# Theory and Computation of Semidefinite Programming for Sensor Network Localization and Other Distance Geometry Problems

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Joint work with Biswas, So, Liang, Toh (Saunders, Jin)

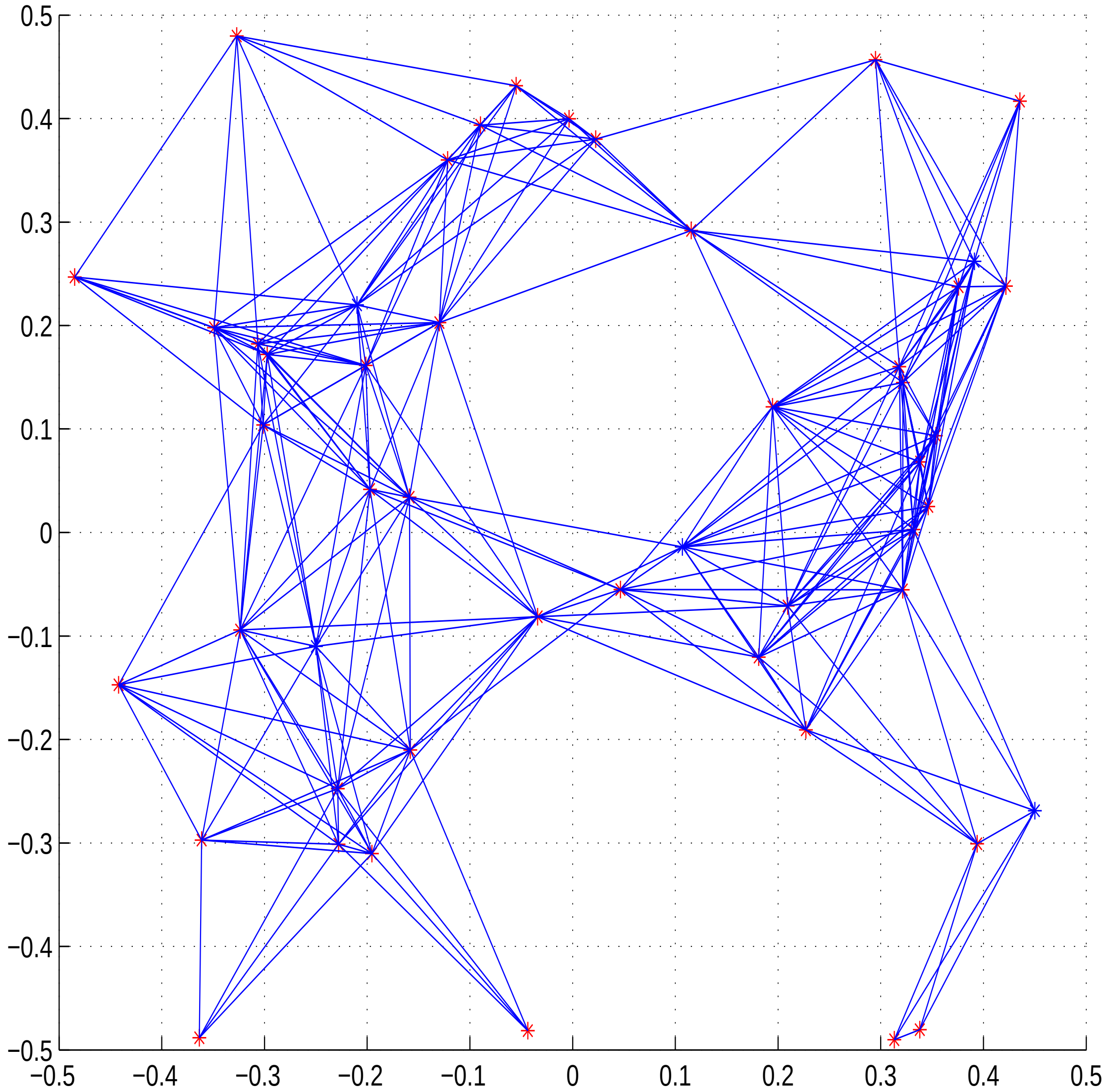
# Outline

- Ad Hoc Wireless Sensor Network Localization: SDP models and analyses
- SDP Computation: decomposition and distribution
- SDP Rounding: improving SDP solution
- More applications

## Ad Hoc Wireless Sensor Network Localization

- **Input** $m$ known points (anchors) $a_k \in \mathbf{R}^2$, $k = 1, ..., m$, and $n$ unknown points (sensors or targets) $x_j \in \mathbf{R}^2$, $j = 1, ..., n$. For some pair of two points, we have a Euclidean distance measure $\hat{d}_{kj}$ between $a_k$ and $x_j$, or distance measure $\hat{d}_{ij}$ between $x_i$ and $x_j$.
- **Output** Position estimation for all unknown points, and confidence measures on reliability of each position estimation.
- **Objective** Robust, fast and accurate.

Figure 1: 50-Sensor Network with Radio Range .3



## Related Work

- FCC requires wireless carriers to provide far more precise location information, within 50 to 100 meters in most cases, of a wireless 911 caller by December 31, 2005.
- A great deal of research has been done on the topic of position estimation in ad-hoc networks, see Hightower and Boriello (2001) and Ganesan et al. (2002); Beacon grid: e.g., Bulusu and Heidemann (2000) and Howard et al. (2001); Distance measurement: e.g., Doherty et al. (2001), Niculescu and Nath (2001), Savarese et al. (2002), Savvides et al. (2001, 2002), Shang et al. (2003), Eren et al. (2004).
- Metric embeddings and Distance geometry problems: Johnson and Lindenstrauss (1984), Bourgain (1985), Barvinok (1995), Moré and Wu (1997), Alfakih et al. (1999), Laurent (2001), etc.

## Euclidean Distance Geometry Model

$$
\begin{aligned}
\|x_i - x_j\|^2 &= d_{ij}^2, \ \forall\, (i,j) \in N_x,\ i < j, \\
\|a_k - x_j\|^2 &= d_{kj}^2, \ \forall\, (k,j) \in N_a, \\
\|x_i - x_j\|^2 &\geq R_{ij}^2, \ \forall (i,j) \notin N_x,\ i < j, \\
\|a_k - x_j\|^2 &\geq R_{kj}^2, \ \forall (k,j) \notin N_a.
\end{aligned}
$$

$d_{ij}^2$ ($d_{kj}^2$) connects $x_i$ to $x_j$ ($a_k$ to $x_j$) with an edge whose length is $d_{ij}$ ($d_{kj}$).

Does the system has a localization or realization of all $x_j$'s? Is the localization unique? Is the localization reliable or trustworthy? Is the system partially localizable?

# Euclidean Distance Geometry Model

Consider a simpler Euclidean Distance Geometry Model:

$$\|x_i - x_j\|^2 = d_{ij}^2, \; \forall \, (i,j) \in N_x, \; i < j,$$
$$\|a_k - x_j\|^2 = d_{kj}^2, \; \forall \, (k,j) \in N_a.$$

## Convex Optimization Method

$$\|x_i - x_j\|^2 \le d_{ij}^2,\ \forall\,(i,j) \in N_x,\ i < j,$$
$$\|a_k - x_j\|^2 \le d_{kj}^2,\ \forall\,(k,j) \in N_a.$$

Doherty et al. (2001)

## Global and Nonlinear Least Squares Method

$$\min \quad \sum_{i,j\in N_x}(\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{k,j\in N_a}(\|a_k - x_j\|^2 - d_{kj}^2)^2$$

$$\min \quad \sum_{i,j\in N_x}(\|x_i - x_j\| - d_{ij})^2 + \sum_{k,j\in N_a}(\|a_k - x_j\| - d_{kj})^2$$

## Matrix Representation

Let $X = [x_1 \; x_2 \; ... \; x_n]$ be the $2 \times n$ matrix that needs to be determined. Then

$\|x_i - x_j\|^2 = e_{ij}^T X^T X e_{ij}$ and $\|a_k - x_j\|^2 = (a_k; e_j)^T [I \;\; X]^T [I \;\; X] (a_k; e_j)$,

where $e_{ij}$ is the vector with $1$ at the $i$th position, $-1$ at the $j$th position and zero everywhere else; and $e_j$ is the vector of all zero except $-1$ at the $j$th position.

$$
\begin{aligned}
& e_{ij}^T Y e_{ij} = d_{ij}^2, \; \forall \, i, j \in N_x, \; i < j, \\
& (a_k; e_j)^T \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} (a_k; e_j) = d_{kj}^2, \; \forall \, k, j \in N_a, \\
& Y = X^T X.
\end{aligned}
$$

where $Y$ denotes the Gram matrix $X^T X$.

