Variance Reduction for Markov Processes and Poisson's Equation

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Abstract

In this second talk we continue the theme of the important role of Poisson's equation in the study of stochastic processes. We turn now to *simulation* of Markov processes in discrete and continuous time on general state spaces. To simplify things, we will often focus on finite state spaces, but generalizations are available. It is not necessary to have attended the first talk in order to understand this one.

Let $X = (X_n : n \geq 0)$ be an irreducible, positive-recurrent Markov chain on a (for now) finite state space S say. Let $f : S \to \mathbb{R}$ be a real-valued function on the state space of the chain X and suppose we are interested in computing $\pi f = E_\pi f(X_0) = \sum_{x \in S} \pi(x) f(x)$. One way to solve this problem is to first compute π and then compute the inner product πf. But for large state spaces this approach is not possible, and instead one might turn to simulation. A reasonable estimator of πf is

$$\alpha(n) = \frac{1}{n} \sum_{i=0}^{n-1} f(X_i),$$

the average of the function values over a long sample path of the chain. Indeed, it can be shown that this estimator is consistent and satisfies a central limit theorem using the martingale central limit theorem and (you guessed it) the solution to Poisson's equation. For this problem, Poisson's equation is of the form

$$Pg(x) - g(x) = -f(x) + \pi f \quad \forall x \in S,$$

where P is the transition matrix of X and

$$Pg(x) = E_x g(X_1) = \sum_{y \in S} P_{xy} g(y).$$

The variance constant in the central limit theorem can be very large, so that $\alpha(n)$ converges very slowly to the true limit πf. In that case, one might be interested in obtaining variance reduction by modifying the estimator $\alpha(n)$ in some way.

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Now, for any function $u : S \to \mathbb{R}$, the *Dynkin* martingale $M(u) = (M_k(u) : k \geq 0)$ is defined by

$$M_k(u) = u(X_k) - u(X_0) - \sum_{i=0}^{k-1} (Pu(X_i) - u(X_i)).$$

This martingale has mean 0 for any choice of u, so that we could also estimate πf by

$$\begin{aligned} \alpha(n) - M_n(u)/n &= \frac{1}{n} \sum_{i=0}^{n-1} [f(X_i) + Pu(X_i) - u(X_i)] + \frac{u(X_0) - u(X_n)}{n} \\ &\approx \frac{1}{n} \sum_{i=0}^{n-1} [f(X_i) + Pu(X_i) - u(X_i)] \\ &= \alpha''(n). \end{aligned}$$

Now, we want to minimize the variance of $\alpha''(n)$ with respect to u. The optimal choice turns out to be the solution to Poisson's equation, since in that case we get zero variance. (There it is again!) In practice we can't solve Poisson's equation exactly, but we can usually make a pretty good guess, and use the guess in place of u above.

This method can be adapted to obtain variance reductions for performance measures other than steady-state means, and for Markov processes other than discrete-time Markov chains on finite state space. I'll give some examples. One can also perform an adaptive search for a good choice of u among a family of functions.

This work was developed in Henderson [1997] and Henderson and Glynn [2002], and an adaptive version that searches for a good choice of u is discussed in Kim and Henderson [2004]. The main tool we apply in this talk is Markov chain theory as can be found in Meyn and Tweedie [1993]. Diffusion processes are also useful tools, and are explained in Karlin and Taylor [1981].

References

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