An Axiomatic Approach to Ranking Systems

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Acknowledgment

- Most work presented in this talk is a joint work with Alon Altman.

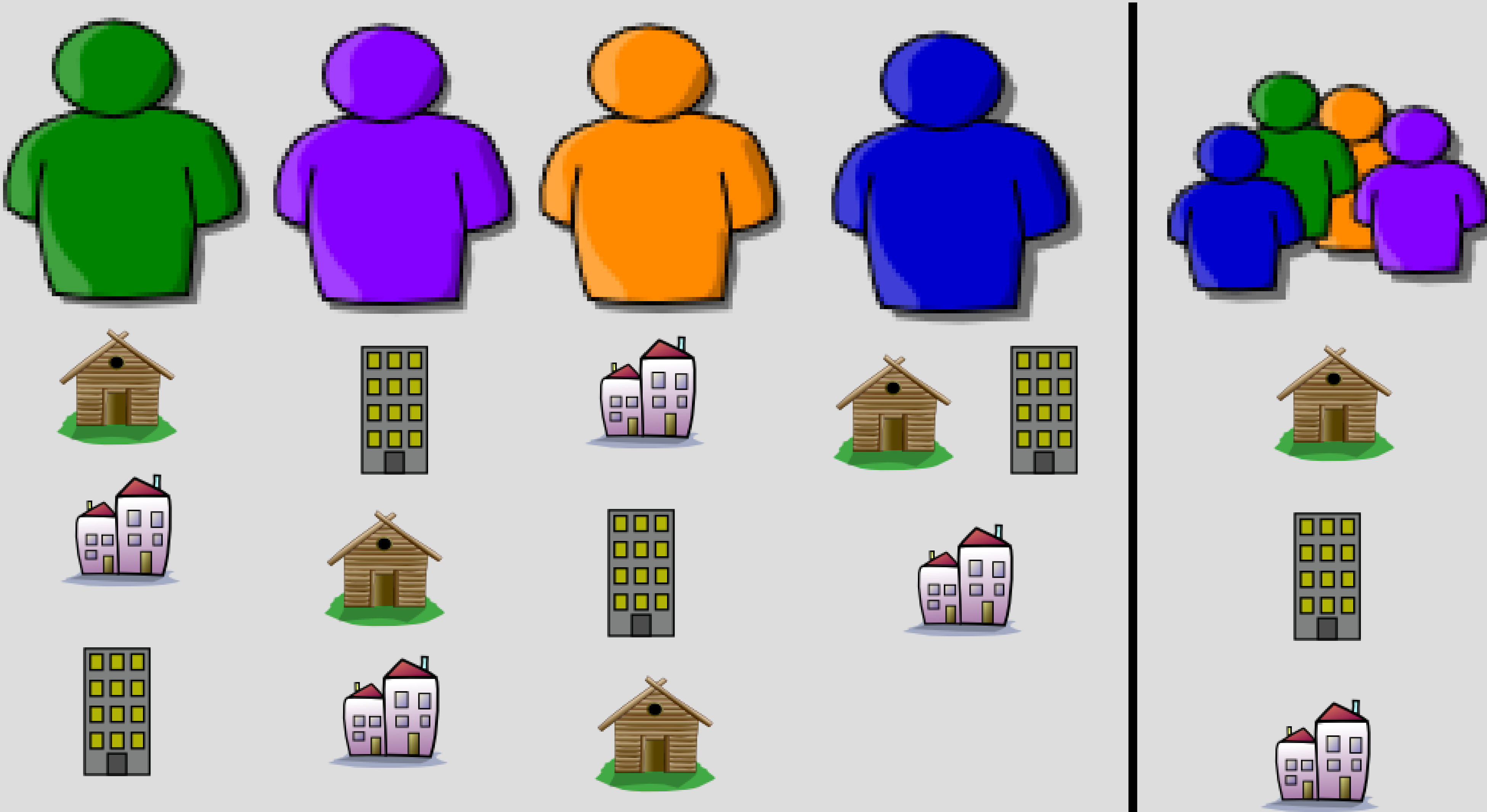
Ranking Systems – Introduction

- Systems in which agents rank for each other are aggregated into a social ranking.
- Ranking systems can be defined in the terms of a *ranking function* combining the individual votes of the agents into a social ranking of the agents.
- Can be seen as a variation of the *social choice* problem where the agents and alternatives coincide.

Social Choice

- The classical *social choice* setting is comprised of:
  - A set of agents
  - A set of alternatives
  - A preference relation for each agent over the set of alternatives.
- A *social welfare function* is a mapping between the agents' individual preferences into a social ranking over the alternatives.
- The goal: produce “good” social welfare functions.

Social Choice - Example



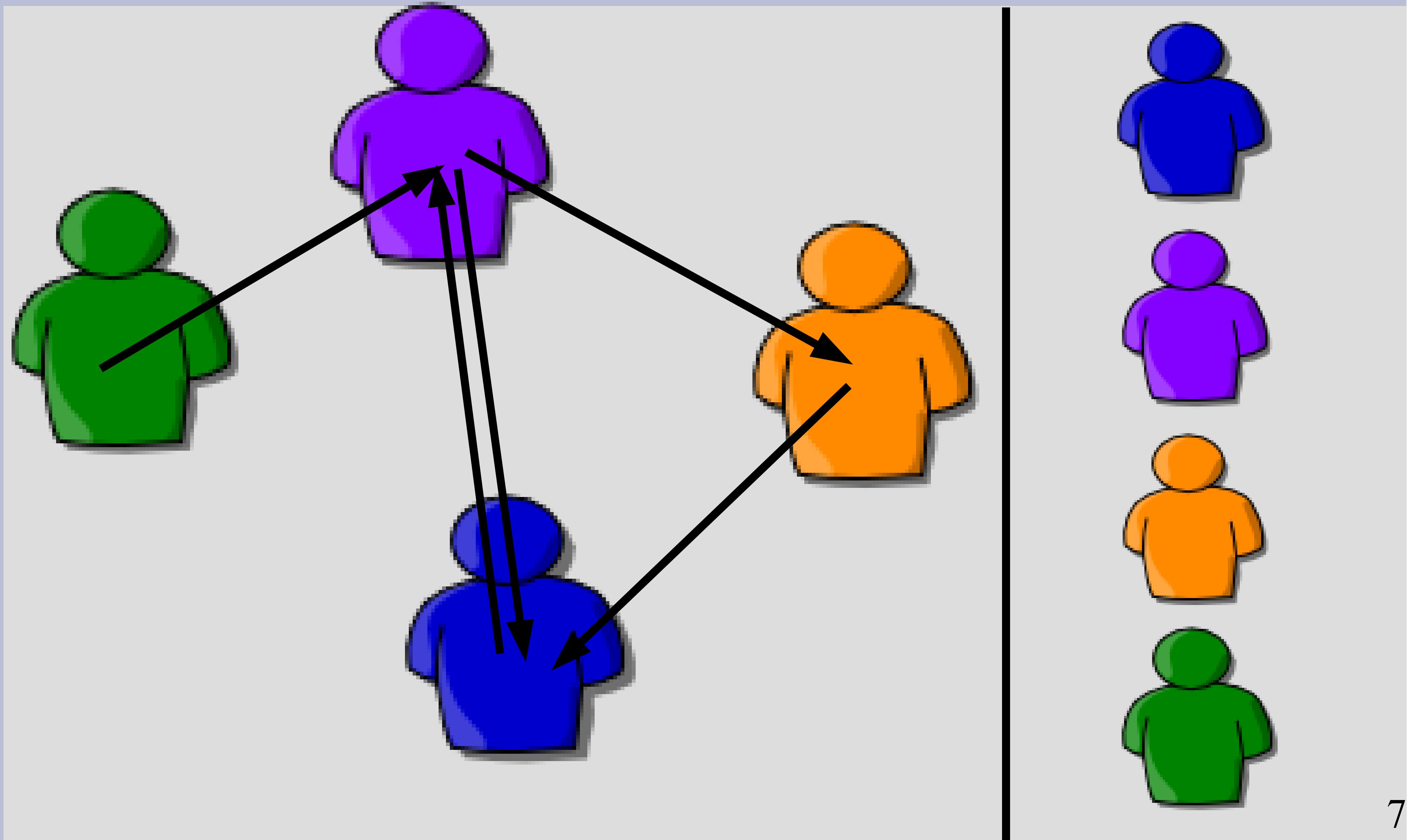
Graph Ranking Systems

- Voters and alternatives are the same set.
- Each agent may only make binary votes: only specify some subset of the agents as “good”.
- Preferences of all the agents may be represented as a graph, where the agents are the vertices and the votes are the edges.
- Applies for ranking WWW pages and eBay traders.

PageRank

Reputation System

Ranking systems



Ranking System – Definition

- Therefore, a (graph) *ranking system* can simply be defined as a functional from the set of all graphs, to the set of linear orderings on the vertices.
- Such a function may be partial. That is, rank only a specific set of graphs, in which case we call it a *partial ranking system*.

The Axiomatic Approach

- We try to find basic properties (**axioms**) satisfied by ranking systems.
- Encompasses two distinct approaches:
  - The normative approach, in which we study sets of axioms that *should* be satisfied by a ranking system; and
  - The descriptive approach, in which we devise a set of axioms that *are* uniquely satisfied by a known ranking system
- We apply both to ranking systems, similarly to seminal studies in the classical social choice setting (Arrow impossibility theorem, May theorem).

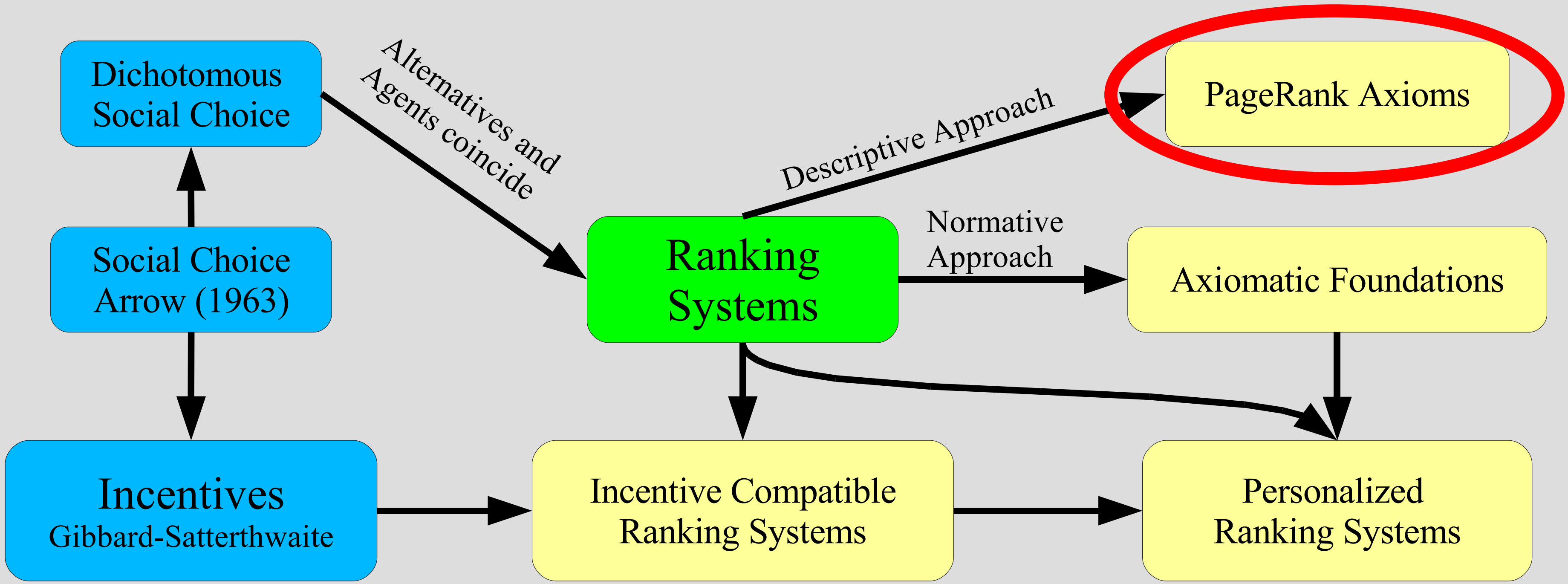
The Normative Approach

- Arrow's(1963) **impossibility theorem** is one of the most important results of the normative approach in Social Choice.
- Does not apply to Ranking Systems.
- In the ranking systems setting, different axioms arise from the fact that the voters and alternatives coincide.

The Descriptive Approach

- In social choice, May's Theorem(1952), provides an axiomatization of the majority rule.
- This approach is useful in ensuring the axioms we suggest are satisfiable.
- We apply this approach towards the axiomatization of the PageRank ranking system.

Research Map



PigeonRank

PageRank

- Simplified version of PageRank
- Ranks according to the stationary probabilities of a random walk.
- We assume the graph $G=(V,E)$ to be strongly connected.
- Let A_G be the following matrix:

$$[A_G]_{i,j} = \begin{cases} 1/|S_G(v_j)| & (v_j, v_i) \in E \\ 0 & \text{Otherwise.} \end{cases}$$

where $S_G(v)$ is the successor set of v .

PageRank (cont.)