# Semidefinite Programming (SDP)

$$(SDP) \quad \text{min} \quad C \bullet Z$$
$$\text{subject to} \quad A_i \bullet Z = b_i, i = 1, 2, ..., m, \; Z \succeq 0,$$

where $C, \; A_i \in \mathcal{M}^n$, the set of $n$-dimension symmetric matrices.

The dual problem to (SDP) can be written as:

$$(SDD) \quad \text{max} \quad b^T y$$
$$\text{subject to} \quad \sum_i^m y_i A_i + S = C, \; S \succeq 0,$$

where $b = (b_1; ...; b_m) \in \mathcal{R}^m$, variables $y \in \mathcal{R}^m$ and $S \in \mathcal{M}^n$.

An generalization of linear programming.

# SDP Relaxation

Change

$$Y = X^T X$$

to

$$Y \succeq X^T X.$$

This matrix inequality is equivalent to (e.g., Boyd et al. 1994)

$$\begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq 0.$$

## SDP standard form

$$Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix}.$$

Find a symmetric matrix $Z \in \mathbf{R}^{(2+n)\times(2+n)}$ such that

$$\begin{aligned} & Z_{1:2,1:2} = I \\ & (\mathbf{0}; e_{ij})(\mathbf{0}; e_{ij})^T \bullet Z = d_{ij}^2, \ \forall\, i, j \in N_x, \ i < j, \\ & (a_k; e_j)(a_k; e_j)^T \bullet Z = d_{kj}^2, \ \forall\, k, j \in N_a, \\ & Z \succeq 0. \end{aligned}$$

Any matrix solution for the SDP relaxation has rank at least $2$. If every sensor point is connected, directly or indirectly, to an anchor point, then the solution set must be bounded.

# The dual of the SDP relaxation

$$\begin{array}{ll}\text{minimize} & I \bullet V + \sum_{i<j\in N_x} w_{ij} d_{ij}^2 + \sum_{k,j\in N_a} w_{kj} d_{kj}^2 \\ \text{subject to} & \begin{pmatrix} V & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} + \sum_{i<j\in N_x} w_{ij} (\mathbf{0}; e_{ij})(\mathbf{0}; e_{ij})^T \\ & + \sum_{k,j\in N_a} w_{kj} (a_k; e_j)(a_k; e_j)^T \succeq 0,\end{array}$$

where variable matrix $V \in \mathcal{M}^2$, varaible $w_{ij}$ is the weight on edge from $x_i$ to $x_j$, and $w_{kj}$ is the weight on edge from $a_k$ to $x_j$.

Note that the dual is always feasible since $V = 0$ and all $w.$ equal $0$ is a feasible solution.

## Localizable problem

A sensor network is localizable if there is a unique localization in $\mathbf{R}^2$ and there is no $x_j \in \mathbf{R}^h,\ j = 1, ..., n$, where $h > 2$, such that

$$\|x_i - x_j\|^2 = d_{ij}^2,\ \forall\, i, j \in N_x,\ i < j,$$
$$\|(a_k; \mathbf{0}) - x_j\|^2 = d_{kj}^2,\ \forall\, k, j \in N_a.$$

The latter says that the problem cannot be localized in a higher dimension space where anchor points are augmented to $(a_k; \mathbf{0}) \in \mathbf{R}^h$, $k = 1, ..., m$.

## When is the problem localizable?

**Theorem 1.** *The following statements are equivalent:*

1. *The sensor network is localizable;*
2. *The max-rank solution of the SDP relaxaion has rank* $2$*;*
3. *The solution matrix has* $Y = X^T X$ *or* Trace$(Y - X^T X) = 0$ *.*

If a localizable problem has nondegenerate solution, then the problem is strongly localizable.

