Call Centers and Poisson's Equation

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Abstract

In inbound call centers the usual goal is to answer a reasonable fraction of calls within a given time limit with the minimum number of staff. For planning purposes, time is broken down into short periods that are usually between 15 minutes and 1 hour long. Let us focus on one of these periods for now, and call it $[0, t]$. One then models the arrival rate of customers during $[0, t]$ as being constant over the whole period. The required staffing level for the period is often obtained from steady-state results for continuous-time Markov chain models. Even if the continuous-time Markov chain is a reasonable model of the call center, there is still some potential error in the use of steady-state quantities to approximate time-dependent quantities. In the first part of this talk we present an approximation for this error, and discuss its ramifications.

The approximation is developed as follows. Let $X = (X(s) : s \geq 0)$ denote the process of the number of customers in the system, so that $X(s) \in \{0, 1, 2, \ldots\}$, and suppose there are c servers. Suppose that the arrival rate is constant and equal to λ. The long-run fraction of calls answered on time in the period $[0, t]$ is then ES/EN, where S and N are the number of calls answered on time and the number of calls received in the period respectively. The denominator EN is easily computed as λt. If we measure performance by the fraction of customers that are served immediately, then by applying a PASTA result [Wolff, 1989, p. 293], we can show that

$$ES = \lambda \int_0^t P_\nu(X(s) \leq c - 1)\, ds,$$

where ν is the distribution of $X(0)$ and P_ν is the distribution on the Markov process X induced by the initial condition ν. If the initial distribution ν is the stationary distribution π, then $ES = \lambda p t$, where $p = P_\pi(X(0) \leq c - 1)$. The error is therefore given by

$$\lambda \int_0^t P_\nu(X(s) \leq c - 1) - p\, ds \approx \lambda \int_0^\infty P_\nu(X(s) \leq c - 1) - p\, ds. \tag{1}$$

It is possible to compute the right-hand side of (1) analytically for several stochastic processes (not just continuous-time Markov chains) that have been applied in the call

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center setting via the solution to *Poisson's equation.* We will discuss the form of Poisson's equation, its solution, the error approximation, and its implications for the use of steady-state approximations.

In the second part of this talk we give a second application of Poisson's equation in the call center setting. For a given staffing level, the long-run fraction of calls is, as noted earlier, given by ES/EN. But for a *single period* there can be significant variation in the fraction of calls S/N answered on time, and managers should be aware of this inherent variation. We give a normal approximation for the distribution of S/N that is justified through a central limit theorem. A similar approximation can be used when the arrival process is, as data often suggests, a doubly-stochastic Poisson process, i.e., a Poisson process with a random arrival rate. Poisson's equation again plays a central role in the analysis.

This work is based on results in Steckley [2005], Steckley et al. [2005] and work that is yet to be published. Gans et al. [2003] is an excellent review of call center research, and Meyn and Tweedie [1993] is a comprehensive discussion of the theory of Markov chains on a general state space, part of which we use here. Brémaud [1999] discusses much of that theory in the context of countable state spaces, which allows the key ideas to be presented more transparently. The central limit theorem applied above can be proved using a martingale central limit theorem that is given, e.g., in Ethier and Kurtz [1986, Chapter 7].

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