- The PageRank of a graph G is defined as the principal eigenvector of the matrix A.
- That is, the PageRank of G is the vector **x** satisfying $A_g\mathbf{x}=\mathbf{x}$.
- The PageRank ranking system *PR* is the ordering on *V* according to **x**:

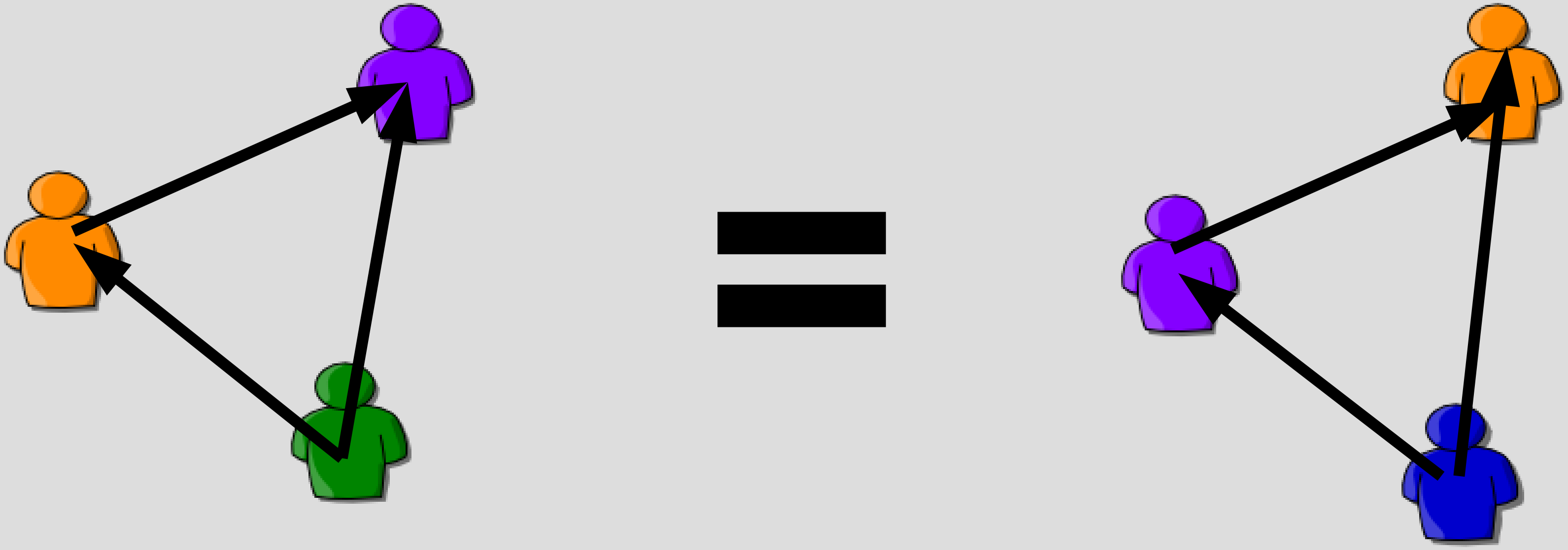
$$v_1 \leqslant_{PR} v_2 \iff x_1 \leq x 2$$

The PageRank Axioms

- Our representation theorem for PageRank requires the following five axioms:
  - Isomorphism;
  - Self-Edge;
  - Vote by Committee;
  - Collapsing; and
  - Proxy

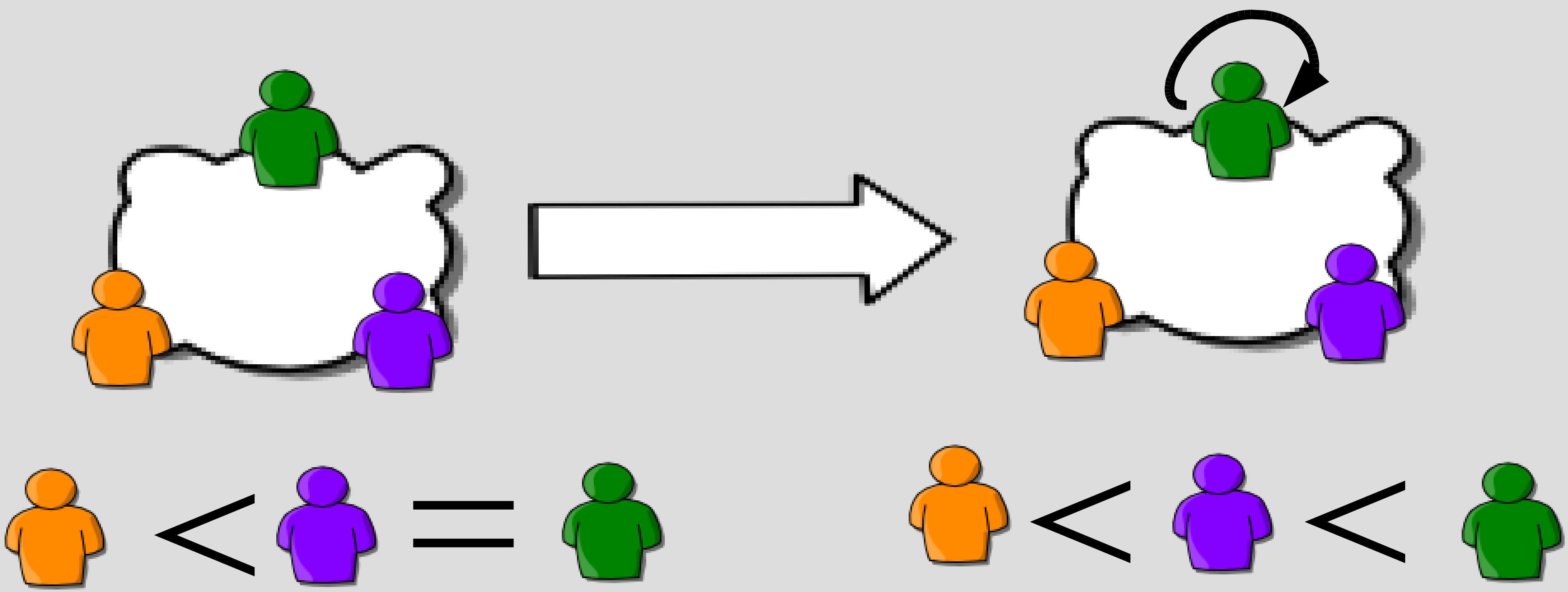
Isomorphism

- A ranking system satisfying *isomorphism* is not sensitive to renaming of the agents, but only to the structure of the graph
- This axiom is similar to the *anonymity* and *neutrality* axioms of classical social choice.

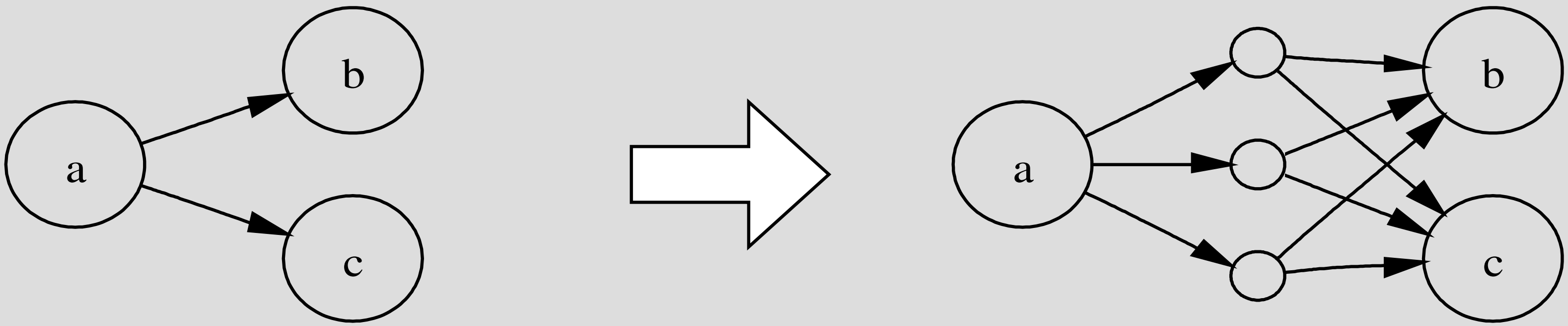


Self-Edge

- This axiom states that adding a self edge on *v* strengthens *v*, but does not change the relative ranking of other vertices.



Vote by Committee



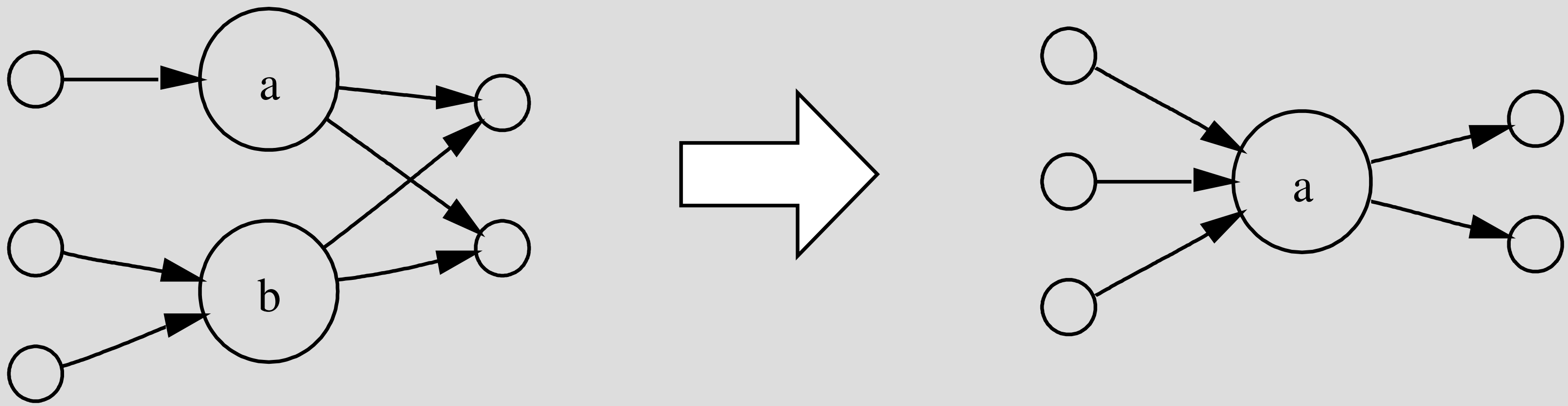
- The Vote by Committee axiom captures the fact that an agent may vote indirectly via any number of intermediate agents, each of which vote to the agent's original preferences.
- This indirect voting does not change the relative ranks of any agents.

Vote by Committee (cont.)

Formal Definition:

Let F be a ranking system. F satisfies *vote by committee* if for every vertex set V, for every vertex $v \in V$, for every graph $G = (V, E) \in \mathbb{G}_V$, for every $v_1, v_2 \in V$, and for every $m \in \mathbb{N}$: Let $G' = (V \cup \{u_1, u_2, \ldots, u_m\}, E \setminus \{(v, x)|x \in S_G(v)\} \cup \{(v, u_i)|i = 1, \ldots, m\} \cup \{(u_i, x)|x \in S_G(v), i = 1, \ldots, m\})$, where $\{u_1, u_2, \ldots, u_m\} \cap V = \emptyset$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

Collapsing



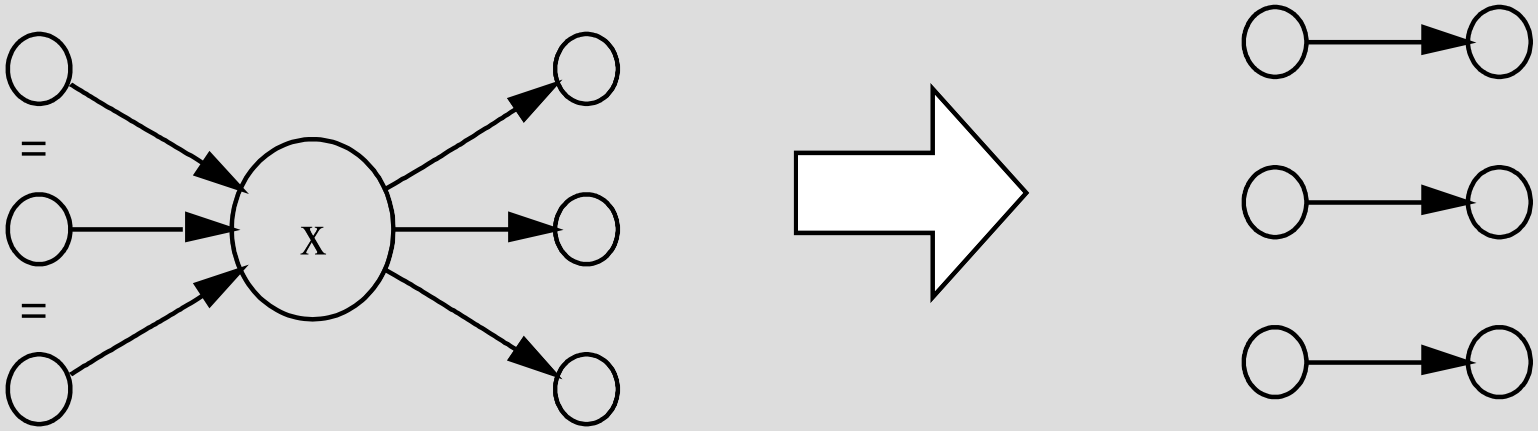
- The collapsing axiom captures the fact that voters which have the same preferences may be collapsed to a single voter with the same preferences, voted by all the voters for both.
- We assume the voter sets for the collapsed vertices are disjoint and do not include *a* or *b*.
- This collapsing only change the rank of *a*.

Collapsing (cont.)

Formal Definition:

Let F be a ranking system. F satisfies *collapsing* if for every vertex set V, for every $v, v' \in V$, for every $v_1, v_2 \in V \setminus \{v, v'\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ for which $S_G(v) = S_G(v')$, $P_G(v) \cap P_G(v') = \emptyset$, and $[P_G(v) \cup P_G(v')] \cap \{v, v'\} = \emptyset$:
Let $G' = (V \setminus \{v'\}, E \setminus \{(v', x) | x \in S_G(v')\} \setminus \{(x, v') | x \in P_G(v')\} \cup \{(x, v) | x \in P_G(v')\})$.
Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

Proxy



- The proxy axiom captures the fact that *n* voters of equal rank who have voted via a proxy (another agent) for *n* alternatives, can achieve the same result by directly voting for one alternative each.

Proxy (cont.)

Formal Definition:

Let F be a ranking system. F satisfies *proxy* if for every vertex set V, for every vertex $v \in V$, for every $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ for which $|P_G(v)| = |S_G(v)|$, for all $p \in P_G(v)$: $S_G(p) = \{v\}$, and for all $p, p' \in P_G(v)$: $p \simeq p'$: Assume $P_G(v) = \{p_1, p_2, \ldots, p_m\}$ and $S_G(v) = \{s_1, s_2, \ldots, s_m\}$. Let $G' = (V \setminus \{v\}, E \setminus \{(x, v), (v, x) | x \in V\} \cup \{(p_i, s_i) | i \in \{1, \ldots, m\}\})$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

Soundess

- **Proposition:** The PageRank ranking system *PR* satisfies isomorphism, self edge, vote by committee, collapsing, and proxy.

- This proposition is proven by a simple application of linear algebra.

Completeness

- The question arises whether PageRank is the *only* ranking system satisfying these axioms.

Completeness

- The question arises whether PageRank is the *only* ranking system satisfying these axioms.

Theorem: Any ranking system that satisfies isomorphism, self edge, vote by committee, collapsing, and proxy is the PageRank ranking system.

Completeness

- In order to prove completeness, we will first show three strong properties that are entailed by our five axioms:
  - **Weak Deletion;**
  - **Strong Deletion; and**
  - **Duplication**

Weak Deletion

- The Weak Deletion property allows us to remove a vertex that has both an in-degree and an out-degree of 1.
- Formally,

  Let V be a vertex set and let $v \in V$ be a vertex. Let $G = (V, E) \in \mathbb{G}_V$ be a graph where $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$. We will use $\mathbf{Del}(G, v)$ to denote the graph $G' = (V', E')$ defined by:

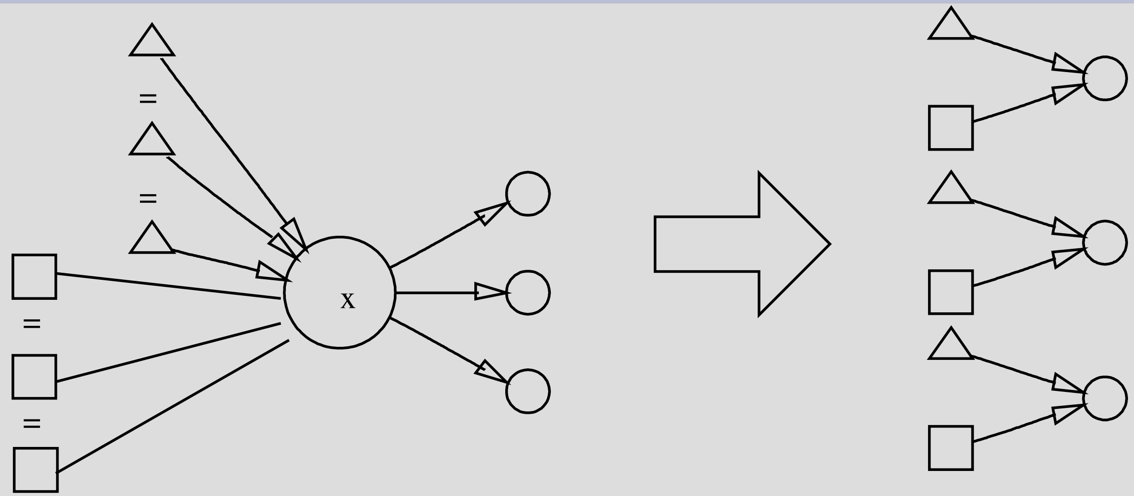
$$\begin{aligned} V' &= V \setminus \{v\} \\ E' &= E \setminus \{(p, v), (v, s)\} \cup \{(p, s)\}. \end{aligned}$$

Weak Deletion (cont.)