Figure 2: One sensor-Two anchors: Not localizable

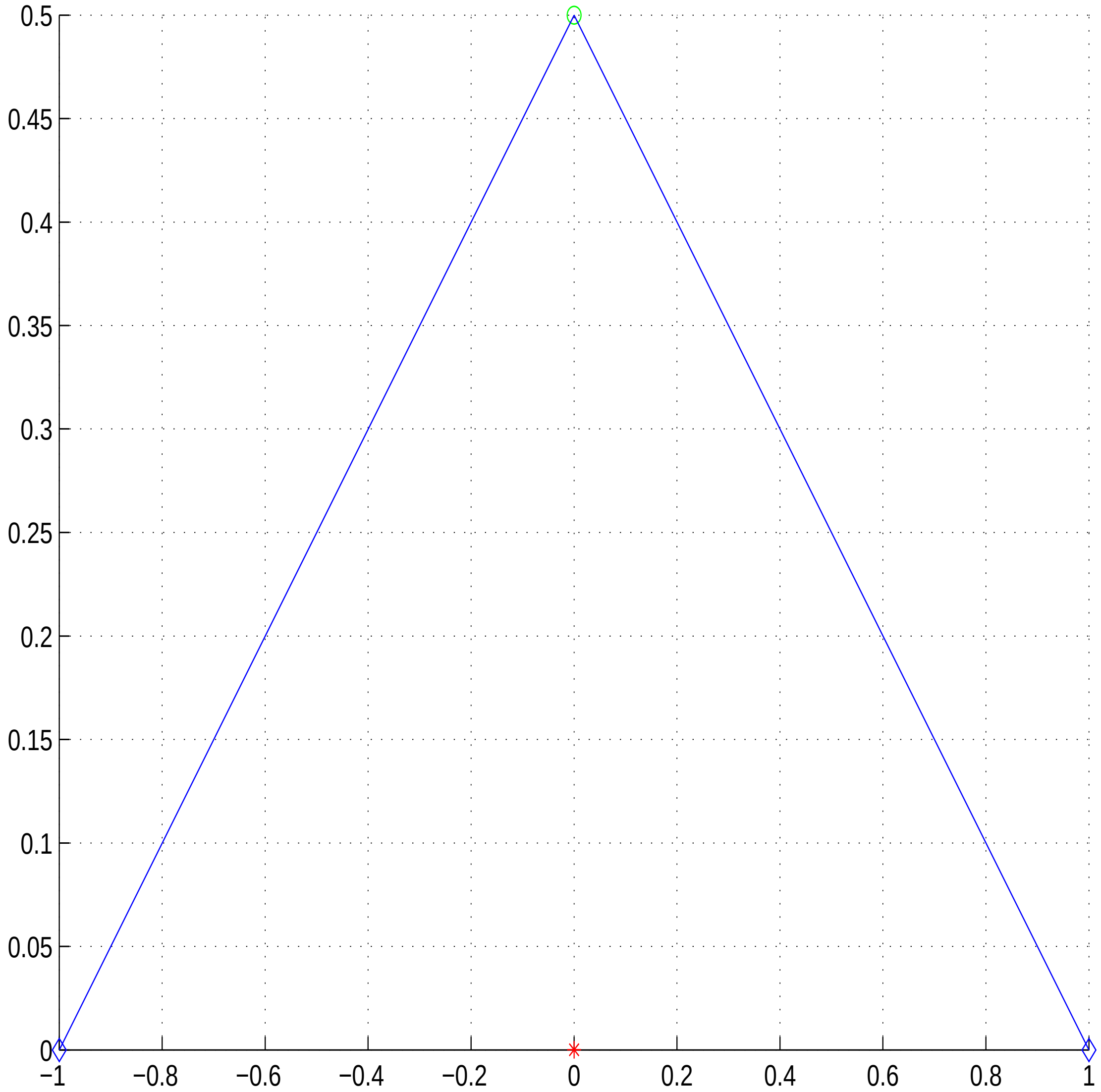


Figure 3: Two sensor-Three anchors: Strongly Localizable

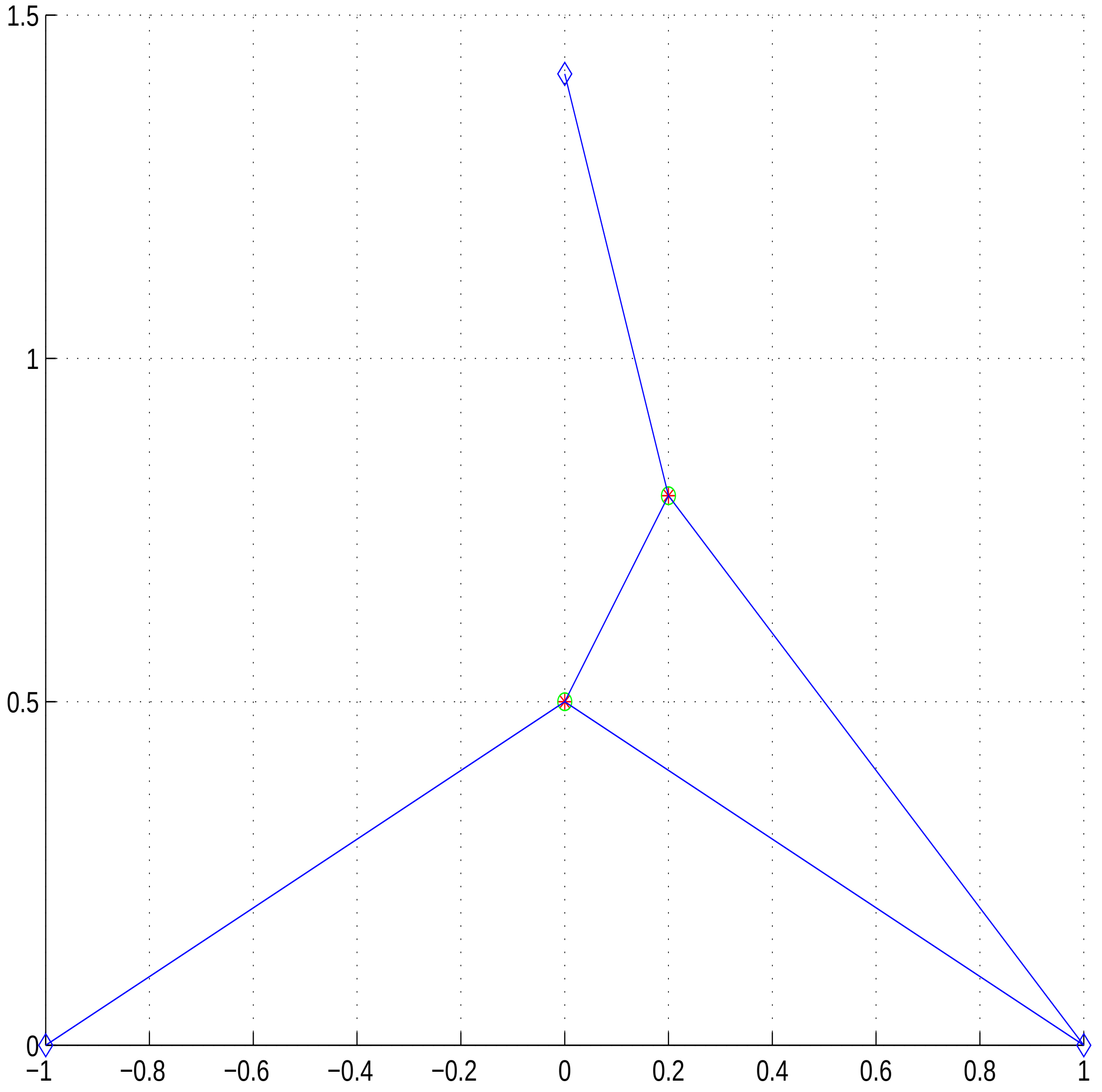


Figure 4: Two sensor-Three anchors: Localizable but not Strongly

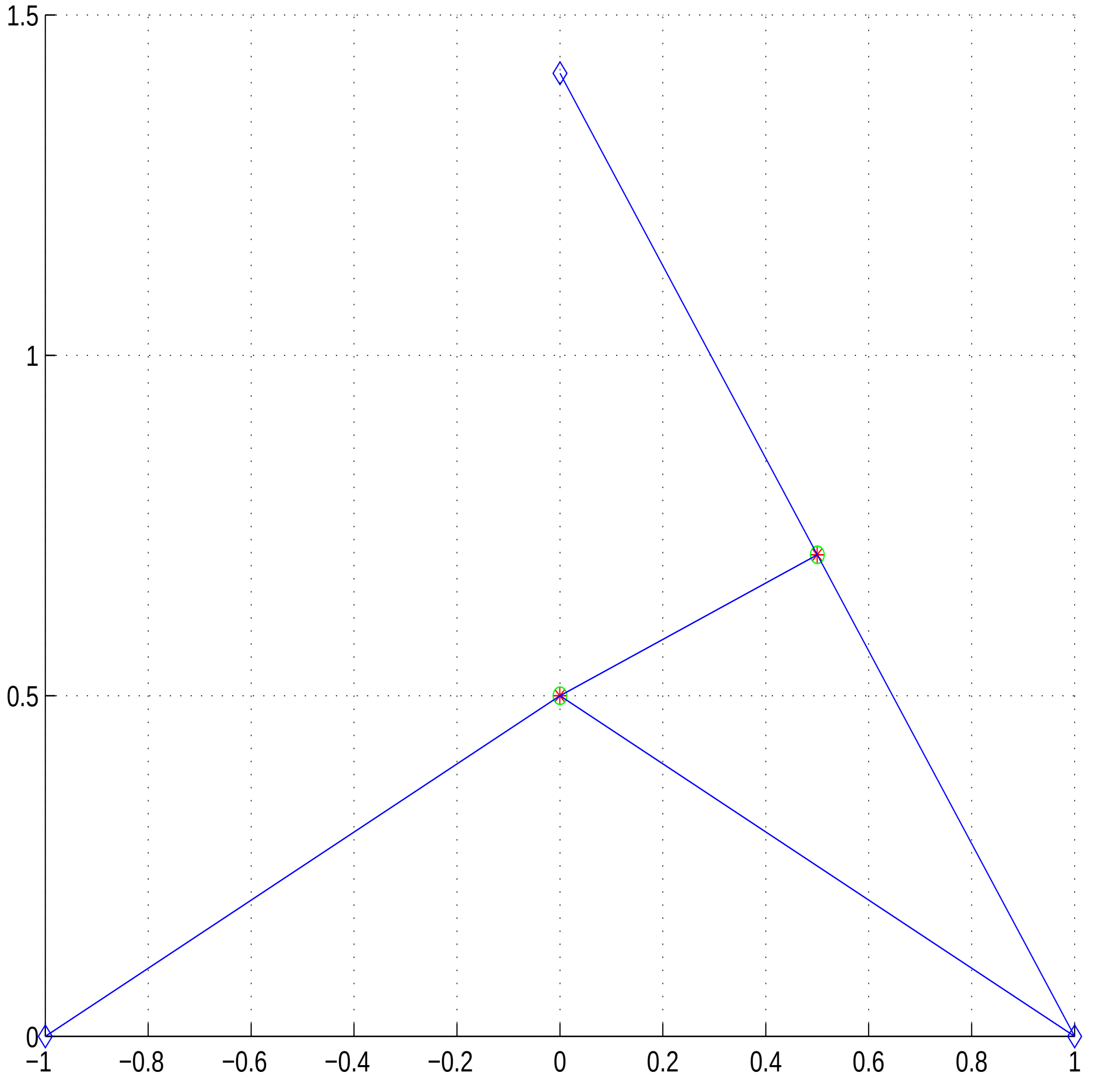


Figure 5: Two sensor-Three anchors: Not localizable

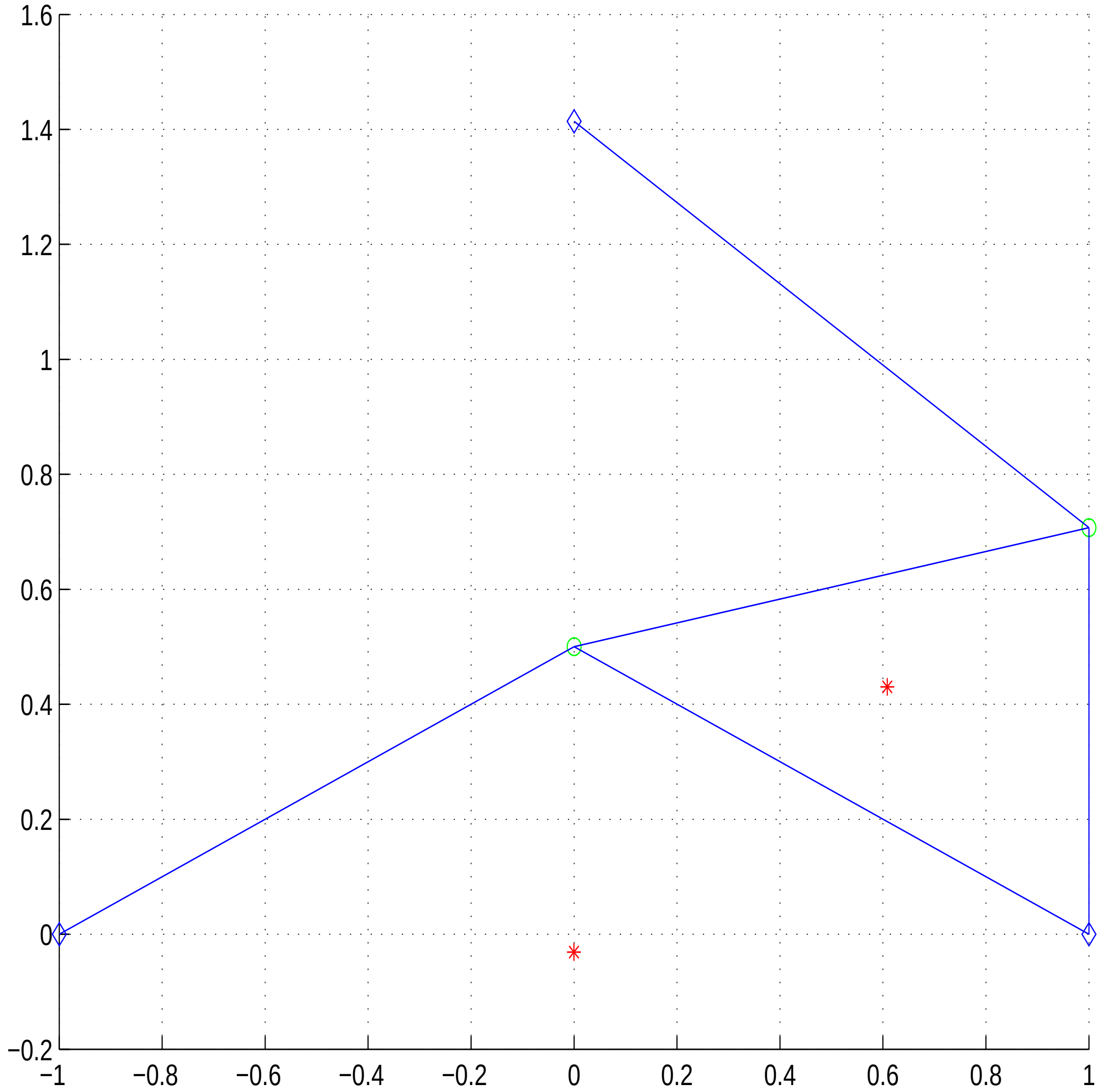
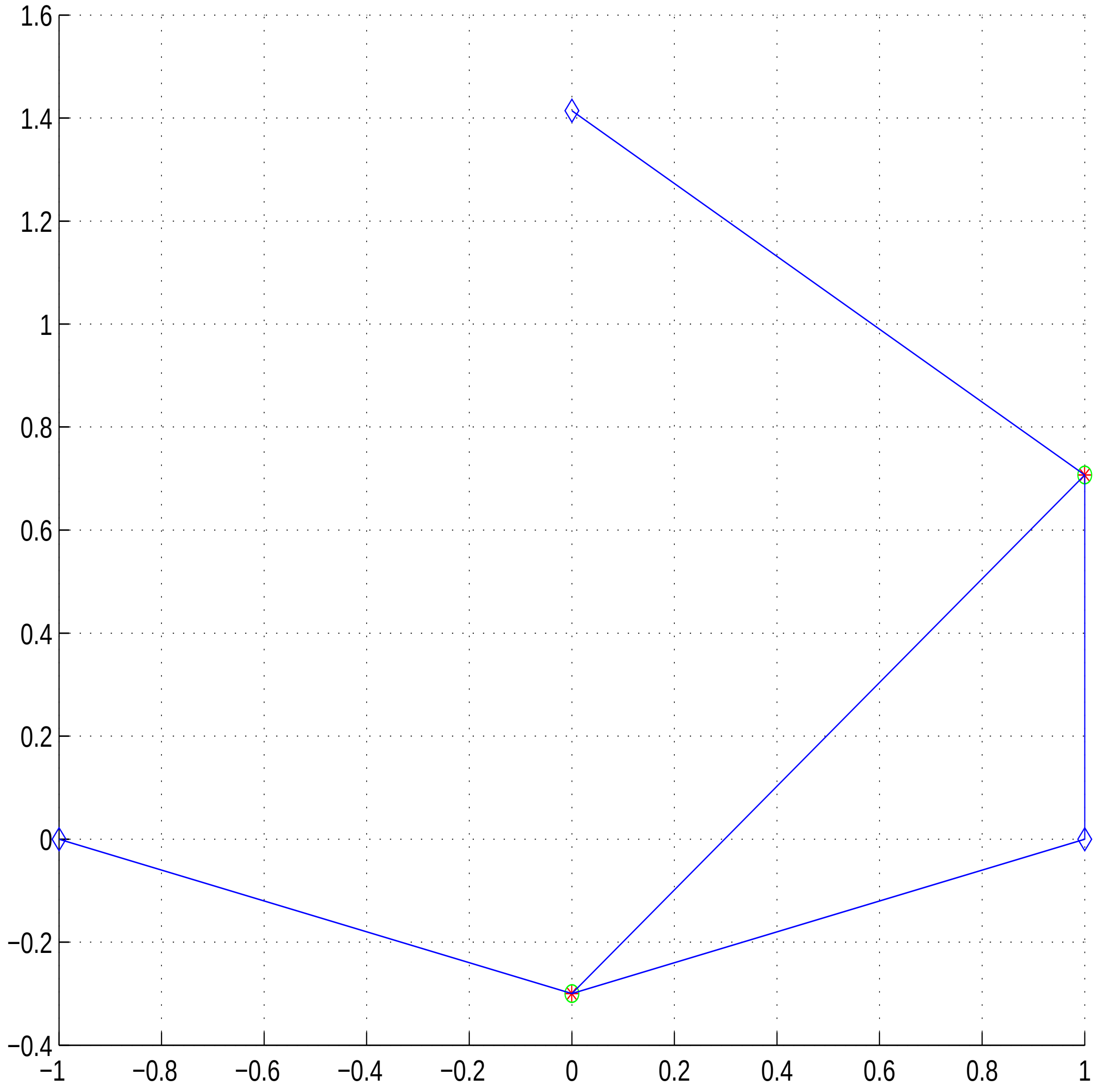


Figure 6: Two sensor-Three anchors: Stronly Localizable



## Localize All Localizable Points

**Theorem 2.** *If a problem (graph) contains a subproblem (subgraph) that is localizable, then the submatrix solution corresponding to the subproblem in the SDP solution has rank* $2$*. That is, the SDP relaxation computes a solution that localize all possibly localizable unknown sensor points.*

Implication: Trace,

$$\text{Trace}(\bar{Y} - \bar{X}^T\bar{X}) = \sum_{j=1}^{n}(\bar{Y}_{jj} - \|\bar{x}_j\|^2)$$

$\bar{Y}_{jj} - \|\bar{x}_j\|^2$ can be used as a measure to see whether $j$th sensor's estimated position is reliable or not.

