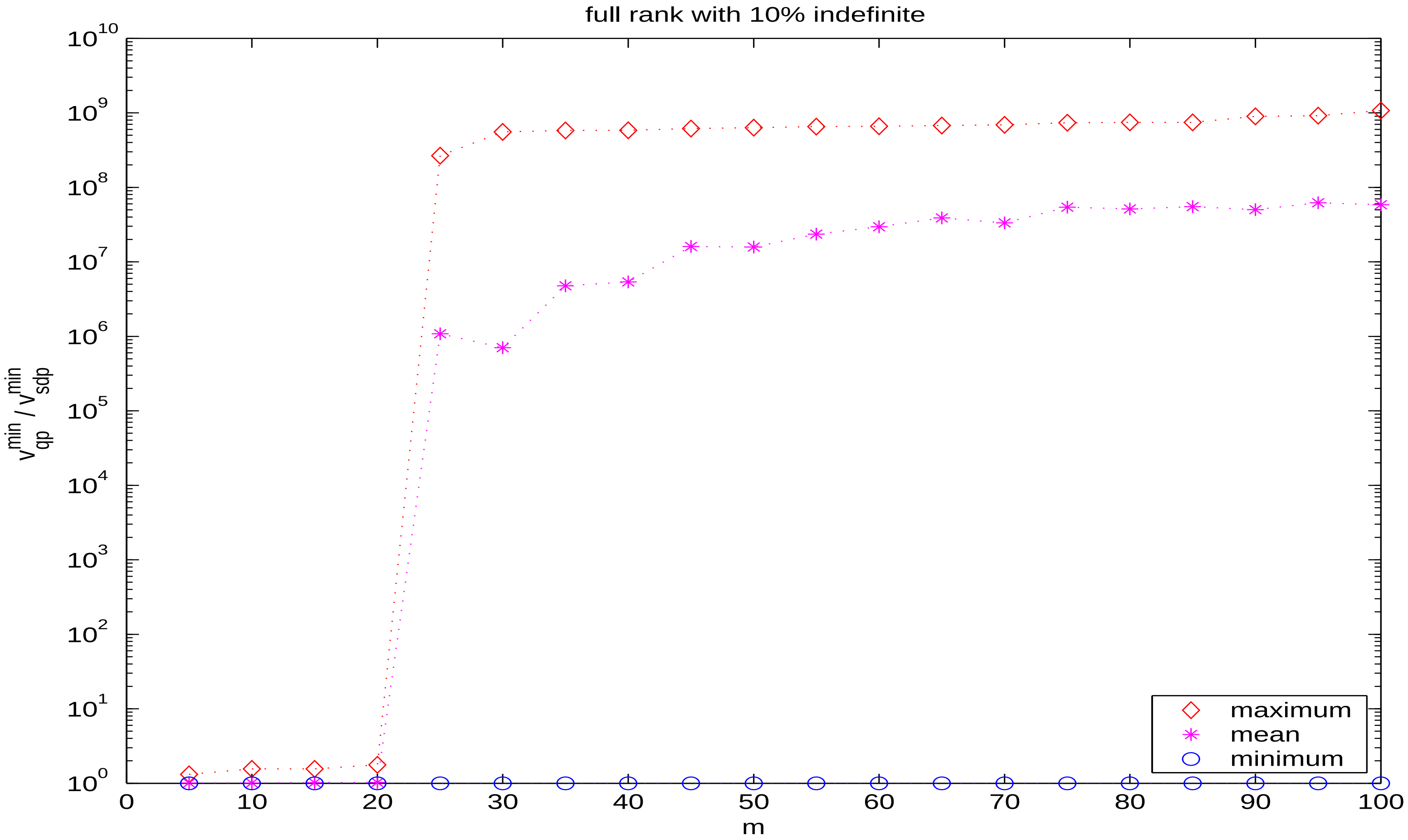
But they only managed to show a lower bound of $\frac{1}{8n^2}$.

We have established a lower bound of $\frac{1}{87}$.