Now we can state the weak deletion property:

Let F be a ranking system. F has the *weak deletion* property if for every vertex set V, for every vertex $v \in V$ and for all vertices $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $S(v) = \{s\}$, $P(v) = \{p\}$, and $(s, p) \notin E$: Let $G' = \mathbf{Del}(G, v)$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

Strong Deletion



- The Strong Deletion property is a generalization of the proxy axiom, allowing removal of a vertex with m sets of t equal predecessors, and t successors.
- One element of each equal sets is set to point to each of the original successors.
- This change does not affect the relative rank of the remaining vertices.

Strong Deletion (cont.)

Formally, the strong deletion operator is defined:

Let V be a vertex set and let $v \in V$ be a vertex. Let $G = (V, E) \in \mathbb{G}_V$ be a graph where $S(v) = \{s_1, s_2, \ldots, s_t\}$ and $P(v) = \{p^i_j | j = 1, \ldots, t; i = 0, \ldots, m\}$, and $S(p^i_j) = \{v\}$ for all $j \in \{1, \ldots t\}$ and $i \in \{0, \ldots, m\}$. We will use $\mathbf{Delete}(G, v, \{(s_1, \{p^i_1 | i = 0, \ldots m\}), \ldots, (s_t, \{p^i_t | i = 0, \ldots m\})\})$ to denote the graph $G' = (V', E')$ defined by:

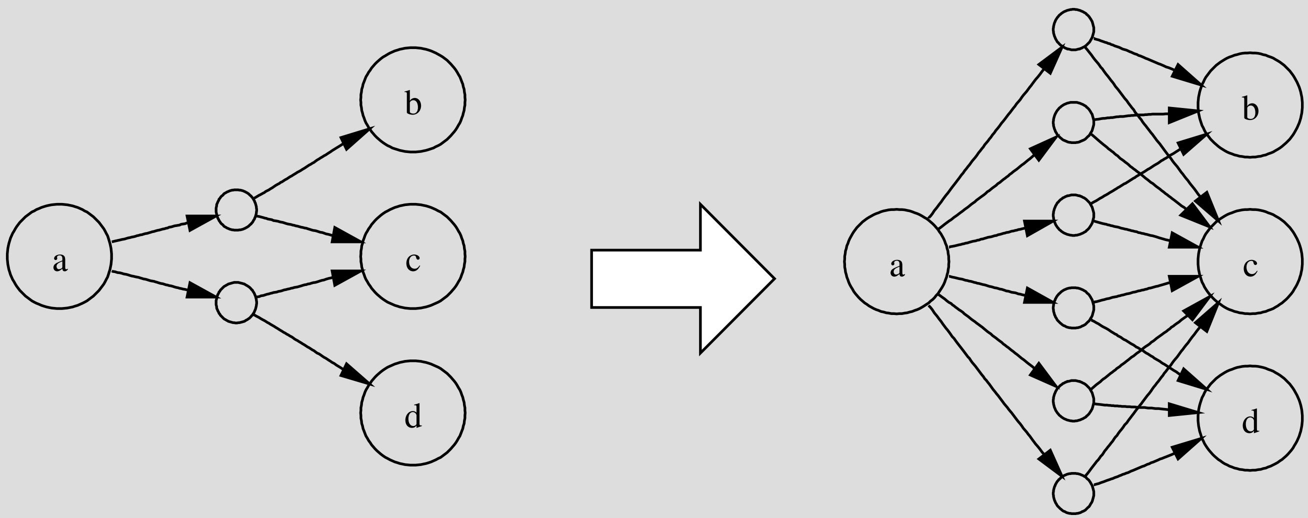
$$\begin{aligned} V' &= V \setminus \{v\} \\ E' &= E \setminus \{(p^i_j, v), (v, s_j) | i = 0, \ldots, m; j = 1, \ldots, t\} \cup \\ &\quad \cup \{(p^i_j, s_j) | i = 0, \ldots, m; j = 1, \ldots, t\}. \end{aligned}$$

Strong Deletion (cont.)

Now we can state the strong deletion property:

Let F be a ranking system. F has the *strong deletion* property if for every vertex set V, for every vertex $v \in V$, for all $v_1, v_2 \in V \setminus \{v\}$, and for every graph $G = (V, E) \in \mathbb{G}_V$ s.t. $S(v) = \{s_1, s_2, \ldots, s_t\}$, $P(v) = \{p^i_j | j = 1, \ldots, t; i = 0, \ldots, m\}$, $S(p^i_j) = \{v\}$ for all $j \in \{1, \ldots t\}$ and $i \in \{0, \ldots, m\}$, and $p^i_j \simeq^F_G p^i_k$ for all $i \in \{0, \ldots, m\}$ and $j, k \in \{1, \ldots t\}$: Let
$G' = \mathbf{Delete}(G, v, \{(s_1, \{p^i_1 | i = 0, \ldots m\}), \ldots (s_t, \{p^i_t | i = 0, \ldots m\})\})$.
Then, $v_1 \preceq^F_G v_2$ iff $v_1 \preceq^F_{G'} v_2$.

Duplication



- The duplication property allows duplication of an agent's successors by any factor.
- The new vertices have the same successors as the old.
- The relative ranking of all vertices except the duplicated successors does not change.

Duplication (cont.)

Formally, the duplication operator is defined:

Let V be a vertex set and let $G = (V, E) \in \mathbb{G}_V$ be a graph. Let $S(v) = \{s_1^0, s_2^0, \ldots, s_t^0\}$. We will use $\mathbf{Duplicate}(G, v, m)$ to denote the graph $G' = (V', E')$ defined by:

$$\begin{aligned} V' &= V \cup \{s_j^i | i = 1, \ldots, m-1; j = 1, \ldots t\} \\ E' &= E \cup \{(v, s_j^i) | i = 1, \ldots, m-1; j = 1, \ldots t\} \cup \\ &\quad \cup \{(s_j^i, u) | i = 1, \ldots, m-1; j = 1, \ldots t; u \in S_G(s_j^0)\}. \end{aligned}$$

Duplication (cont.)

- Now we can state the duplication property:

  Let F be a ranking system. F has the *edge duplication* property if for every vertex set V, for all vertices $v, v_1, v_2 \in V$, for every $m \in \mathbb{N}$, and for every graph $G = (V, E) \in \mathbb{G}_V$: Let $S(v) = \{s_1^0, s_2^0, \ldots, s_t^0\}$, and let $G' = \mathbf{Duplicate}(G, v, m)$. Then, $v_1 \preceq_G^F v_2$ iff $v_1 \preceq_{G'}^F v_2$.

Satisfication

- The three properties are entailed by our axioms:
  - **Lemma:** Let F be a ranking system that satisfies isomorphism, vote by committee, and proxy. Then, F has the weak deletion property.
  - **Lemma:** Let F be a ranking system that satisfies collapsing and proxy. Then, F has the strong deletion property.
  - **Lemma:** Let F be a ranking system that satisfies isomorphism, vote by committee, collapsing, and proxy. Then, F has the edge duplication property.

Completeness

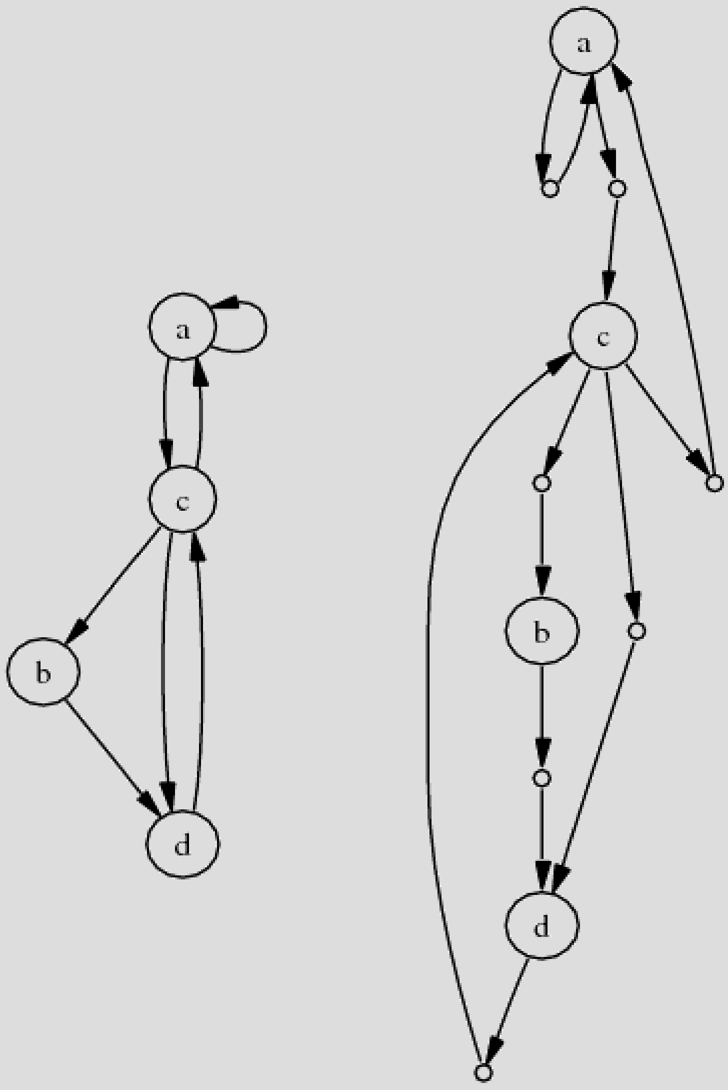
- Do other systems satisfy these five axioms?
- No! PageRank is the only ranking system satisfying all 5 axioms.
- The completeness proof is a constructive one.
- We suggest a (grossly inefficient) algorithm for computing relative PageRank.

Completeness Proof Algorithm

- Fix two vertices *a* and *b*
- Manipulate the graph preserving their relative ranking.
- Apply further manipulation to modify the relative ranking of *a* and *b* in one direction.
- *a* and *b* can then now be proven of equal rank.
- The relative ranking of *a* and *b* in the original graph can now be deduced.

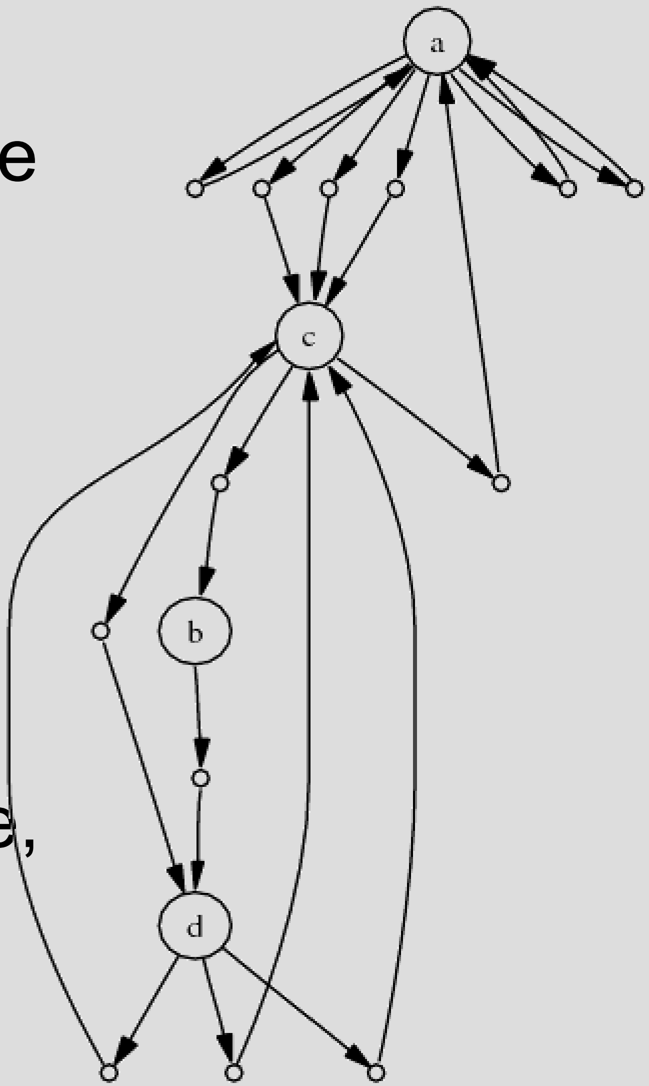
Demonstration of Proof

- Start with the input graph and two vertices *a* and *b* to be compared.
- Add a vertex on each edge.
- The relative ranking of *a* and *b* does not change because of the weak deletion property



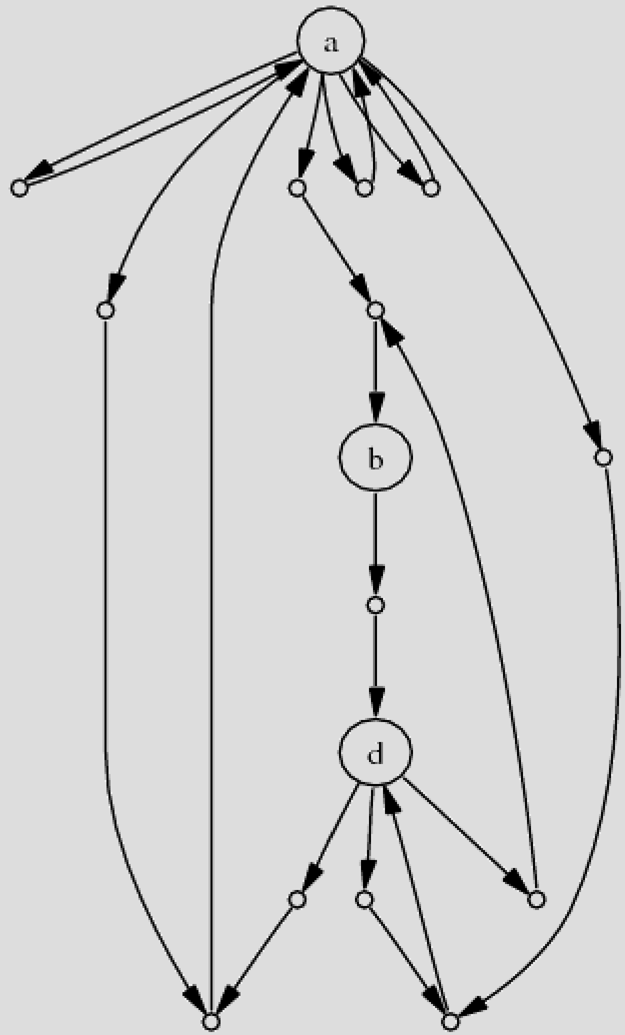
Demonstration of Proof (cont.)

- Select an original vertex except *a* and *b* (*c* in our example), and delete all its self edges with a vertex on them.
- This does not change the relative ranking of *a* and *b* due to the self-edge and weak deletion axioms.
- Next, we use the duplication property to duplicate the predecessors of *c* by *c*'s out degree, without changing the relative ranking of *a* and *b*.