## Probabilistic Analysis and Confidence Measure

Alternatively, each $x_j$'s can be viewed as random points from the distance measures containing random errors. Then the solution to the SDP problem provides the first and second moment sample estimation (Bertsimas and Ye 1998).

Generally, $\bar{x}_j$ is a point estimate of $x_j$ and $\bar{Y}_{ij}$ is a point estimate $x_i^T x_j$.

Consequently,

$$\bar{Y}_{jj} - \|\bar{x}_j\|^2,$$

which is the individual variance estimation of sensor $j$, gives an interval estimation for its true position.

## SDP Relaxation of Least Squares

Find a symmetric matrix $Z \in \mathbf{R}^{(2+n)\times(2+n)}$ and $\alpha_{ij}$ and $\alpha_{kj}$ such that

$$
\begin{array}{ll}
\text{minimize} & \sum_{(ij)} \alpha_{ij}^2 + \sum_{(kj)} \alpha_{kj}^2 \\
\text{subject to} & Z_{1:2,1:2} = I \\
 & (\mathbf{0}; e_{ij})(\mathbf{0}; e_{ij})^T \bullet Z + \alpha_{ij} = d_{ij}^2, \ \forall\, i,j \in N_x,\ i < j, \\
 & (a_k; e_j)(a_k; e_j)^T \bullet Z + \alpha_{kj} = d_{kj}^2, \ \forall\, k,j \in N_a, \\
 & Z \succeq 0.
\end{array}
$$

The SDP objective value is a lower bound on the original least squares problem.

# Simulation and Experiment Results

SDP solvers used were SeDuMi (Sturm, 2001) and DSDP2.0 (Benson et al. 1998).

$$(SDP) \quad \text{min} \quad C \bullet Z$$
$$\text{subject to} \quad A_i \bullet Z = (\leq, \geq) b_i, i = 1, 2, ..., m, \; Z \succeq 0,$$

where $A_i = a_i a_i^T$.

In our computational experiments:

$$d_{ij} = \text{true}d_{ij} \cdot (1 + randn(1) \cdot nf)$$

# Rounding the SDP solution

- When measurement noises exist, the SDP solution almost always has a high rank. How to round the high-rank solution into a low rank?

- Gradient-based local search: using the SDP relaxation solution as the initial point, we apply the steepest descent method to further reducing the estimation error:

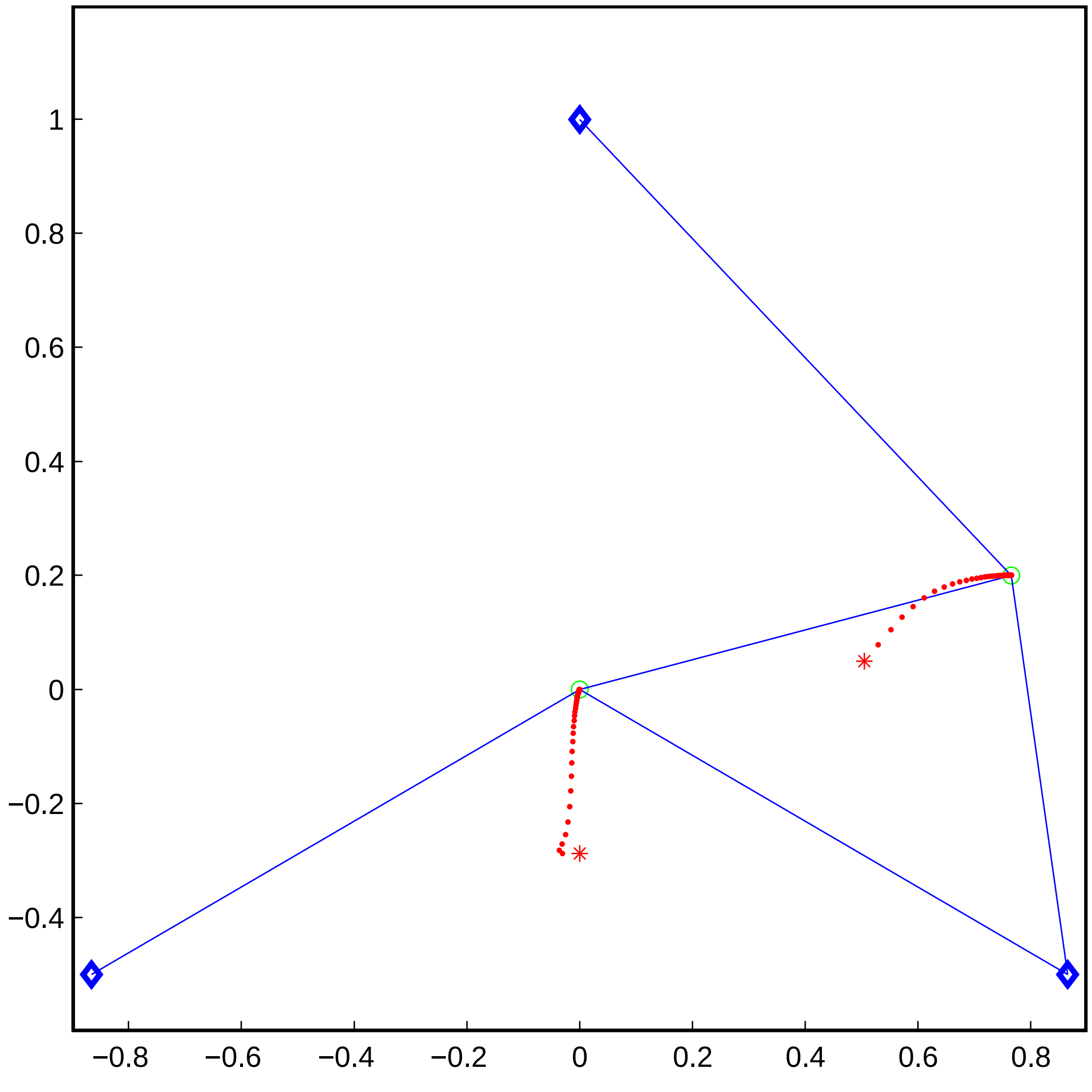
$$\sum_{i,j \in N_x} (\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{k,j \in N_a} (\|a_k - x_j\|^2 - d_{kj}^2)^2$$

or

$$\sum_{i,j \in N_x} (\|x_i - x_j\| - d_{ij})^2 + \sum_{k,j \in N_a} (\|a_k - x_j\| - d_{kj})^2$$

- A checkable bound of suboptimality can be used to ensure the solution quality.