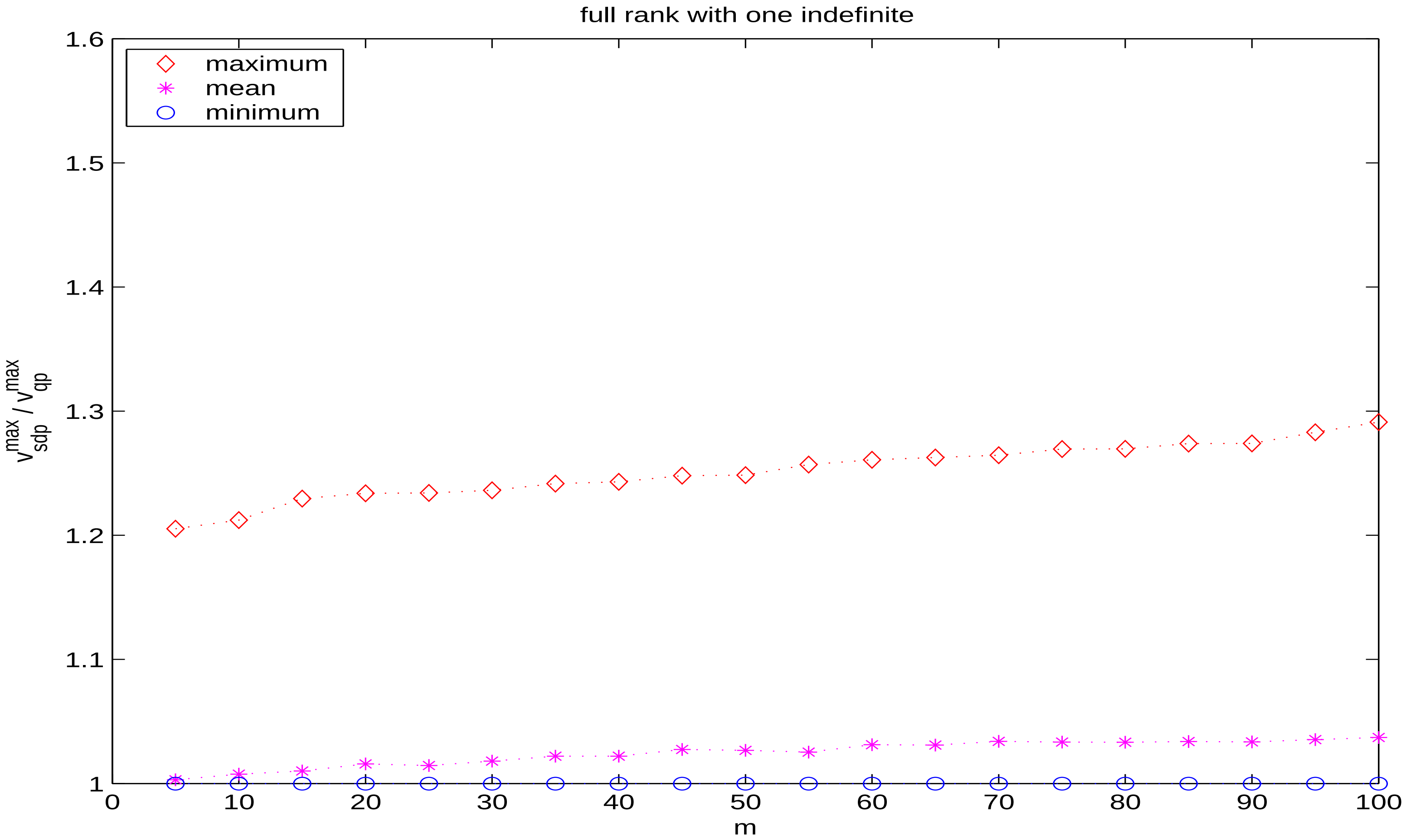
# Put Theory to Work: Simulation Results