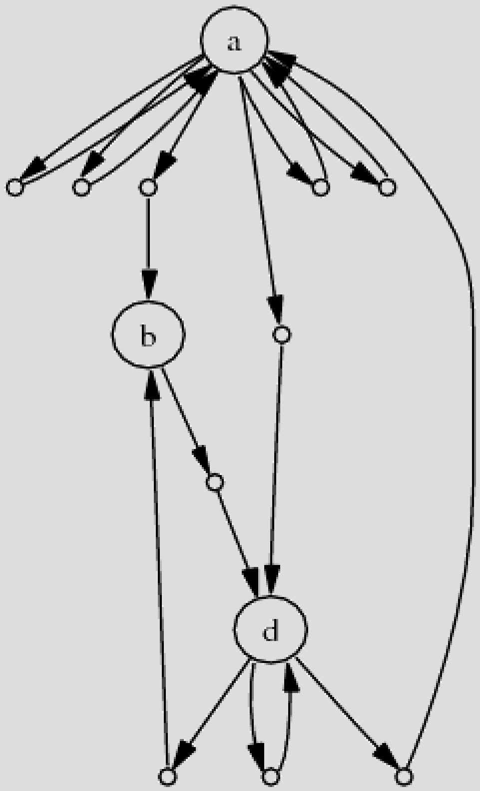
Demonstration of Proof (cont.)

- The isomorphism axiom guarantees that *c* satisfies the conditions of the strong deletion property.
- Thus, we can apply Strong Deletion.
- Due to the strong deletion property, this does not change the relative ranking of *a* and *b*.



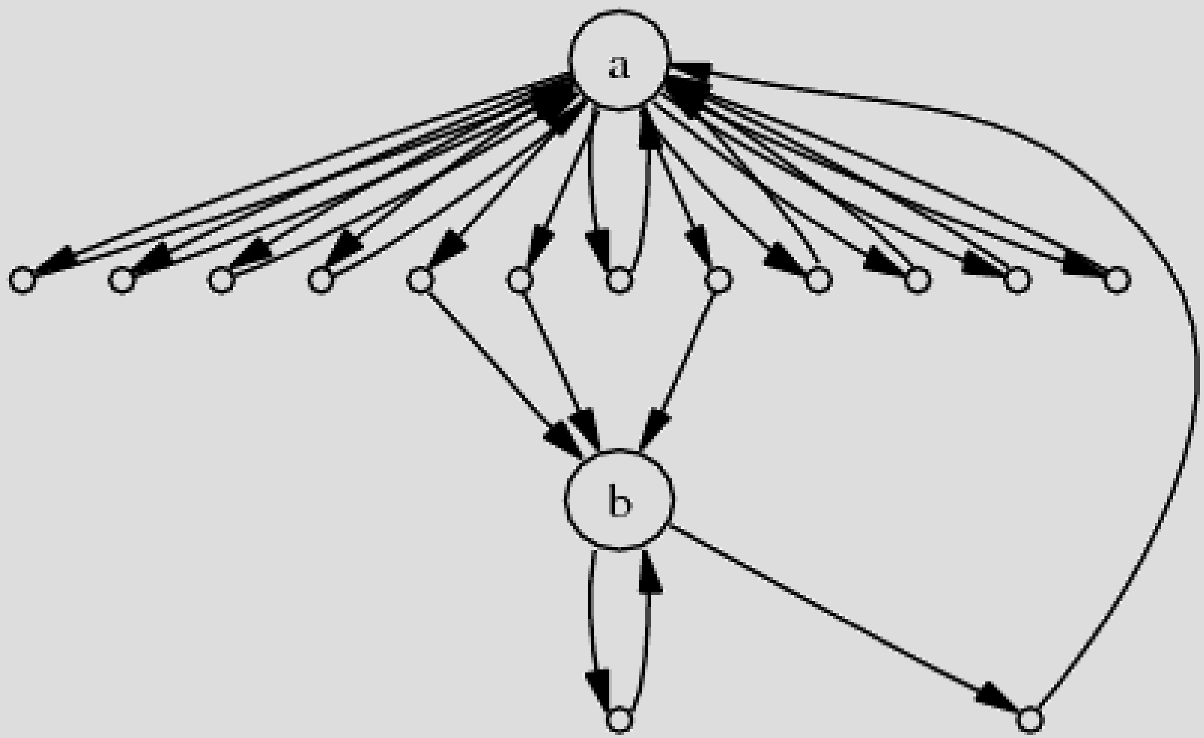
Demonstration of Proof (cont.)

- We apply the strong deletion property again to delete the new vertices that were successors of *c*.
- Again, this does not change the relative ranking of *a* and *b*.
- Note that now again all successors and predecessors of the original vertices are new vertices, and all successors and predecessors of the new vertices are original vertices.



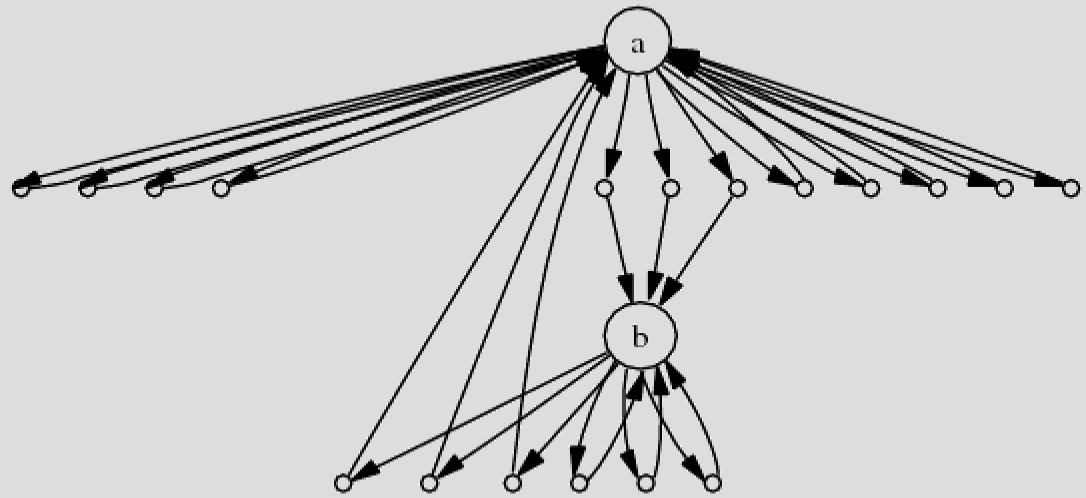
Demonstration of Proof (cont.)

- Repeat the previous steps, selecting a different vertex each time, until the only remaining original vertices are *a* and *b*.



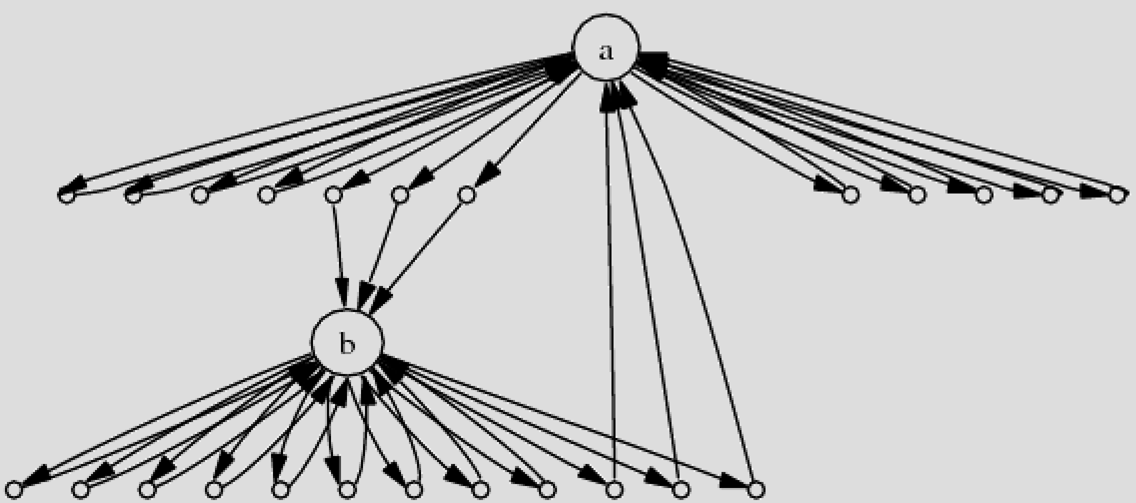
Demonstration of Proof (cont.)

- Equalize the number of edges with vertices from *a* to *b* to the number of edges with vertices from *b* to *a* by duplicating *a* by the number edges with vertices from *b* to *a* and vice versa.
- In our example *b* is duplicated by 3.



Demonstration of Proof (cont.)

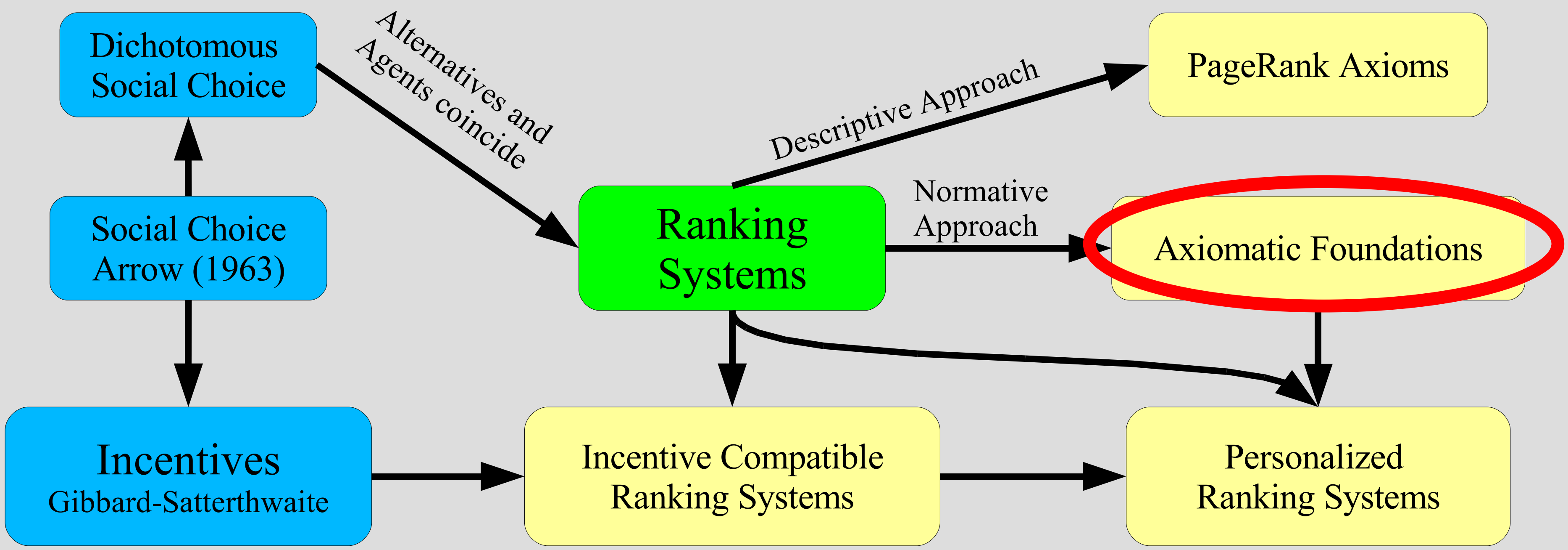
- Assume without loss of generality that *b* has fewer self edges with vertices than *a*.
- Add self edges with vertices to *b*, until *a* and *b* have the same number of self edges (with vertices).



Completeness Proof (cont.)

- Now, *a* and *b* are equally ranked according to the isomorphism axiom.
- But, according to the self edge axiom we increased the relative rank of *b* compared to *a*, so we conclude that in the original graph, *b* was ranked lower than *a*.
- This unique outcome is general, and thus the axioms guarantee a unique ranking, and thus exactly represent PageRank.
- QED

Research Map

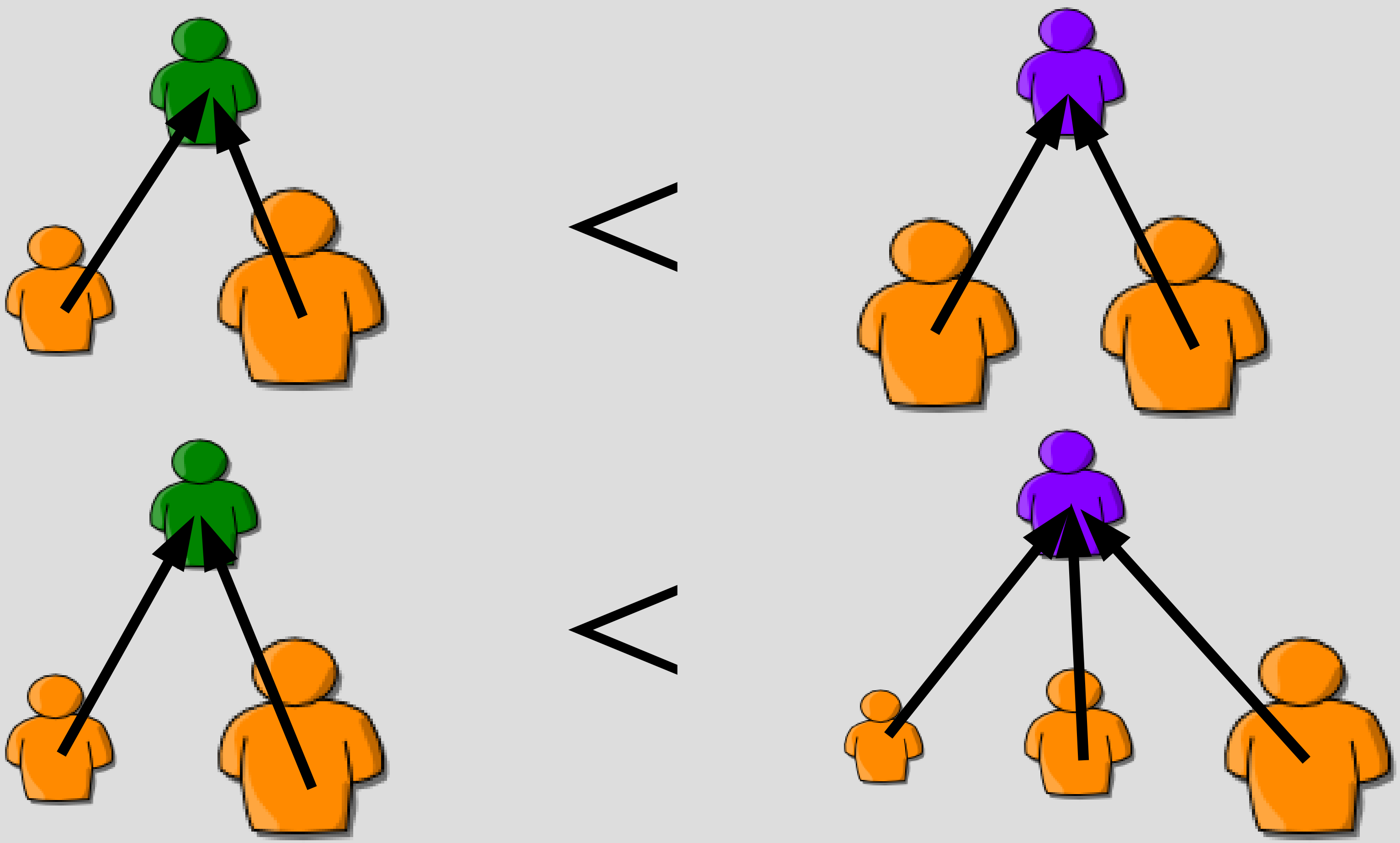


Comment

- We henceforth assume arbitrary graphs, with no self-edges.

