Stochastic Networks with Concurrent Resource Occupancy *

David D. Yao

http://www.columbia.edu/∼yao

*Joint work with Xuan Li of Columbia University

Outline

- Model Description
- Fixed-point approximation
- Two revenue optimization problems: control and pricing
- Asymptotic optimality
- Conclusions

Model Description

- A network with a set of links (flights), denoted $\mathcal{K}$, with each link connecting a pair of nodes (cities).
- Each link k has a capacity limit C_k, number of seats on the flight.
- A set of routes, denoted $\mathcal{R}$; each route r is a subset of links; $k \in r$ if link k is part of route r.
- Demand on each route r follows an independent Poisson process with rate λ_r.
- Planning horizon is a given time T; assume $T \equiv 1$.

The Fixed-Point Approximation

- For each route r, we want to set a booking limit y_r. An order (demand) for a route r will be rejected once y_r is reached or C_k is reached for some $k \in r$, whichever happens first.

- Let A_r be the total number of orders accepted (up to $T = 1$). Let N_r denote the Poisson variate with mean λ_r. Then,

$$A_r \le N_r \wedge y_r, \quad r \in \mathcal{R}; \qquad \sum_{r \ni k} A_r \le C_k, \quad k \in \mathcal{K}.$$

Hence, $A_r = N_r \wedge y_r \wedge \min_{k \in r} \left\{ C_k - \sum_{s \neq r, s \ni k} A_s \right\}$.

- Approximate $\mathsf{E}(A_r)$ as follows:

$$\mathsf{E}(A_r) \approx \mathsf{E}\left[N_r \wedge y_r \wedge \min_{k \in r}\left\{C_k - \sum_{s \neq r, s \ni k} \mathsf{E}(A_s)\right\}\right].$$

- To simplify notation, write $x_r := \mathsf{E}(A_r)$,

$$m_r := \min_{k \in r}\left\{C_k - \sum_{s \neq r, s \ni k} x_s\right\},$$

and

$$h(\lambda, n) := \mathsf{E}[N(\lambda) \wedge n] = n - \sum_{k=0}^{n}(n-k)\frac{\lambda^k}{k!}e^{-\lambda}.$$

- Then, we can express the mean number of accepted orders for route r, $\mathsf{E}(A_r)$ (denoted x_r), as the solution to the following fixed-point problem:

$$x_r = h(\lambda_r, y_r \wedge m_r), \qquad r \in \mathcal{R}.$$

  Recall: $m_r := \min_{k \in r} \{C_k - \sum_{s \neq r, s \ni k} x_s\}$.

- Reminiscent of the Erlang fixed-point approximation (Kelly, Whitt). Differences: (a) not a steady-state result, (b) mean versus probability; and (c) excellent accuracy, no independence assumption (among links).

The Booking Conrol Problem

- Let w_r be the per unit revenue obtained from supplying a route r order. We want to set the booking limit $\mathbf{y} := (y_r)_{r\in\mathcal{R}}$ so as to maximize the expected revenue:

$$\max_{\mathbf{y}} \sum_{r\in\mathcal{R}} w_r x_r \qquad \text{s.t.} \quad x_r = h(\lambda_r, y_r \wedge m_r), \quad r \in \mathcal{R}. \tag{1}$$

- Clearly, only need to consider $y_r \le m_r$ for all r, in which case the constraints above become $x_r = h(\lambda_r, y_r)$ for all r. Hence, the problem becomes

$$\max_{\mathbf{x}} \sum_{r\in\mathcal{R}} w_r x_r \quad \text{s.t.} \quad h^{-1}(\lambda_r, x_r) + \sum_{s\neq r, s\ni k} x_s \le C_k, \quad k \in r,\ r \in \mathcal{R}.$$

Repeat:

$$\max_{\mathbf{x}} \sum_{r \in \mathcal{R}} w_r x_r \quad \text{s.t.} \quad h^{-1}(\lambda_r, x_r) + \sum_{s \neq r, s \ni k} x_s \leq C_k, \quad k \in r, \ r \in \mathcal{R}.$$

Note the constraint above is simply $y_r \leq m_r$:

$$\left(h^{-1}(\lambda_r, x_r) = \right) y_r \leq m_r := \min_{k \in r} \{ C_k - \sum_{s \neq r, s \ni k} x_s \}.$$

- Therefore, equivalently, we can solve the following problem:

$$\max_{\mathbf{y}} \sum_{r \in \mathcal{R}} w_r h(\lambda_r, y_r) \quad \text{s.t.} \quad y_r + \sum_{s \neq r, s \ni k} h(\lambda_s, y_s) \leq C_k, \quad k \in r, \ r \in \mathcal{R}.$$

- Let η_{rk} be the Lagrangian multipliers. The optimality equations are:

$$w_r \frac{\partial h(\lambda_r, y_r)}{\partial y_r} - \sum_{k \in r} \eta_{rk} - \Big(\sum_{s \neq r} \sum_{k \in s \cap r} \eta_{sk} \Big) \frac{\partial h(\lambda_r, y_r)}{\partial y_r} = 0.$$

That is,

$$\frac{\partial h(\lambda_r, y_r)}{\partial y_r} = \frac{\sum_{k \in r} \eta_{rk}}{w_r - \sum_{s \neq r} \sum_{k \in s \cap r} \eta_{sk}}.$$

- Write

$$\pi_r := \sum_{k \in r} \eta_{rk}, \qquad \pi_{\bar{r}} := \sum_{s \neq r} \sum_{k \in s \cap r} \eta_{sk}.$$

π_r is the cost to route r for using the links, and $\pi_{\bar{r}}$ is the indirect cost (or "penalty") incurred by route r in terms of its impact on other routes that share links with r.