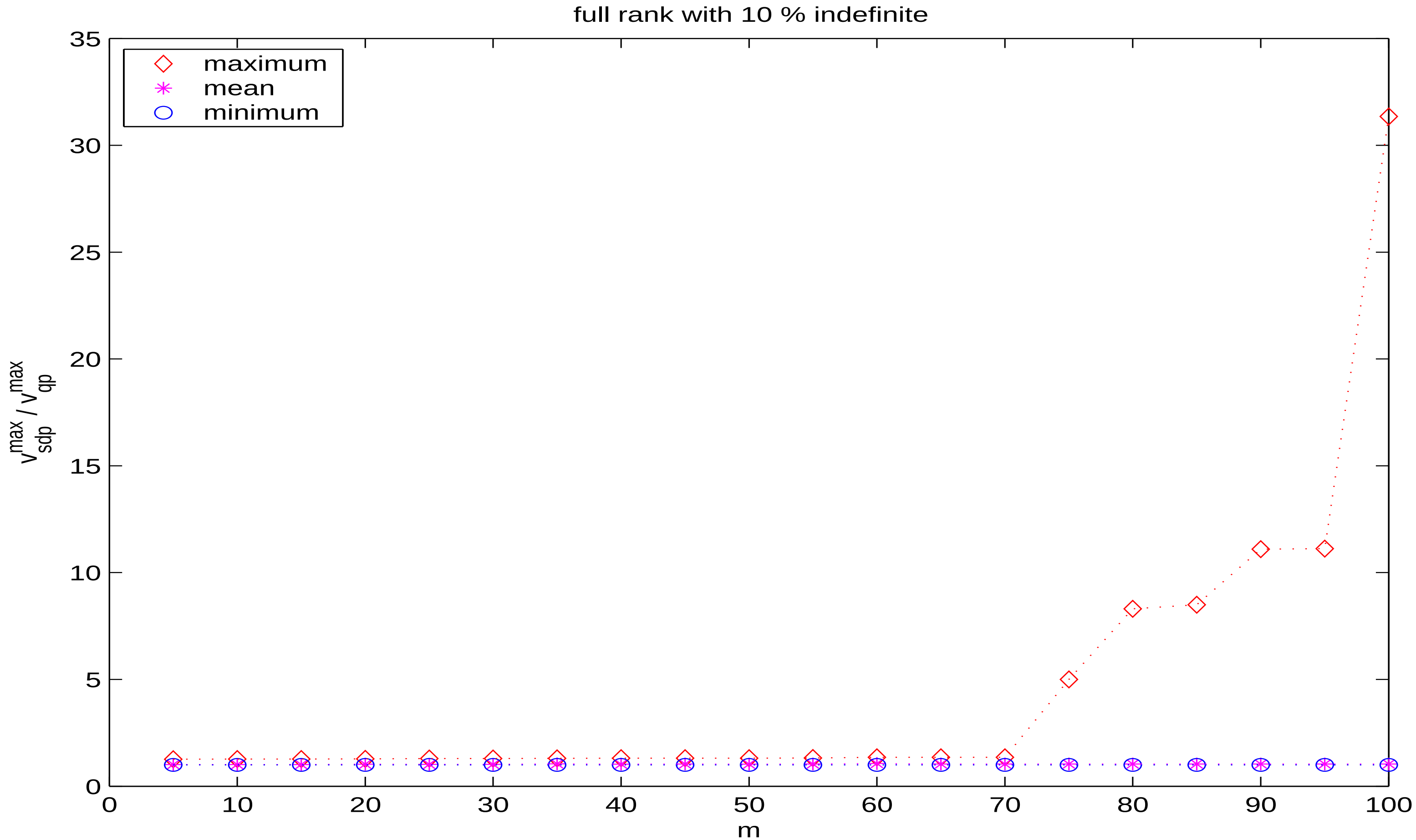

Minimization model: one indefinite constraint




Minimization model: many indefinite constraints




# Maximization model: one indefinite constraint




Maximization model: many indefinite constraints

# Randomized Investment: Theory and Practice

*Portfolio Selection:*

Among $n$ assets, select a set of good ones with right amounts.

Classical notions (due to Markowitz, 1952):

- Return of assets → random variables
- Gain on investment → mean of the portfolio
- Risk on investment → variance of the portfolio

# The Mathematical Model

Let there be $n$ assets, each with return rate $\xi_i$, $i = 1, ..., n$.

Let the mean of $\xi_i$ be $r_i = \mathsf{E}[\xi_i]$, $i = 1, ..., n$.

Let the covariance matrix of $\xi_i$, $i = 1, ..., n$, be $Q$.

Let the initial budget be \$1, and the target gain to be $\mu$. Then the model could be

$$
\begin{array}{ll}
\text{minimize} & x^T Q x \\
\text{subject to} & r^T x \geq \mu \\
& e^T x \leq 1 \\
& x \in \text{“a certain desirable constraint set”}
\end{array}
$$

where $e$ is the vector of all one's.

# A Practical Issue

In real life, $n$ can be a large number. A reasonable investor may only wish to handle a small set of assets; that is,

*"a small portfolio"* — terminology used by Blog, Van der Hoek, Rinnooy Kan, and Timmer, 1983.

*How about we explicitly require to choose $k$ out of $n$ assets?*

The problem can be formulated as

$$
\begin{array}{lll}
(MV_s) & \text{minimize} & x^T Q x \\
 & \text{subject to} & r^T x \geq \mu, \\
 & & e^T x \leq 1, \\
 & & \sum_{i=1}^{n} |\text{sign}(x_i)| \leq k.
\end{array}
$$

But this is an extremely difficult problem to solve!

It is NP-hard: the only guaranteed method to solve the problem to optimality is basically to enumerate all the possibilities.

If $n = 100$ and $k = 50$ then there are more than $10^{29}$ possible combinations.

Remember that there are only about $10^{21}$ stars in the entire universe!

# Randomization Approach (I)

Let $x_i$ to be the quantity invested in asset $i$, $i = 1, ..., n$.

Then, consider

$$\eta_i := \begin{cases} x_i, & \text{with probability } \frac{k}{n} \\ 0, & \text{with probability } 1 - \frac{k}{n} \end{cases}$$

The portfolio is now essentially $\xi^T \eta$, with its variance being

$$\left(\frac{k}{n}\right)^2 \times \left(\frac{n-k}{k} \sum_{i=1}^{n} (q_{ii} + r_i^2) x_i^2 + x^T Q x\right)$$

The variable $x_i$ can be regarded as the *seeds of randomization*.

Therefore, the *seeds optimization* problem can be cast as

$$
\begin{array}{lll}
(RP1) & \text{minimization} & \frac{n-k}{k}\sum_{i=1}^{n}(q_{ii}+r_i^2)x_i^2 + x^TQx \\
& \text{subject to} & r^Tx \geq \mu, \\
& & e^Tx \leq 1.
\end{array}
$$

This is a nice and solvable convex optimization problem.