Transitive Effects

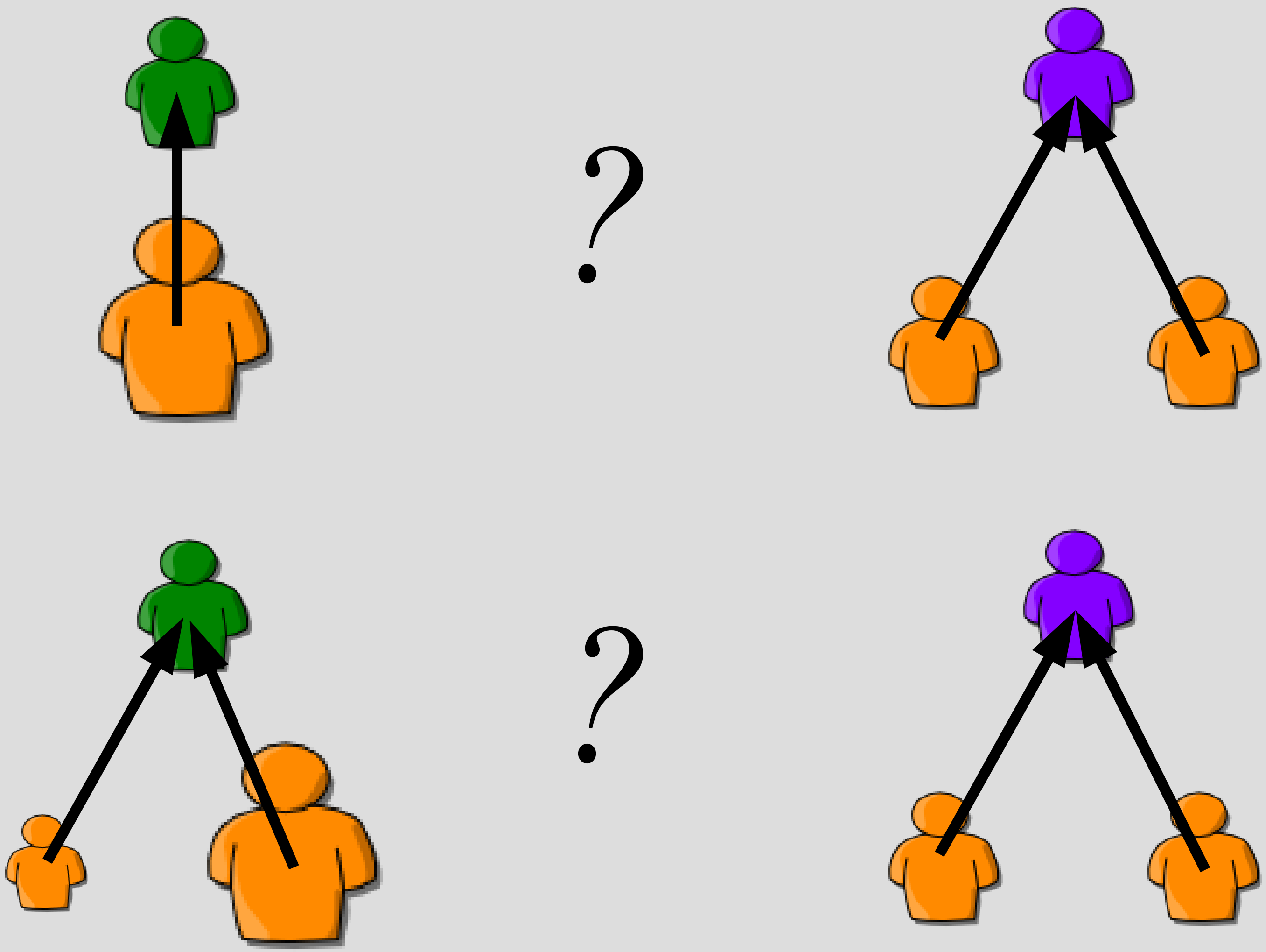
The rank of your voters should affect your own.



Strong Transitivity

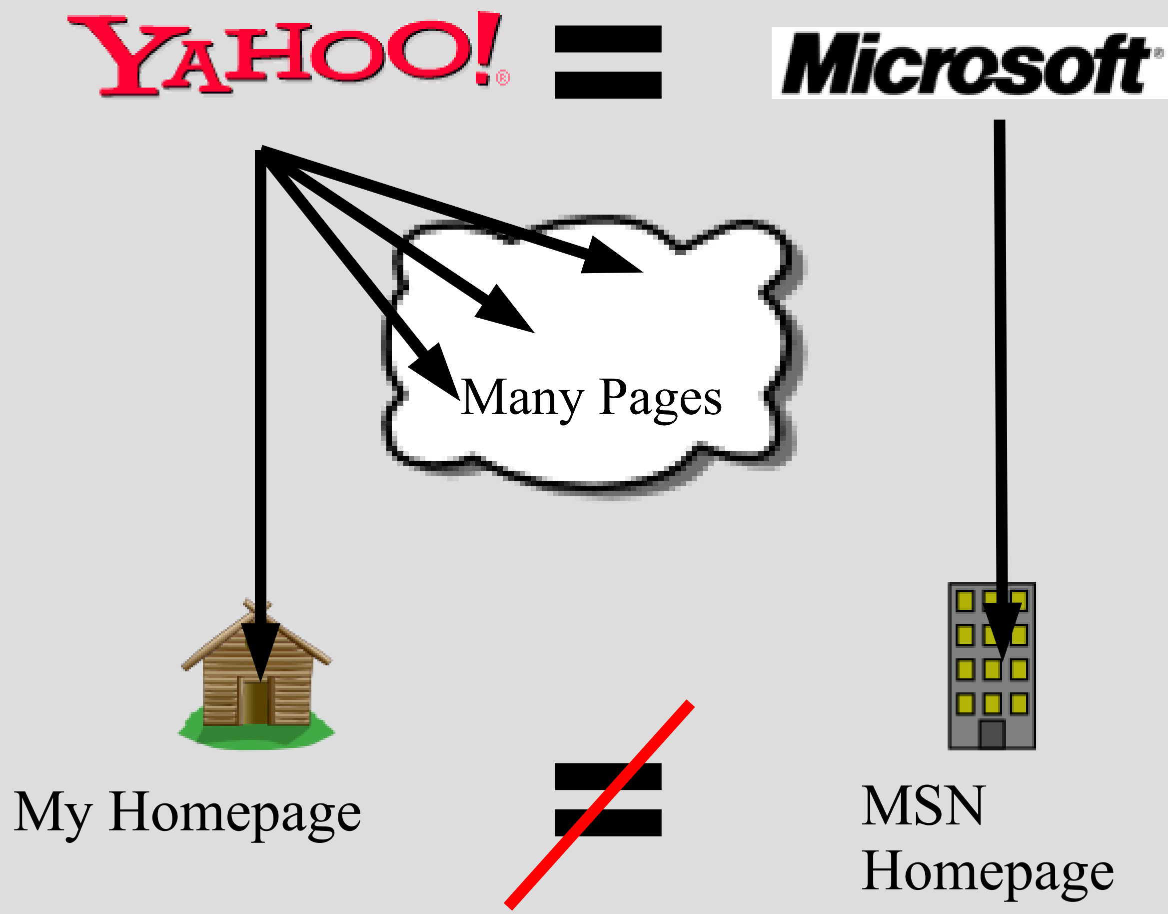
- Formally, a ranking system *F* satisfies *strong transitivity* if for every two vertices *x,y* where *F* ranks *x*'s predecessor set *P(x)* (strictly) weaker than *P(y)*, then *F* must rank *x* (strictly) weaker than *y*.
- We define a predecessor set *P(x)* as being weaker than *P(y)* as the existence of a 1-1 mapping between *P(x)* and *P(y)* where every vertex in *P(x)* is mapped to a stronger or equal vertex in *P(y).* Moreover, *P(x)* is strictly weaker if at least one of the comparisons is strict, or the mapping is not onto.

Strong Transitivity Doesn't always apply



Strong Transitivity too Strong?

Assume:

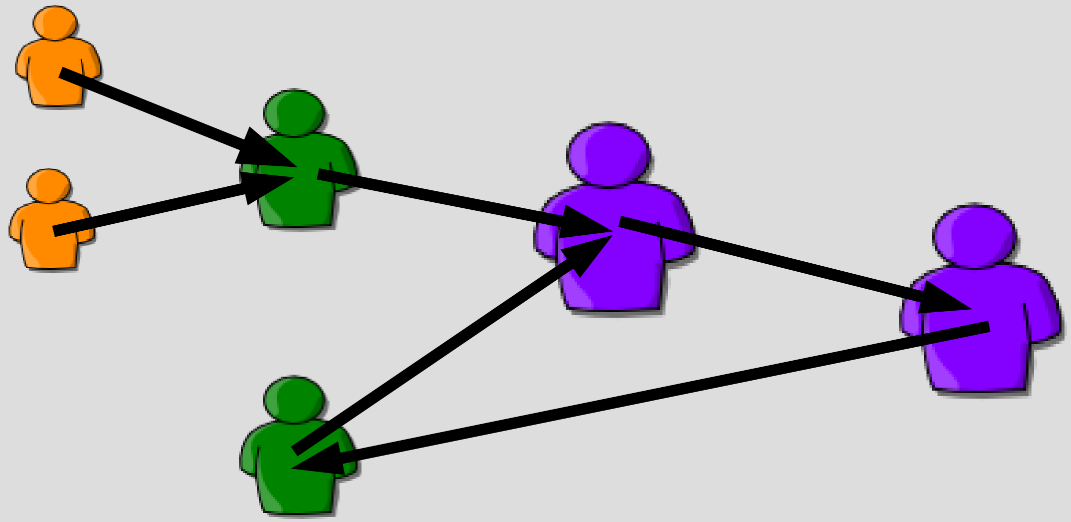


More about Transitivity

- Weak Transitivity
  - The idea: Only match predecessors with equal out-degree.
  - We assume nothing about predecessors of different out-degrees.
  - Otherwise, same as Strong Transitivity.
- PageRank satisfies Weak Transitivity but not Strong Transitivity.
- Strong Transitivity can be satisfied by a nontrivial Ranking System [Tennenholtz 2004]

Ranked IIA

- Consider the statement: “An agent with votes from two weak agents should be ranked the same as one with a vote from one strong agent”.



Ranked IIA

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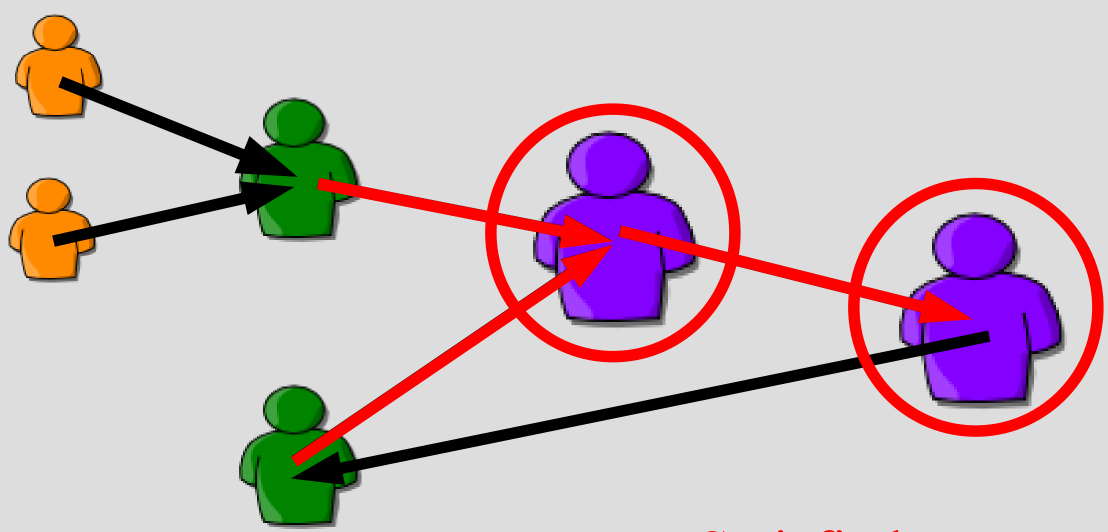

Satisfied

Ranked IIA

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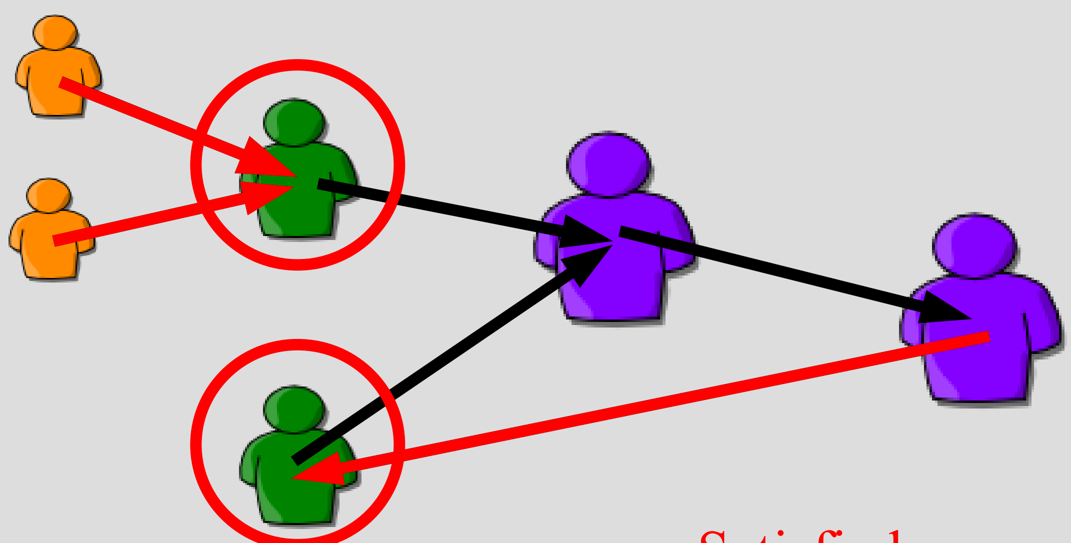