Figure 7: Gradient search trajectories: Two sensor-Three anchor Example



# Figure 8: SDP/Gradient search trajectories: 10% Noise Example

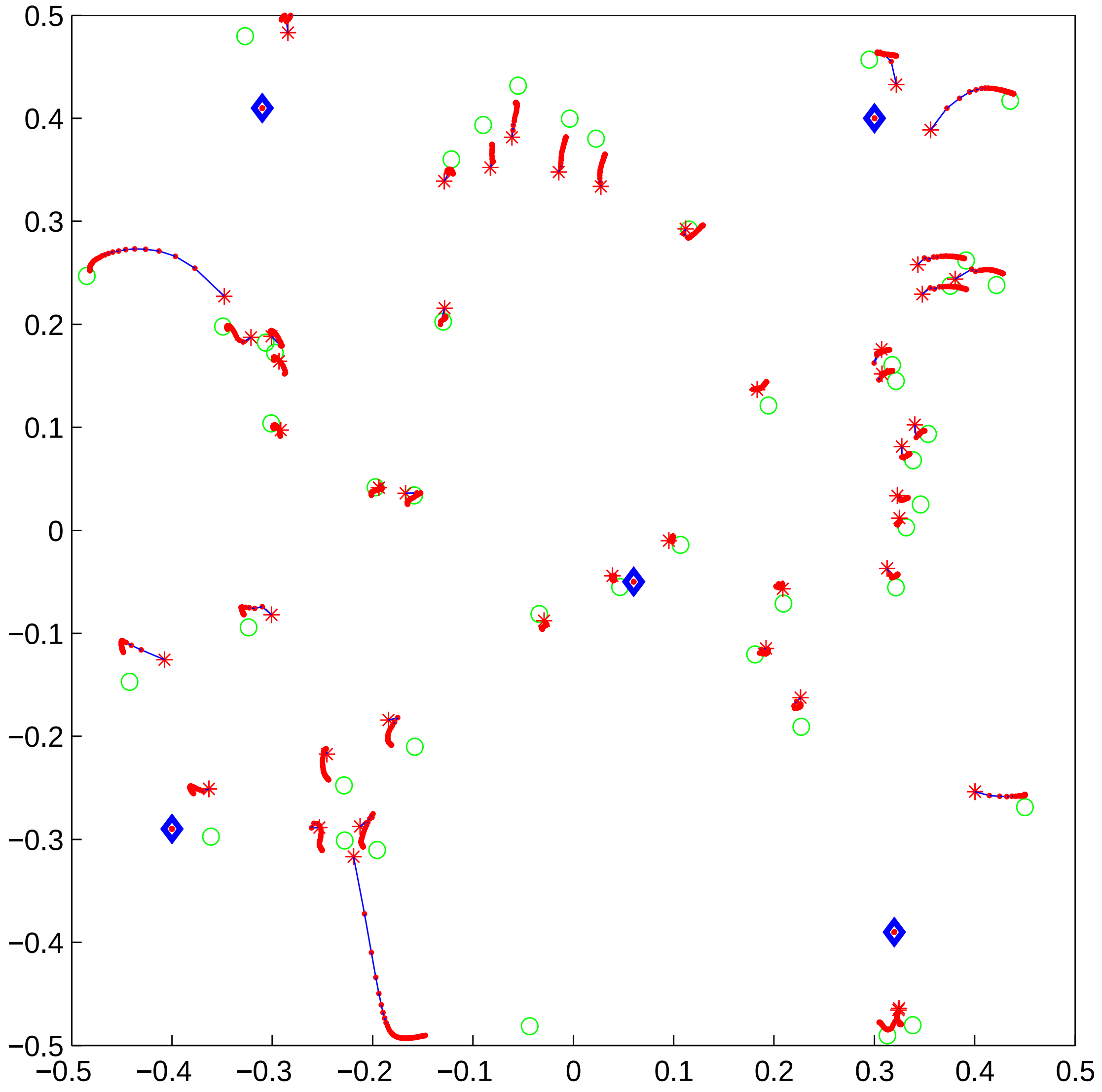


Figure 9: The objective value reduction: 10% Noise Example

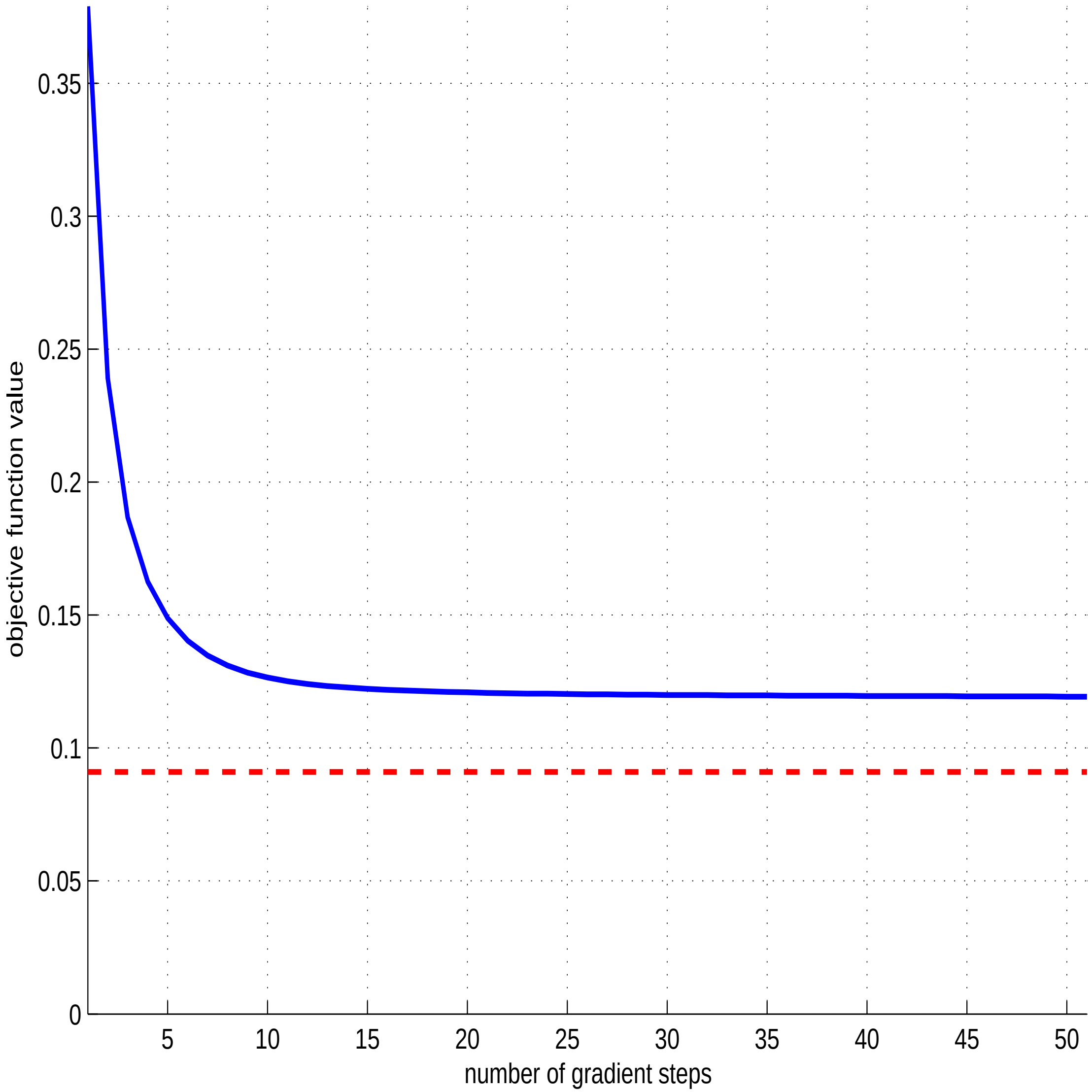


Figure 10: No-SDP/Gradient search trajectories: 10% Noise Example

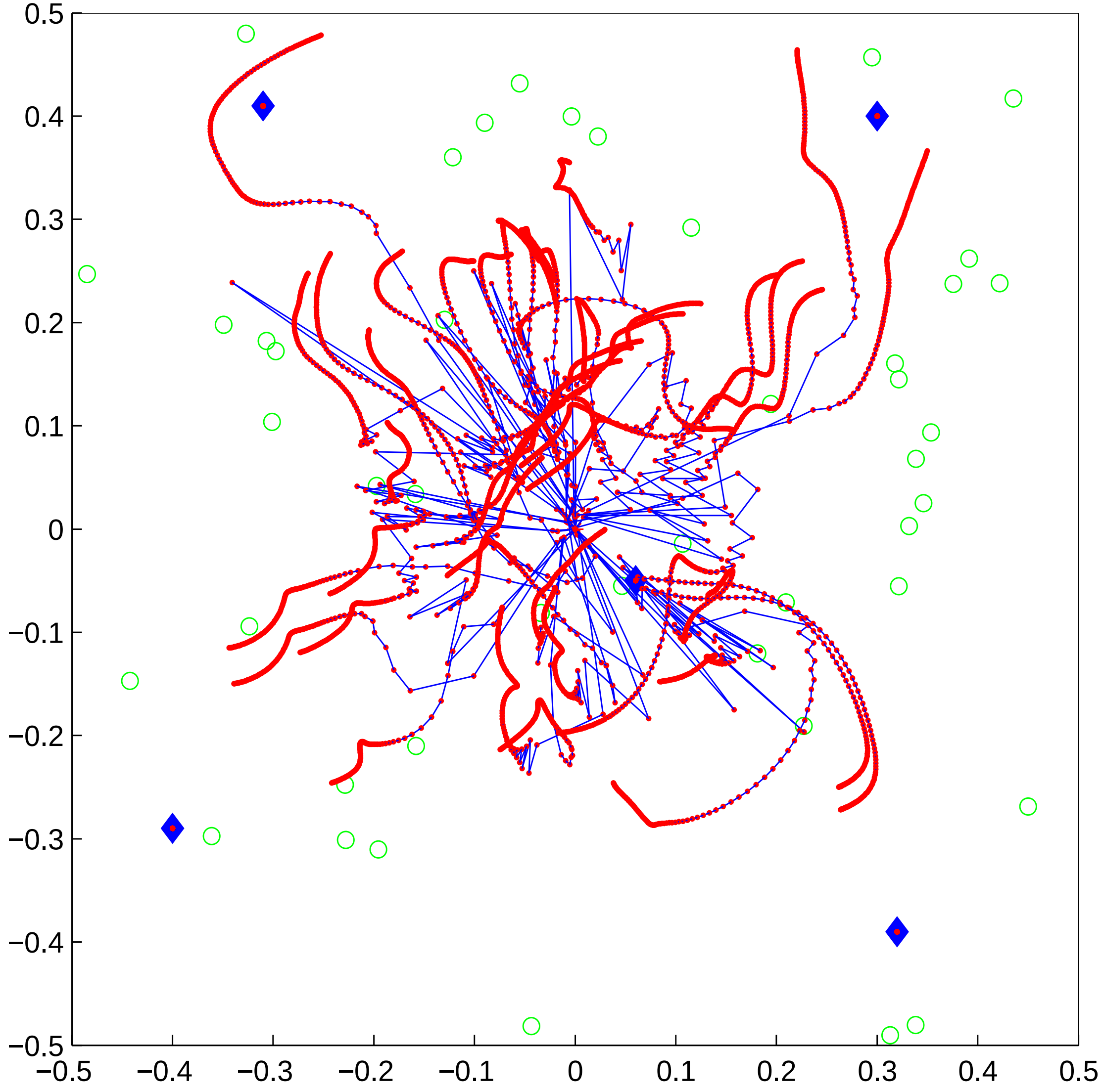
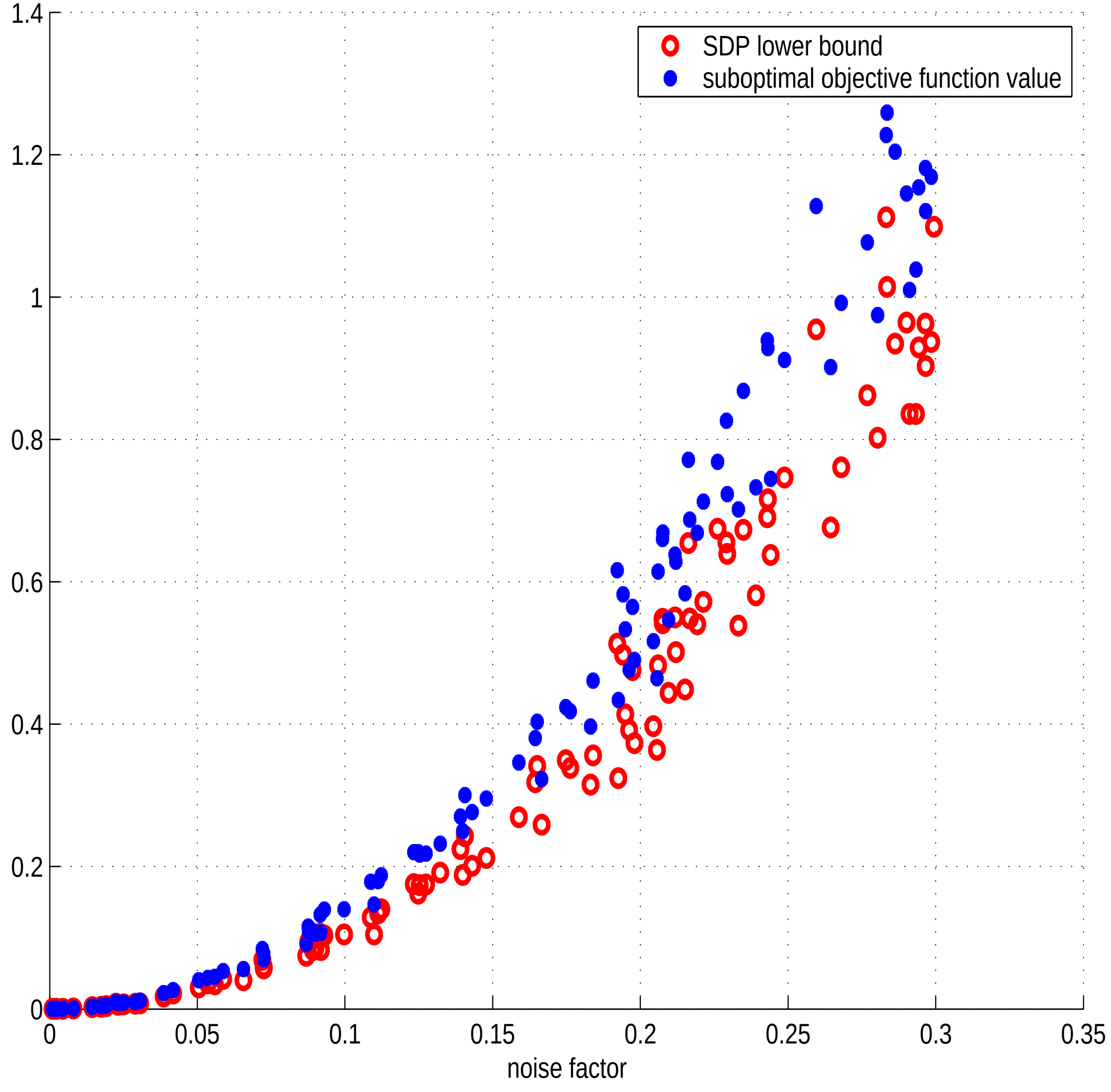


Figure 11: SDP lower bound and suboptimal objective function value vs noisy factor



# A Distributed SDP Method

1. Partition the anchors into a number of clusters according to their geographical positions. In our implementation, we partition the entire sensor area into a number of equal-sized squares and those anchors in a same square form a regional cluster.

2. Each (unpositioned) sensor sees if it has a direct connection to an anchor (within the communication range to an anchor). If it does, it becomes an unknown sensor point in the cluster to which the anchor belongs. Note that a sensor may be assigned into multiple clusters and some sensors are not assigned into any cluster.

3. For each cluster of anchors and unknown sensors, formulate the error minimization problem for that cluster, and solve the resulting SDP model if the number of anchors is more than $2$. Typically, each cluster has less than $100$ sensors and the model can be solved efficiently.

4. After solving each SDP model, check the individual trace for each unknown sensor in the model. If it is below a predetermined small tolerance, label the sensor as *positioned* and its estimation $\bar{x}_j$ becomes an “ anchor”. If a sensor is assigned in multiple clusters, we choose the $\bar{x}_j$ that has the smallest individual trace. This is done so as to choose the best estimation of the particular sensor from the estimations provided by solving the different clusters.

5. Consider positioned sensors as anchors and return to Step 1 to start the next round of estimation.

Figure 12: SDP localization, $2,000$ sensors, radiorange=.05, noise=10%.