*Strategy*: Solve (RP1) and obtain $x$. Then, run $\xi$ for several times. Pick up the best run as our actual investment policy!

# Randomization Approach (II)

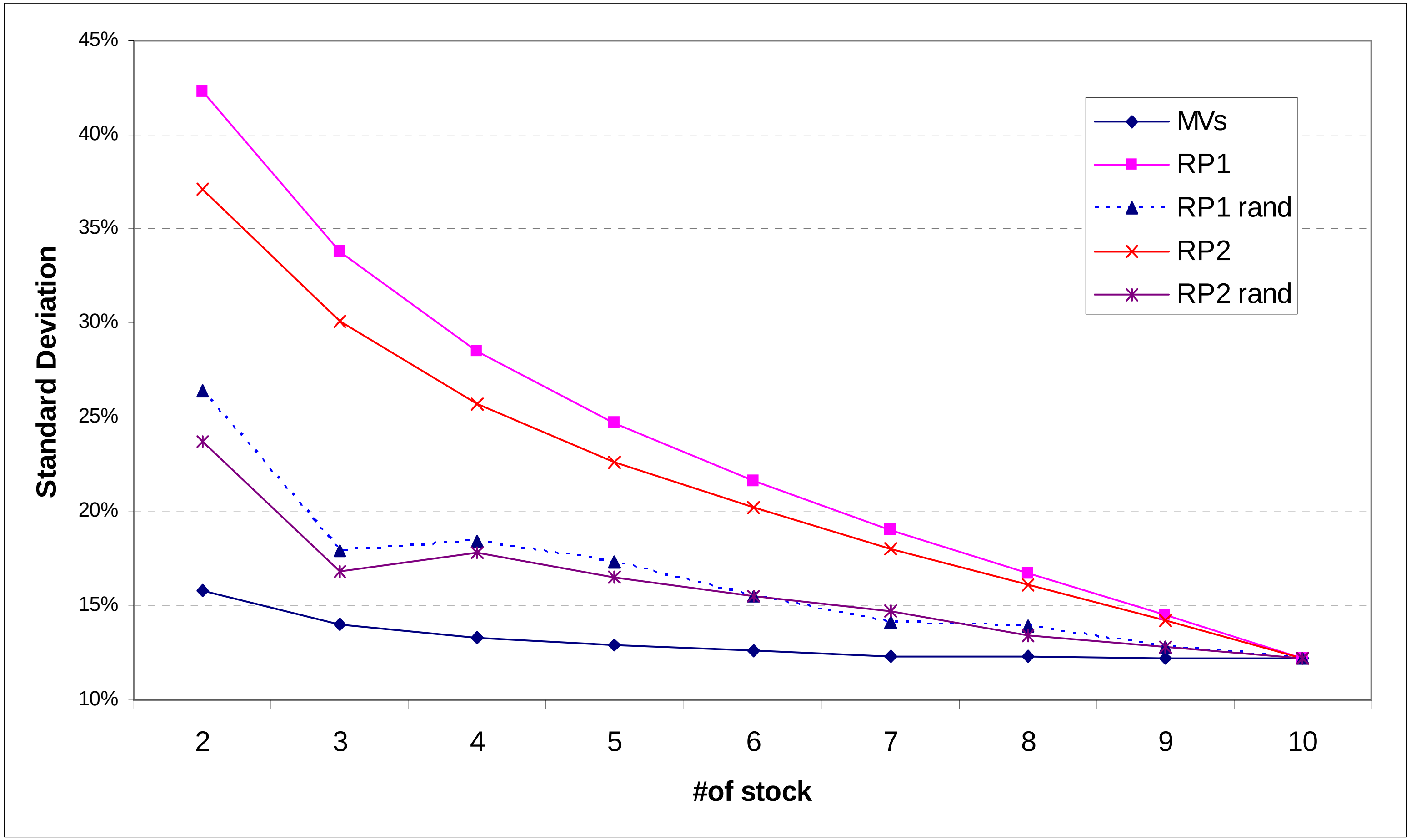
The previous approach will generate a portfolio of about $k$ assets with high probability, but may not be exactly $k$.

Another approach is to only consider in the sample space of choosing $k$ out of $n$ assets.