Satisfied

Ranked IIA

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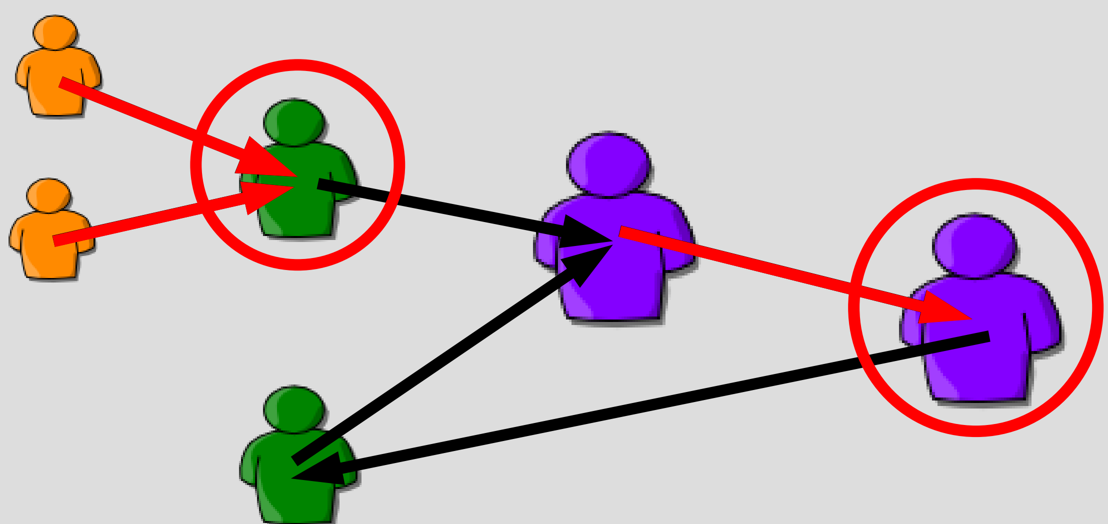


Not Satisfied

Ranked IIA

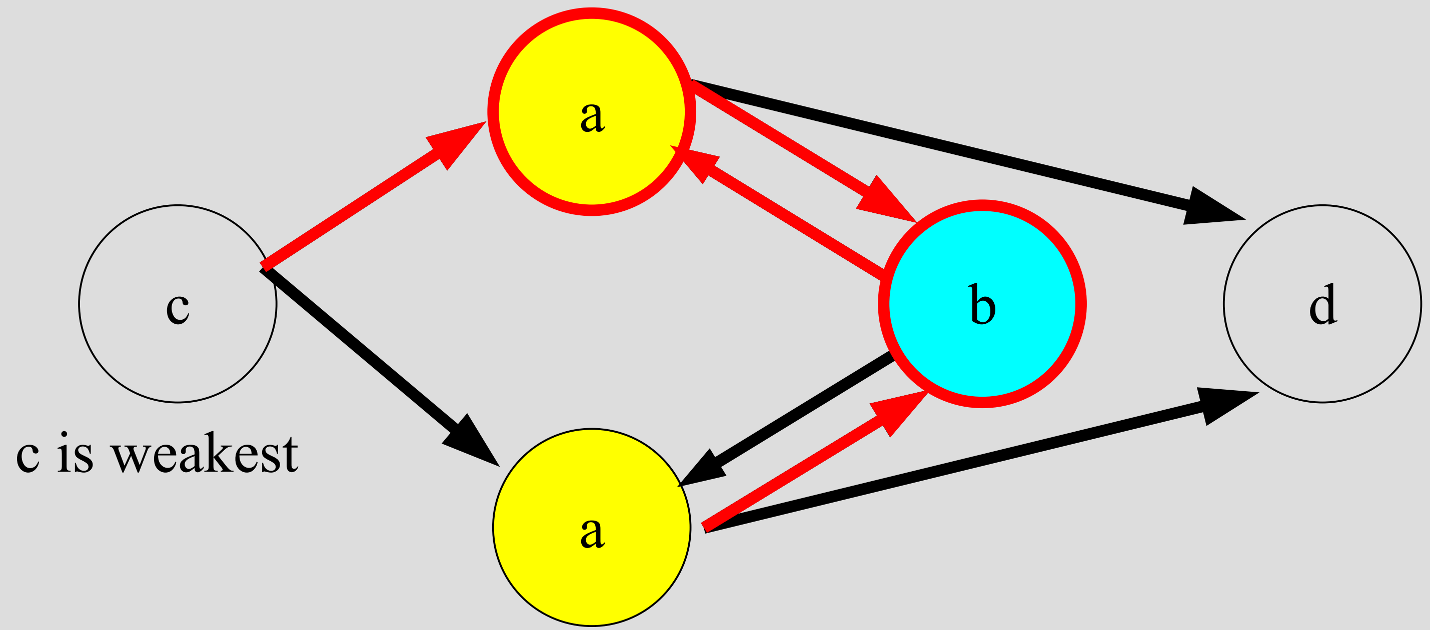
- We would like such comparisons to be *consistent*.
- That is, in every *profile* such as the one described in the previous slide we should decide >/</= consistently.
- This captures the Independence of Irrelevant Alternatives (IIA) for ranking systems.
- Can be seen as an ordinality requirement.
- Compare to Arrow's IIA axiom, which considers the name but not rank of the agents.

Impossibility

- **Theorem**: There exists no general Ranking System that satisfies Weak Transitivity and Ranked IIA.
- Proof: Constructive.
  - We assume existence of such ranking system and see graphs it cannot rank consistently.

Impossibility Proof – Part 1

Assume b ≤ a

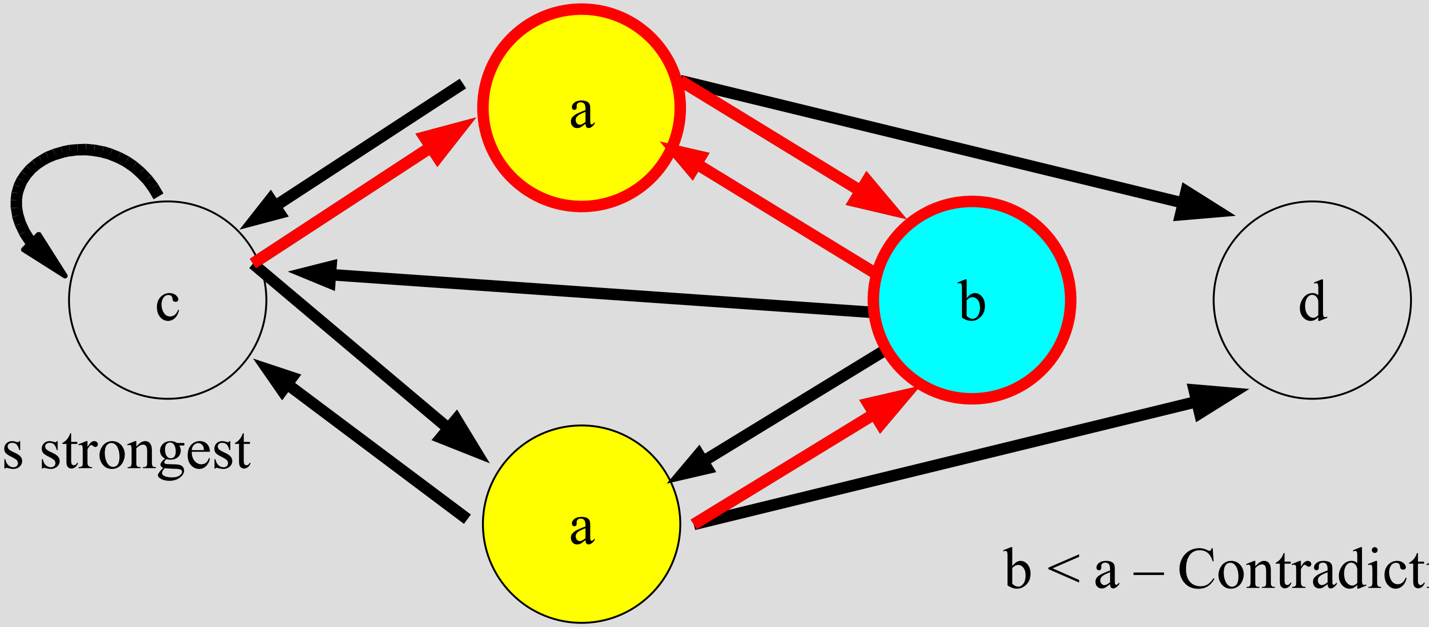


a < b – Contradiction!

→ A vertex with two equal predecessors is stronger than one with one weaker and one stronger predecessor.

Impossibility Proof – Part 2

Assume a ≤ b



c is strongest

b < a – Contradiction!

→ A vertex with two equal predecessors is ***weaker*** than one with one weaker and one stronger predecessor.

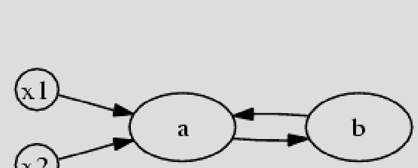
→ Contradiction to part 1. **QED**

Stronger Impossibility Results

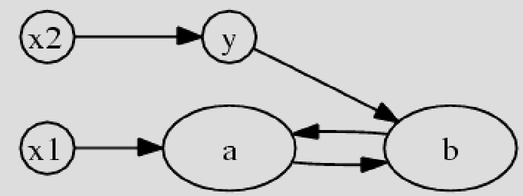
- Our impossibility result exists even in very limited domains:
  - Small graphs (4 agents are enough with Strong Transitivity).
  - Strongly connected graphs (as with PageRank).
  - Bipartite (buyer/seller) graphs.
  - Single vote per agent

One Vote Bipartite Proof

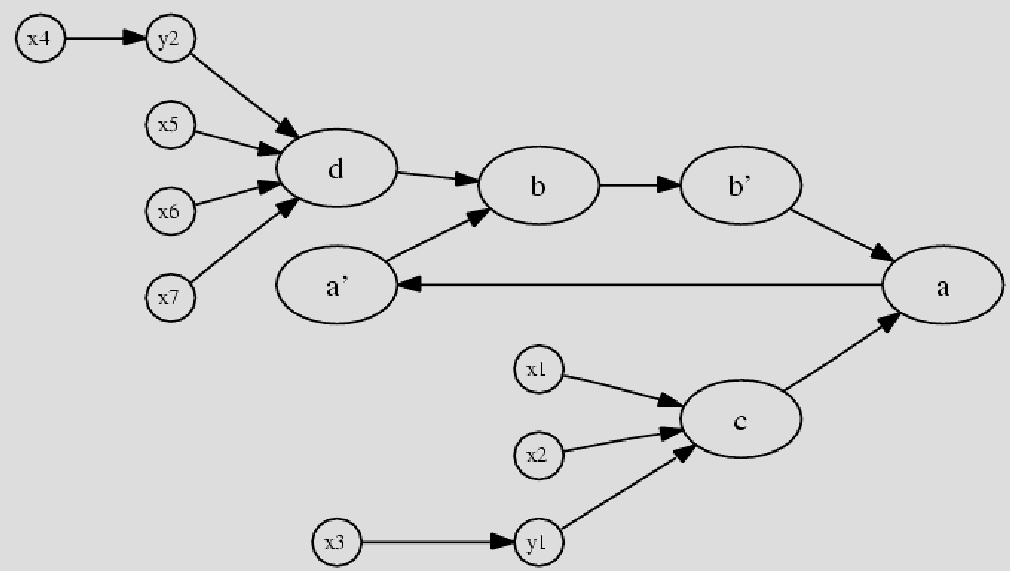
- In G_1: a(3) < b(1,1,2)
- In G_2: a(1,4) < b(2,3)
- In G_3: b(2,3) < a(1,4)
- Contradiction!



(a) Graph G_1



(b) Graph G_2



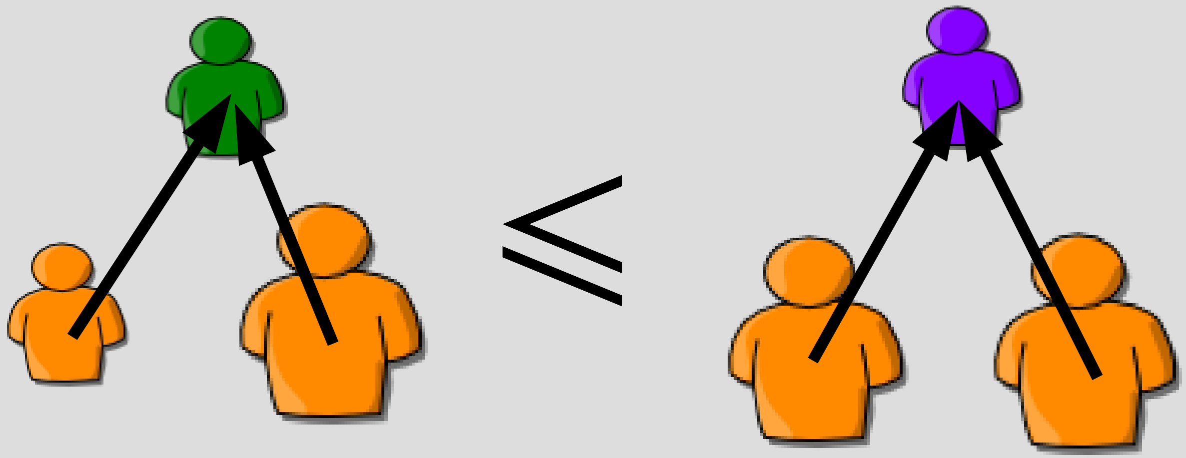
(c) Graph G_3

Transitive effects and IIA?

- We have proven that transitive effects and ranked IIA are incompatible.
- However, it turns out that under a different notion of transitivity these properties can be satisfied together.
- Moreover, the proposed ranking system is nontrivial and interesting.

Quasi-Transitivity

- We define the notion of *quasi-transitivity* as requiring only non-strict comparisons.
- A ranking system *F* satisfies *quasi- transitivity* if for every two vertices *x,y* where *F* ranks *x*'s predecessor set *P(x)* weaker or equal to *P(y)*, then *F* must rank *x* weaker or equal to *y*.