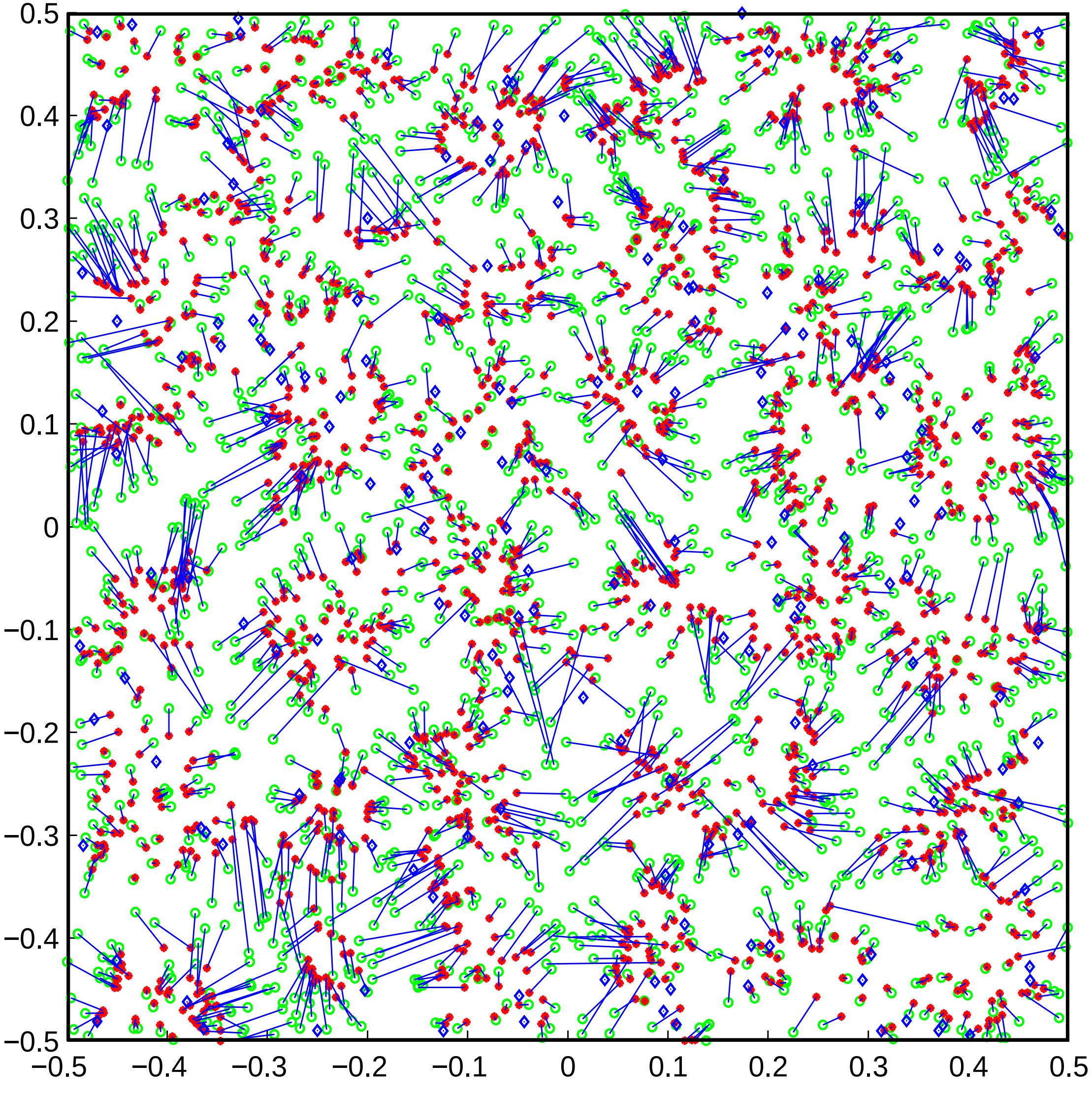
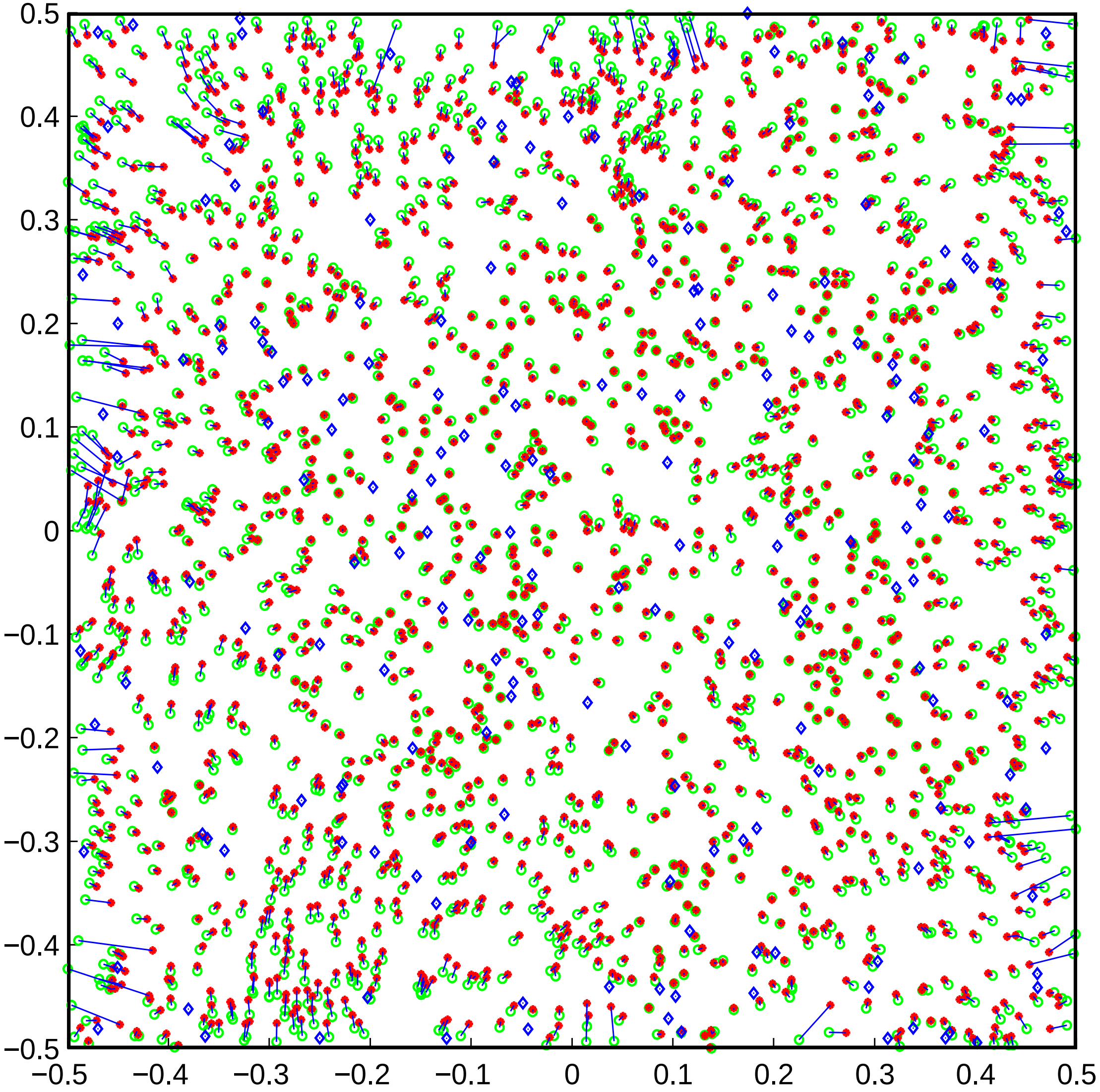


Figure 13: Localization after 50 gradient search steps, $2,000$ sensors.



## More applications: Data dimensionality reduction

Given $P$, a data point set of $p_1, ..., p_n \in R^d$, a fundamental question is how to embed $P$ into $Q$ of $q_1, ..., q_n \in R^k$, where $k \ll d$, such that $q_j$s keep all essential information of $P$, such as the norms, distances and angles between $p_j$s.

In other words, find a $d - k$-dimension subspace such that the projections of $p_j$s onto the subspace has a minimal “information loss.”

## Euclidean ball packing

The Euclidean ball packing problem is an old mathematical geometry problem with plenty modern applications.

## Molecular confirmation

3-D Localization.

## More research topics

- Are there necessary and sufficient conditions for all sensors being localizable? More on rounding the SDP solution matrix into a lower rank matrix?
- What is the best objective function in the SDP model such that the position errors, resulted from the noise in distance measures, are minimal?
- Incorporate other information, angle etc., into the model?
- Design: how many anchors need to be used? Where to place them? What is the best topology?