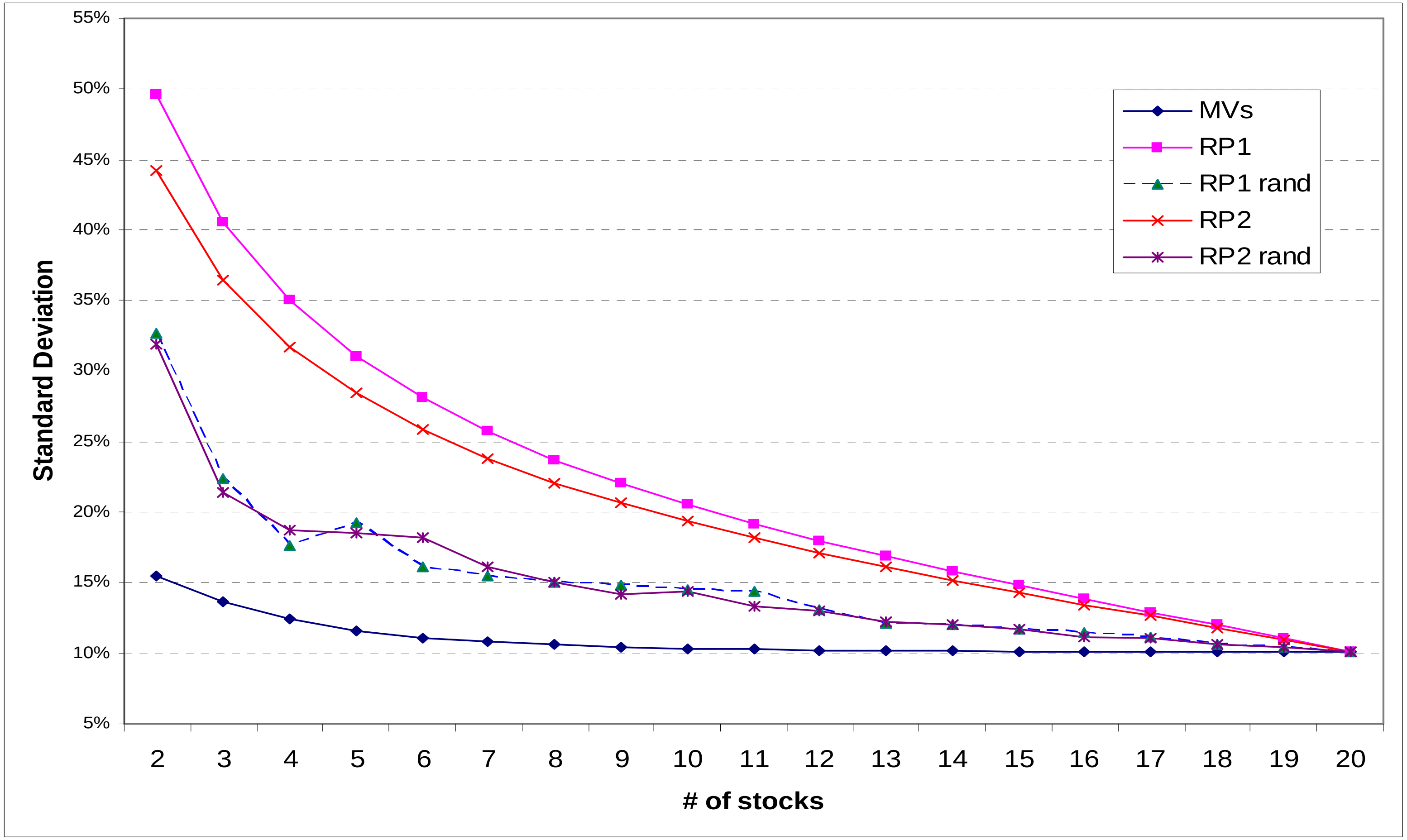
The random seeds optimization problem is now

$$
\begin{array}{lll}
(RP2) & \text{minimize} & \frac{n(n-k)}{k(n-1)} \sum_{i=1}^{n} (q_{ii} + r_i^2) x_i^2 + \frac{n(n-k)}{k(n-1)} x^T Q x \\
& & - \frac{n-k}{k(n-1)} (r^T x)^2 \\
& \text{subject to} & r^T x \geq \mu \\
& & e^T x \leq 1.
\end{array}
$$