Positive Result

- **Proposition:** There exists a nontrivial ranking system satisfying Ranked IIA and Quasi-Transitivity.
- The *recursive-indegree* ranking system can be defined using a simple and efficient algorithm:

$$v_1 \preceq_G^{RID_r} v_2 \Leftrightarrow \text{value}(v_1, r, \mathbf{0}) \geq \text{value}(v_2, r, \mathbf{0})$$

The *value* function

Procedure value(x, r, h) – returns numeric rank of node x under weight function r given previously seen nodes h:

1. Let $d := \begin{cases} 0 & |P(x)| = 0 \\ r(|P(x)|) & \text{Otherwise.} \end{cases}$

2. Let $h'(y) := \begin{cases} 0 & h(y) = 0 \wedge y \neq x \\ (n+1) \cdot h(y) + d & \text{Otherwise.} \end{cases}$

3. If $h(x) = 0$:

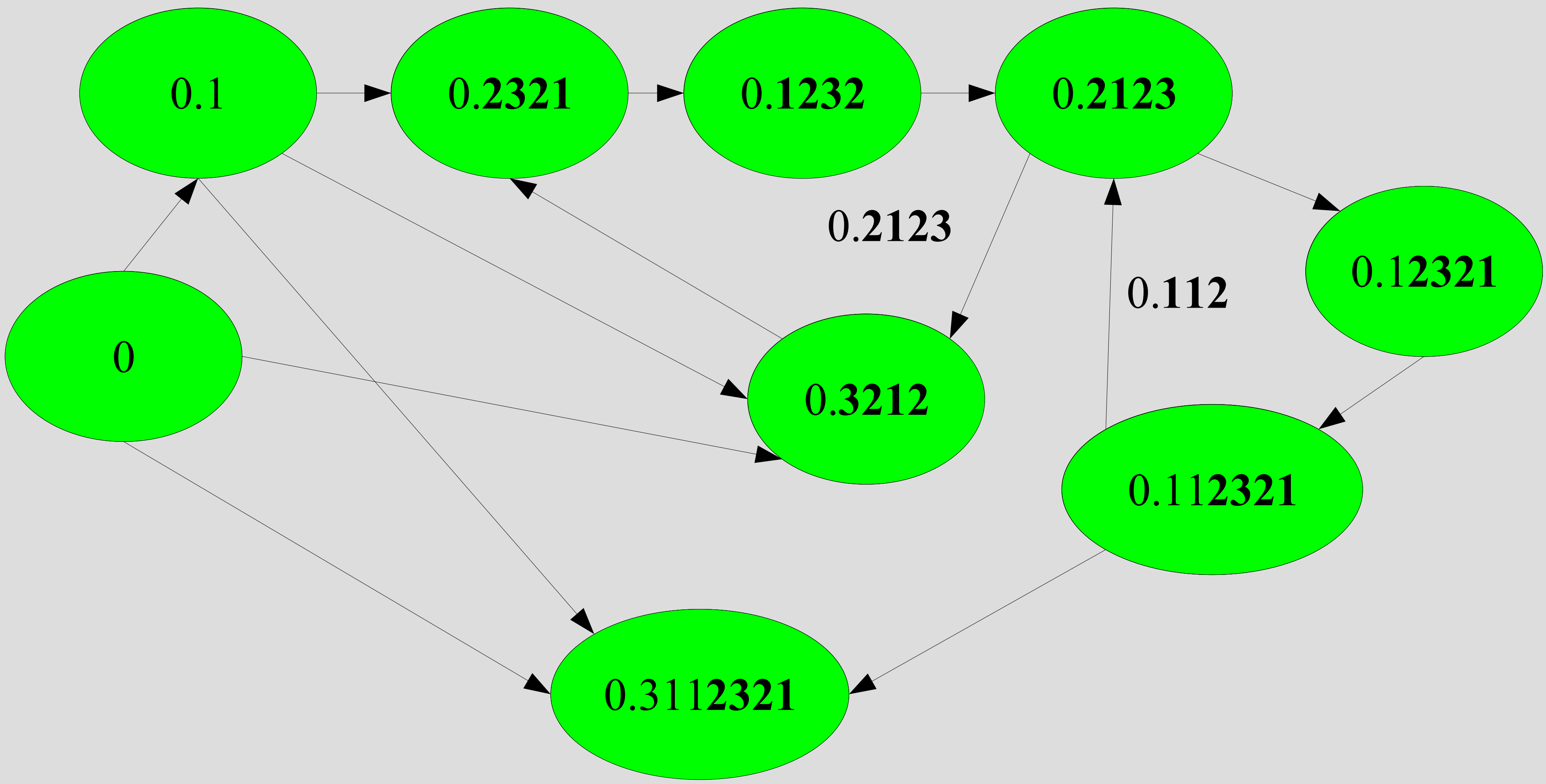
   (a) Return $\frac{1}{n+1}\left[d + \max(\{\text{value}(x, h', r) | p \in P(x)\} \cup \{0\})\right]$

4. Otherwise:

   (a) Let $m = \min\{(n+1)^k - 1 | (n+1)^k > h'(x)\}$.

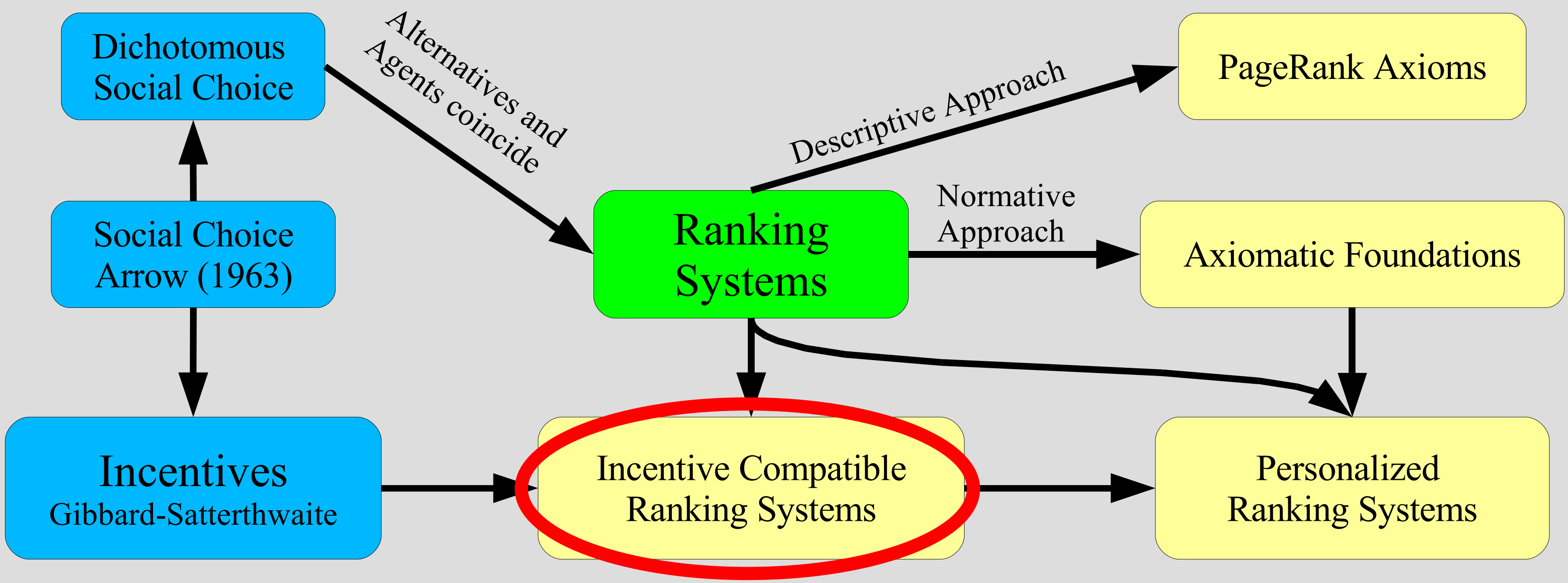
   (b) Return $h'(x)/m$.

Example



0.1
0.2321
0.1232
0.2123
0.2123
0.12321
0.112
0
0.3212
0.112321
0.3112321

Research Map



Incentives

- Agents may choose to cheat and not report their real preferences, in order to improve their position.
- Utility of the agents only depends on their own rank, not on the rank of other agents.
- Utility is nonincreasing in rank.
- Ties are considered a uniform distribution over pure rankings.

Utility Function

- Formally, the utility function *u* for the agents maps for each agent count the number of agents ranked lower than the agent to a utility for that ranking:

$$u_n : \mathbb{N} \rightarrow \mathbb{R}$$

- The expected utility of an agent with *k* agents ranked strictly below it and *m* agents ranked the same is:

$$E[u_n] = u_n^*(k, m) = \frac{1}{m} \sum_{i=k}^{k+m-1} u_n(i)$$

Utility of a ranking

- Let $\preccurlyeq$ be the ordering of the agents of some ranking system F on some graph G=(V,E).
- The utility of agent *v* in graph *G* under ranking system *F* is:

$$u_G^F(v) = u_n^*(|\{u : u \prec v\}|, |\{u : u \simeq v\}|)$$

Incentive Compatibility

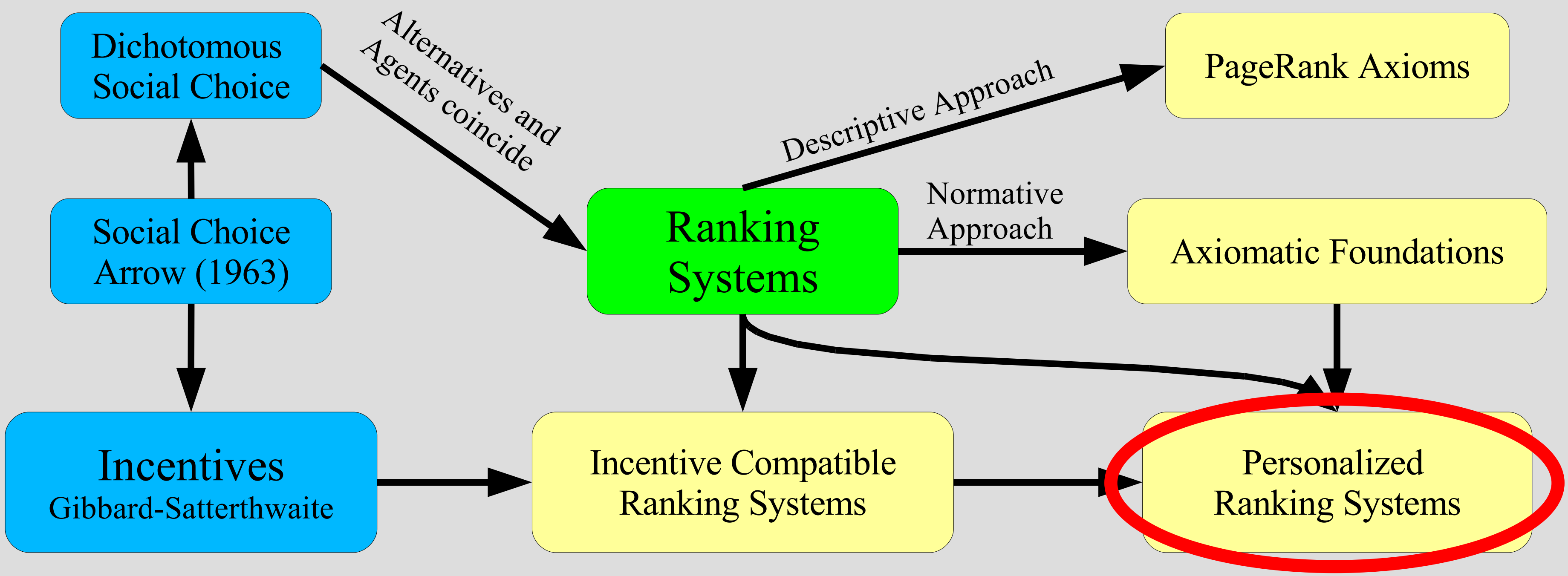
- Let *G=(V,E)* and *G'=(V,E')* be graphs that differ only in the outgoing edges from vertex *v*.
- A ranking system is *strongly incentive compatible*, if for every utility function *u*:

$$u_G^F(v) = u_{G'}^F(v)$$

Results

- We have **classified** several types of incentive compatible ranking systems, under a wide range of axioms.
  - This classification has shown that full incentive compatibility is impossible for any practical purpose.
- We have also **quantified** the incentive compatibility of known ranking systems, and suggested useful new ranking systems that are almost incentive compatible.
- Due to lack of time, these results will not be presented in this talk.

Research Map



Personalized Ranking Systems

- The “client” of the ranking system may also be a participant.
- Examples:
  - Social Networks
  - C2C commerce sites (eBay)
  - Trust (PGP).
- It is useful to generate a personalized ranking for each individual.
- Many impossibility results are reversed.

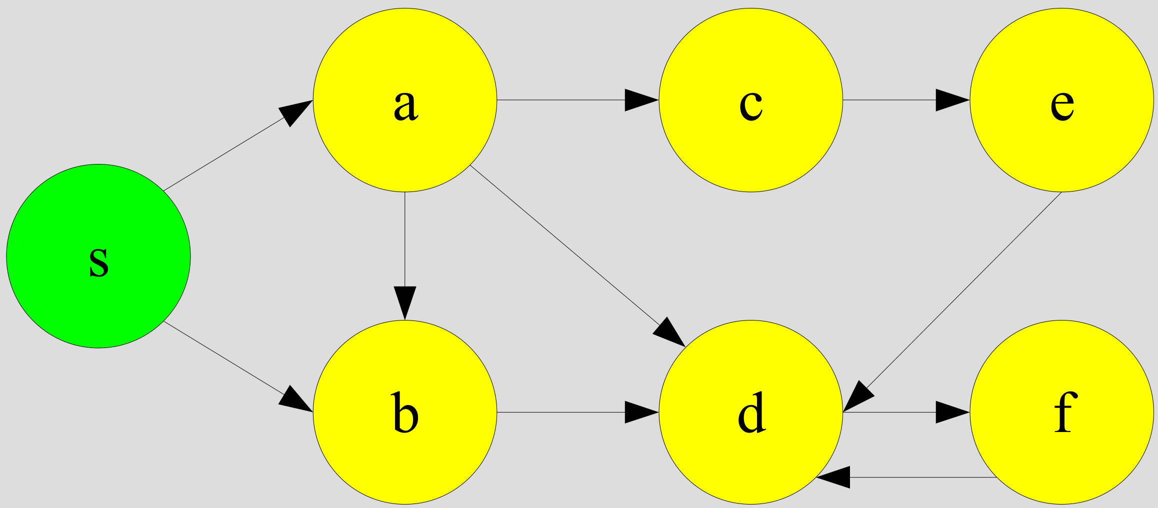
What is a personalized ranking system?

- A personalized ranking system is like a general ranking system, except:
  - Additional parameter: the *source*, i.e. the agent under whose perspective we're ranking.
  - Defined only on the graphs where the source *s* is a *root*, that is there is a directed path from *s* to all vertices.
    - Usually we simply assume the graph is strongly connected.

Examples of PRSs

- **Distance rule** - rank agents based on length of shortest path from s.
- **Personalized PageRank** with damping factor *d* - The PageRank procedure with probability *d* of restarting at vertex *s*.
- **α-Rank –** Rank based on fixed point values when every vertex is valued at α times the sum of its predecessors' value and s is defined as 1, where $\alpha=1/n^2$.