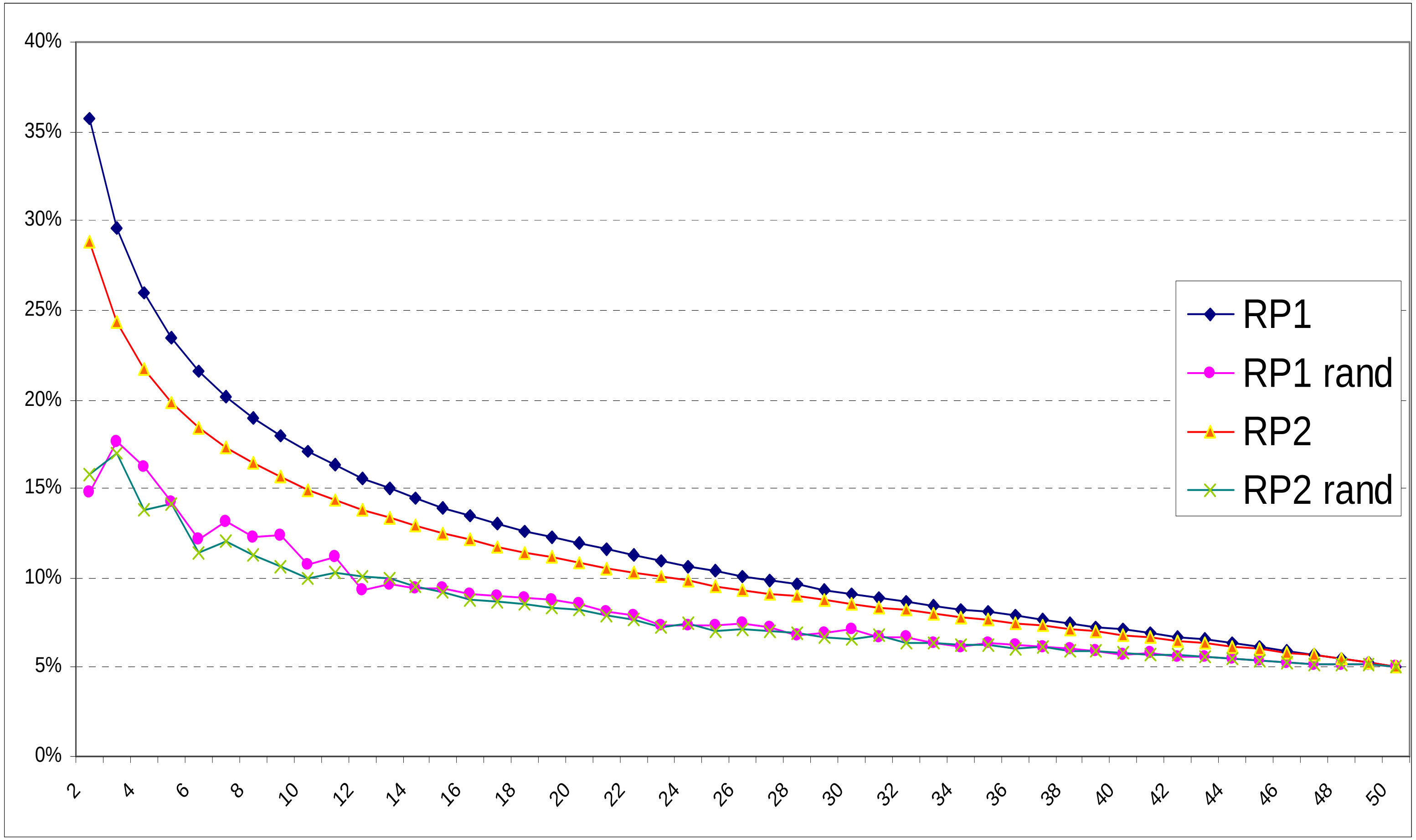
# Actual Performances



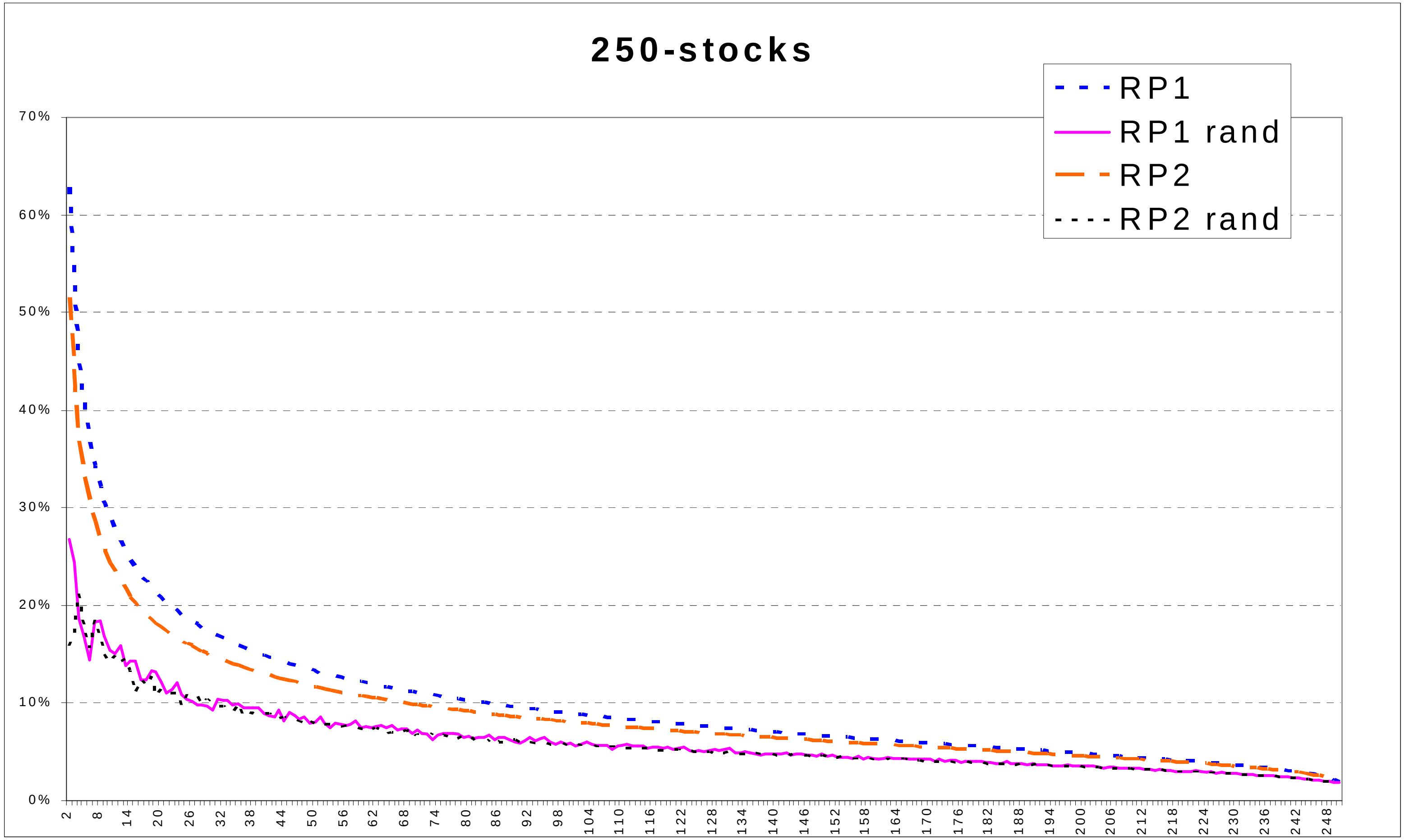

Number of assets $n = 10$.




Number of assets $n = 20$.




Number of assets $n = 50$.




Number of assets $n = 250$.

## Do It Not or Do It Well!

Consider decision problem

$$
\begin{array}{lll}
(MV_q) & \text{minimize} & x^T Q x \\
 & \text{subject to} & x^T Q_i x = 0,\ i = 1, ..., s \\
 & & x^T Q_i x = \begin{cases} 0 & \text{or} \\ \geq 1 & \end{cases} \quad i = s+1, ..., k \\
 & & x^T Q_i x \geq 1,\ i = k+1, ..., m
\end{array}
$$

This is an extremely difficult combinatorial problem.

A relaxed version of the problem is

$$
\begin{array}{lll}
(DSDP) & \text{minimize} & Q \bullet X \\
& \text{subject to} & Q_i \bullet X = 0 \text{ for } i = 1, ..., s, \\
& & Q_i \bullet X = \left\{ \begin{array}{ll} 0 & \text{or} \\ \geq 1 & \end{array} \right. , \quad i = s+1, ..., k, \\
& & Q_i \bullet X \geq 1 \text{ for } i = k+1, ..., m, \\
& & X \succeq 0
\end{array}
$$

This is still a hard combinatorial problem.

But it has a much better structure.

## A Quality Assurance

**Theorem.** (He, Xie, and Z.; 2006)
There is an algorithm that solves $(DSDP)$ such that its objective value is no ore than $k-s$ times the optimal value.

Let such a solution be $\hat{X}$.

We can view this as a randomization seed.

*What do we do next?*

# A Randomized Rounding Method

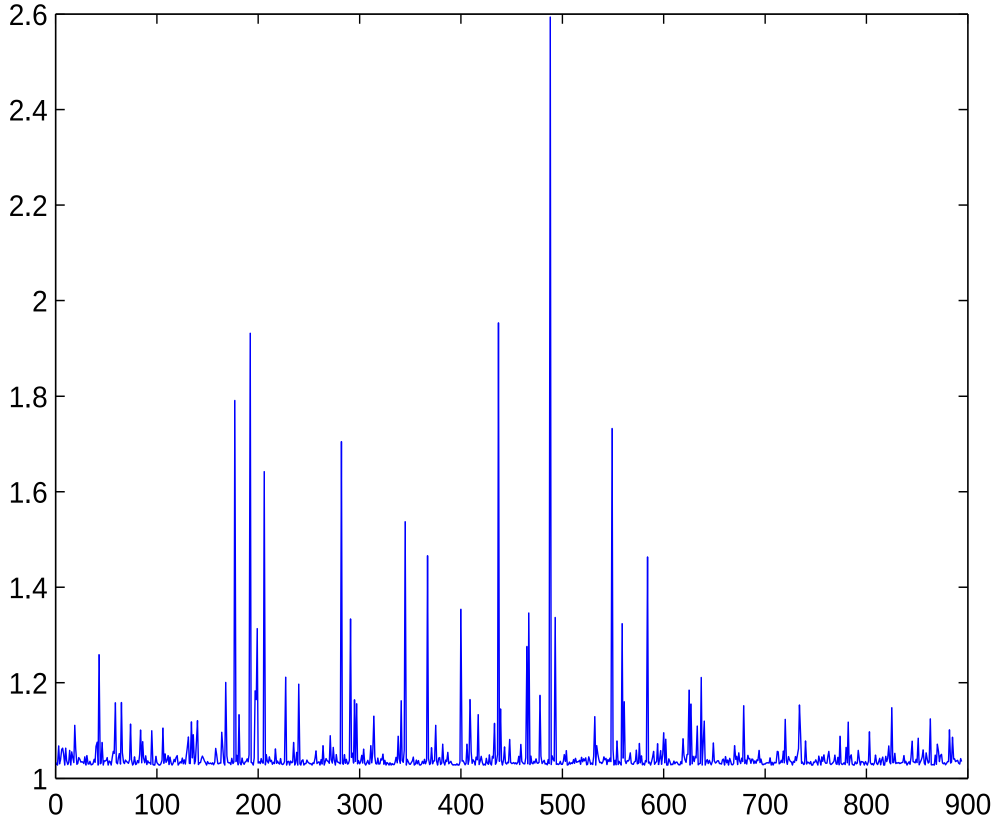
Step 1. Generate $\xi \to \mathcal{N}(0, \hat{X})$.