Example of Ranking



<u>Distance</u>
s, a=b, c=d, e=f

<u>α-Rank</u>
s, b, a, d, c, f, e

<u>Personalized PageRank</u>
<u>(d=0.2)</u>
d, **s**, f, b, a, c, e
<u>(d=0.5)</u>
s, b, a, d, f, c, e

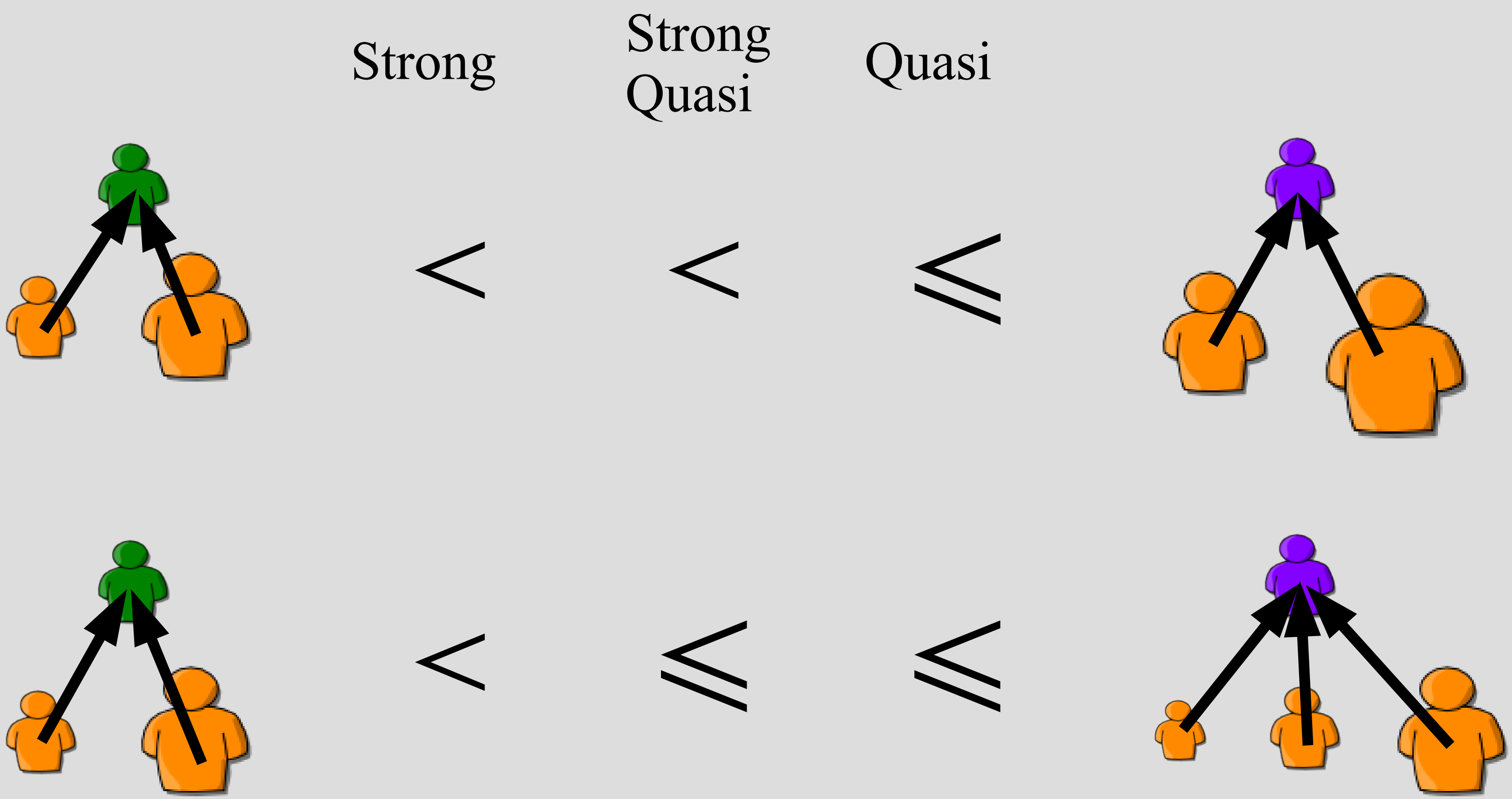
Properties of PRSs

- A PRS satisfies *self-confidence* if the source *s* is ranked stronger than all other vertices.

- The following properties from general ranking systems could be adapted to PRSs.
  - Strong/Quasi/Weak transitivity
  - Ranked IIA
  - Strong Incentive Compatibility

- In every case, we require the property to be satisfied by all vertices except *s*.

Types of Transitivity



New type of Transitivity

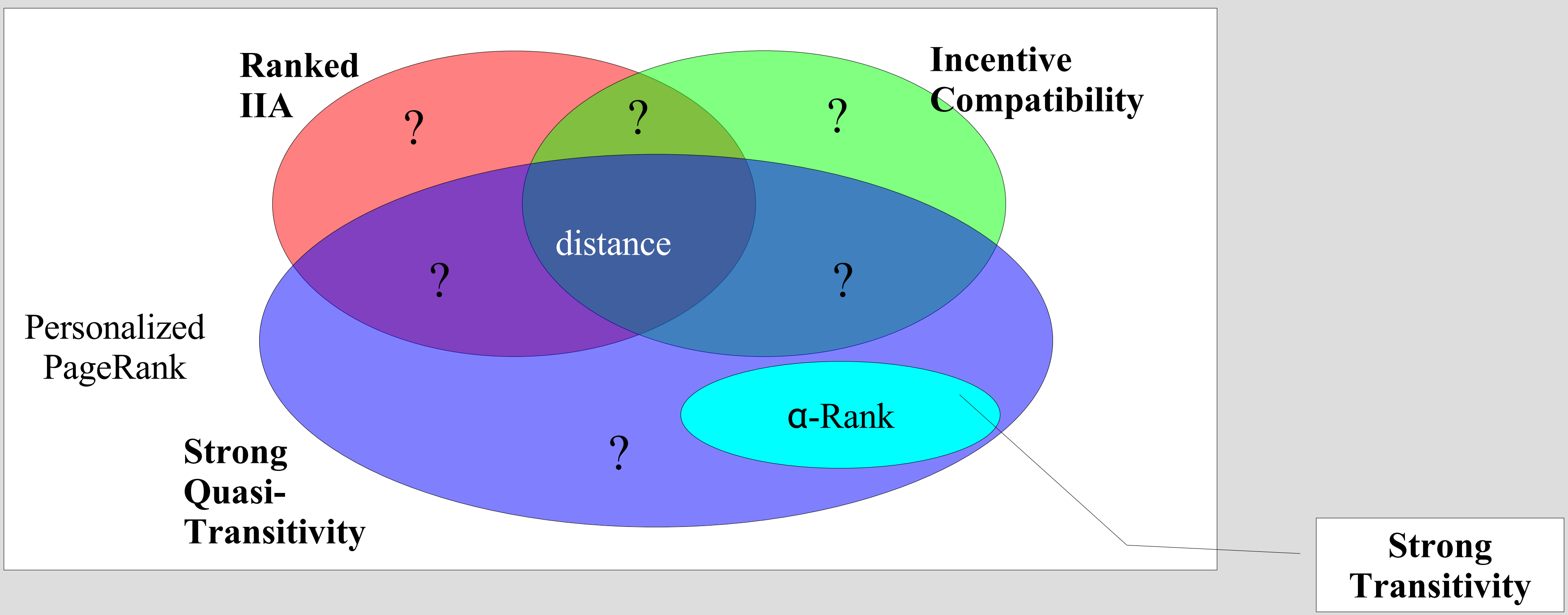
- Assume a ranking system F and two vertices x,y (excluding the source) with a mapping f from P(x) to P(y) that maps each vertex in P(x) to one at least as strong in P(y).
  - **Quasi-transitivity**: $x \precsim y$.
  - **Strong Quasi transitivity**: Furthermore, if *all* of the comparisons are strict: $x \prec y$.
  - **Strong transitivity**: Furthermore, if *at least one* of the comparisons is strict or f is not onto: $x \prec y$.

Classification of PRSs

- **Proposition:** The **distance PRS** satisfies self confidence, ranked IIA, strong quasi transitivity, and strong incentive compatibility, but does not satisfy strong transitivity.
- **Proposition:** The **Personalized PageRank** ranking systems satisfy self confidence iff $d>1/2$. Moreover, Personalized PageRank does not satisfy quasi transitivity, ranked IIA or incentive compatibility for any damping factor.
- **Proposition:** The **α-Rank PRS** satisfies self confidence and strong transitivity, but does not satisfy ranked IIA or incentive compatibility.

Summary

PRS	Distance	P. PageRank	α-Rank
Self Confidence	YES	for d> 1/2	YES
Ranked IIA	YES	NO	NO
Transitivity	strong quasi	none	strong
Incentive Comp.	strong	none	none



The Strong Count System

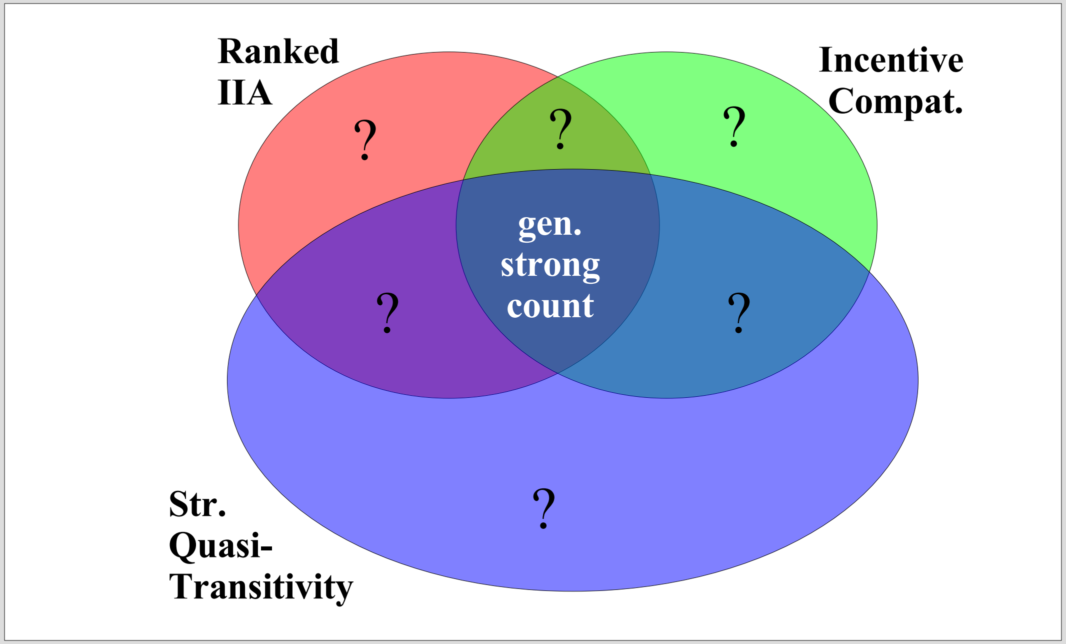
- The strong count PRS sets s to be the top ranked vertex, and then ranks by comparing the strongest predecessors, and when equal ranks based on the *number* of strongest predecessors.
- **Proposition:** The strong count PRS satisfies Self Confidence, Ranked IIA, Strong Quasi Transitivity and Strong Incentive Compatibility.

Generalizing Strong Count

- The Strong Count system can be generalized to systems where some ranges of strongest predecessor counts are considered equivalent.
- For example, such a system can consider one and two strong votes as equivalent, and consider three or more strong votes as equivalent but strictly stronger.
- Specifically, the distance rule arises when all predecessor counts are considered equivalent.
- We will call such systems Generalized Strong Count systems.

Classification Theorem

- **Theorem:** A PRS satisfies self confidence, strong quasi transitivity, RIIA and strong incentive compatibility if and only if it is a generalized strong count system.



Relaxing the Axioms

- All axioms are required for the previous result.
- If we relax any axiom, the system no longer a generalized strong count system.
- In particular there are artificial systems with the following properties:

Self Confidence	YES	NO
Ranked IIA	YES	YES
Str.Quasi-Trans	NO	YES
Inc. Comp	YES	YES

Relaxing Ranked IIA

- The Path Count PRS ranks vertices based on distance, breaking ties by the number of shortest directed paths each vertex has from the source.
- **Proposition:** The path count PRS has the following properties:

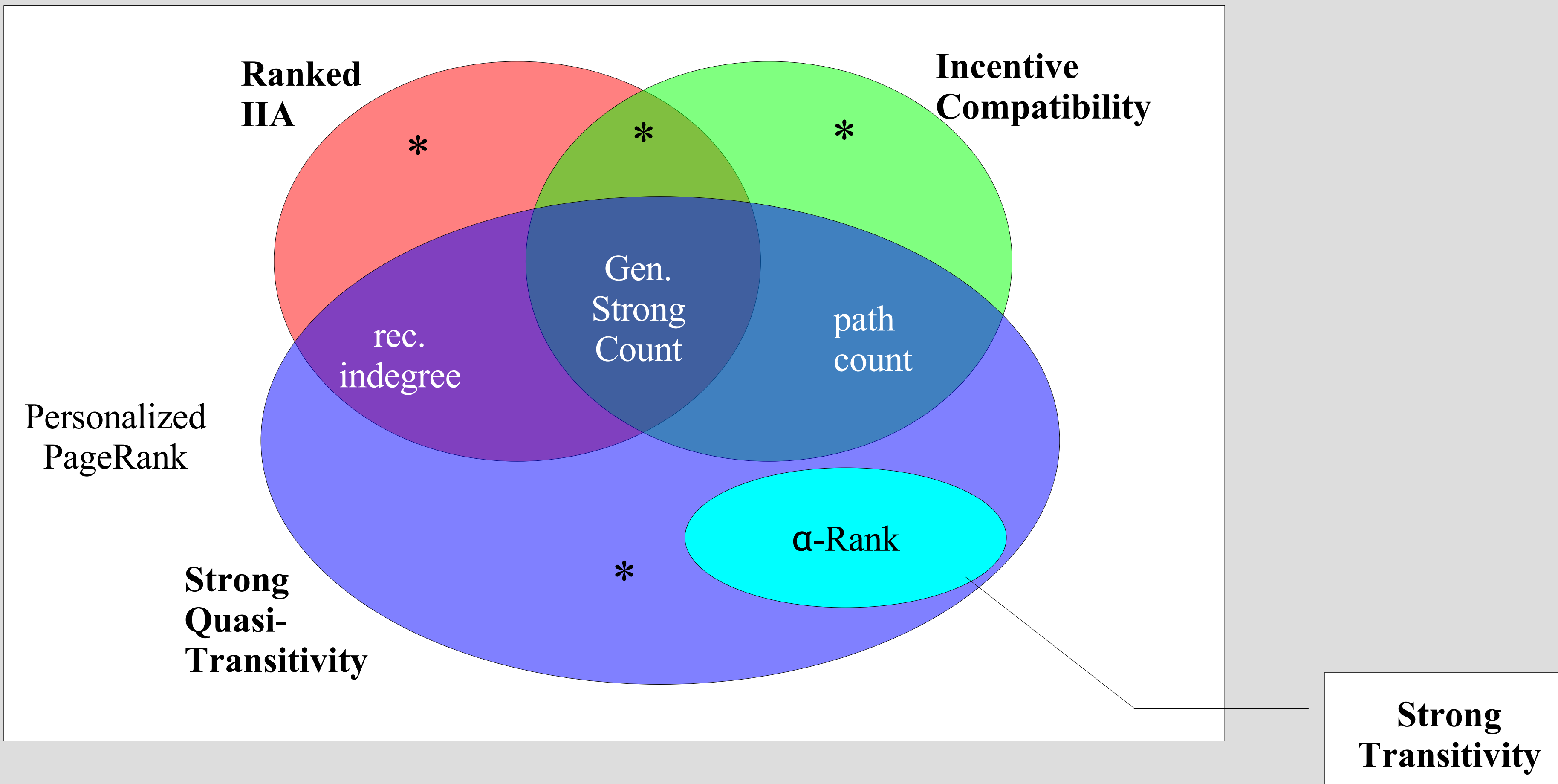
Self Confidence	YES
Ranked IIA	NO
Str.Quasi-Trans	YES
Inc. Comp	YES

Relaxing Incentive Compatibility

- The recursive in-degree ranking system can be adapted to the personalized setting by giving the source vertex a maximal value, as if it has in-degree n+1.
- **Proposition:** The recursive in-degree PRS has the following properties:

Self Confidence	YES
Ranked IIA	YES
Str.Quasi-Trans	YES
Inc. Comp	NO

Personalized Ranking Systems -- Summary



* Artificial Ranking Systems

Summary

- We have shown and proven a representation theorem for PageRank.
- In the Normative Approach, we have seen both impossibility and possibility results.
- We have applied this approach to personalized ranking systems, with very positive results.

Further Research

- New Settings
  - Ternary votes (good/bad/none).
  - Ranking systems over complete preferences.
  - Probabilistic Ranking Systems.
- Descriptive Approach
  - Prove PageRank axioms' independence.
  - Axiomatization for PageRank with damping factor.
  - Axiomatization for Hubs&Authorities.
  - Representation theorems in personalized setting.

Further Research (cont.)

- Normative Approach
  - Explore new axioms and prove possibility or impossibility results.
- Personalized Ranking Systems
  - Consider non-connected case.
  - Reputation systems (ternary votes).

Thank You!