Step 2. Let

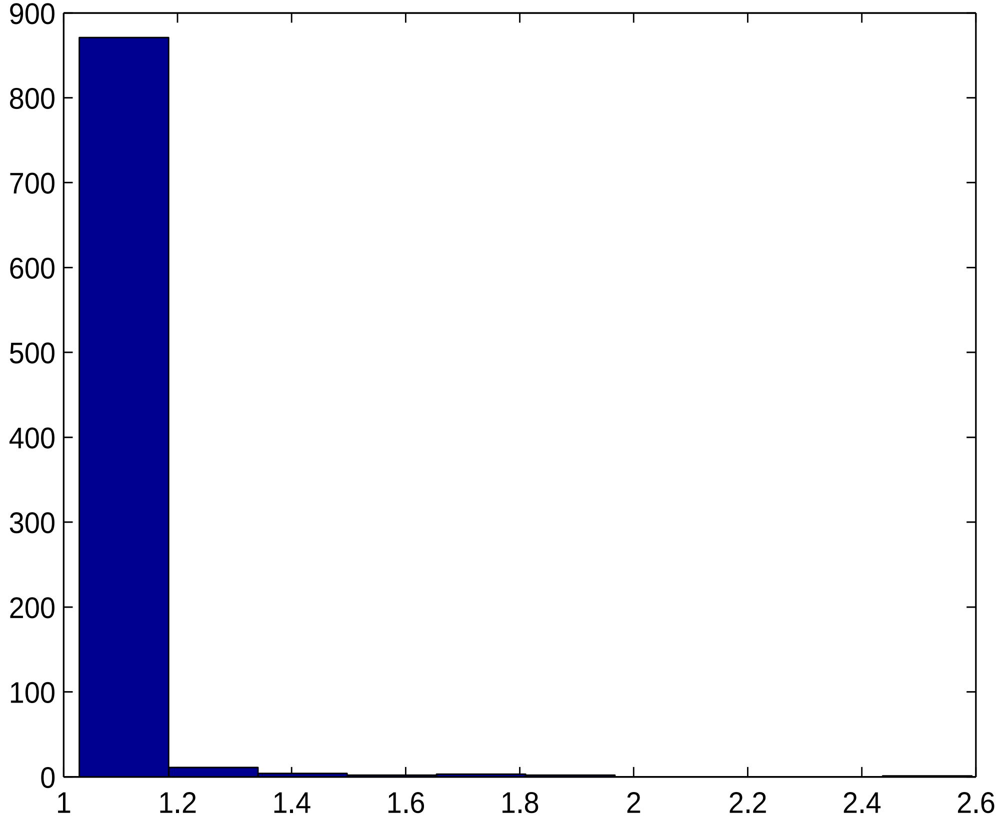
$$\eta := \xi / \sqrt{\min_{s+1 \le i \le m} \xi^T Q_i \xi > 0}.$$

**Theorem.** (He, Xie, and Z.; 2006)
The above algorithm yields an $O(m^3)$ approximation with probability of at least 7.5%.

# A Simulation Test




Upper bound on $v(MV_q)/v^*$, $n = 33$, $m = 34$, 1000 realizations.




The histogram of the previous figure.

## Conclusions

- Randomness is a force of nature
- Beyond Natural Science, so is the case in Management Science
- Randomization can be controlled and used in making decisions
- Theory embraces practice in this endeavor

## URL of the reports

`http://www.se.cuhk.edu.hk/~zhang/#workingpaper`

- *Semidefnite Relaxation Bounds for Indefinite Homogeneous Quadratic Optimization*, Technical Report SEEM2007-01, Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong, 2007 (with Simai He, Zhi-Quan Luo, and Jiawang Nie).

- *Approximation Bounds for Quadratic Optimization with Homogeneous Quadratic Constraints*, Technical Report SEEM2005-07, Department of Systems Engineering & Engineering Management, The Chinese University of Hong Kong, 2005 (with Z.Q. Luo, N.D. Sidiropoulos, and P. Tseng).