- Simple derivation establishes $\partial h(\lambda_r, y_r)/\partial y_r = \bar{P}_{\lambda_r}(n)$, where $P_\lambda(\cdot)$ denotes the Poisson distribution function with parameter λ. Hence, the optimality equation takes the following form:

$$P_{\lambda_r}(\lfloor y_r \rfloor) = \frac{w_r - \pi_r - \pi_{\bar{r}}}{w_r - \pi_{\bar{r}}}.$$

- That is, the optimal booking limit for route r should be set at y_r, such that the proportion of orders accepted on that route satisfies a "critical ratio," which is profit (revenue minus both direct and indirect costs) over profit plus penalty (or, indirect cost).

Linearization

- To solve the nonlinear optimization problem, first ignore the integrality of the variables, by extending the domain of the function $h(\lambda, n)$ to $\mathbf{R}_+$ by linear interpolation:

$$h(\lambda, y) = h(\lambda, n) + (y - n)[h(\lambda, n+1) - h(\lambda, n)], \qquad y \in (n, n+1).$$

Hence,

$$h^{-1}(\lambda, x) = n + \frac{x - h(\lambda, n)}{h(\lambda, n+1) - h(\lambda, n)}, \qquad x \in (h(\lambda, n), h(\lambda, n+1)).$$

- From the above linearization, we can express x_r as:

$$x_r = h(\lambda_r, y_r) = \sum_{n=0}^{\hat{C}_r} \theta_{rn}[h(\lambda_r, n) - h(\lambda_r, n-1)] = \sum_{n=0}^{\hat{C}_r} \theta_{rn} \bar{P}_{\lambda_r}(n),$$

where θ_{rn} is a set of parameters, and $\hat{C}_r := \min_{k \in r}\{C_k\}$.

- For instance, when $x_r \in [h(\lambda_r, n_0 - 1), h(\lambda_r, n_0)]$ for some n_0, then
  - $\theta_{rn} = 1$, for $n \leq n_0 - 1$;
  - $\theta_{rn_0} \in [0, 1]$; and
  - $\theta_{rn} = 0$, for $n \geq n_0 + 1$.
- Applying the inverse $h(\lambda_r, \dot{})^{-1}$, we have

$$y_r = h^{-1}(\lambda_r, x_r) = \sum_{n=1}^{\hat{C}_r} \theta_{rn}.$$

Solution via Linear Program

- Hence, we can re-express the original nonlinear control problem as a linear program, with $\theta := (\theta_{rn})$ as decision variables:

$$\begin{aligned}
\max_{\theta} \quad & \sum_{r \in \mathcal{R}} w_r \sum_{n=1}^{\hat{C}_r} \theta_{rn} \bar{P}_{\lambda_r}(n) \\
\text{s.t.} \quad & \sum_{n=1}^{\hat{C}_r} \theta_{rn} + \sum_{s \neq r, s \ni k} \sum_{n=1}^{\hat{C}_s} \theta_{sn} \bar{P}_{\lambda_s}(n) \leq C_k, \quad k \in r,\ r \in \mathcal{R}; \\
& \theta_{rn} \in [0,1], \quad n = 0, 1, \ldots, \hat{C}_r,\ r \in \mathcal{R}.
\end{aligned}$$

- Let η_{rk} be the dual variables corresponding to the link capacity constraints, and ξ_{rn} the dual variables corresponding to the constraint $\theta_{rn} \leq 1$. Then, complementary slackness requires

$$w_r \bar{P}_{\lambda_r}(n) - \sum_{k \in r} \eta_{rk} - \Big(\sum_{s \neq r} \sum_{k \in s \cap r} \eta_{sk} \Big) \bar{P}_{\lambda_r}(n) = \xi_{rn},$$

for any (r, n) such that $\theta_{rn} > 0$.

- This leads to, in terms of the π notation defined earlier,

$$w_r \bar{P}_{\lambda_r}(n) - \pi_r - \pi_{\bar{r}} \bar{P}_{\lambda_r}(n) = \xi_{rn}.$$

That is,

$$\bar{P}_{\lambda_r}(n) = \frac{\xi_{rn} + \pi_r}{w_r - \pi_{\bar{r}}};$$

or

$$P_{\lambda_r}(n-1) = \frac{w_r - \xi_{rn} - \pi_r - \pi_{\bar{r}}}{w_r - \pi_{\bar{r}}}.$$

- Repeat:

$$P_{\lambda_r}(n-1) = \frac{w_r - \xi_{rn} - \pi_r - \pi_{\bar{r}}}{w_r - \pi_{\bar{r}}}.$$

Recall, from the original problem, the optimal solution is:

$$P_{\lambda_r}(\lfloor y_r \rfloor) = \frac{w_r - \pi_r - \pi_{\bar{r}}}{w_r - \pi_{\bar{r}}}.$$

- Here, the optimal booking limit, following the LP solution, is

$$y_r^* = \Big(\sum_{n=1}^{\hat{C}_r} \theta_{rn} = \Big)(n_r - 1) + \theta_{rn_r}, \qquad r \in \mathcal{R};$$

where n_r is such that $\theta_{rk} = 1$ for $k \leq n_r - 1$, $\theta_{rk} = 0$ for $k \geq n_r + 1$, and $\theta_{rn_r} \leq 1$. If $\theta_{rn_r} < 1$, then $\xi_{rn_r} = 0$; i.e., the LP generates the same solution as the original problem.

Comparison against Existing Models

- Existing approaches start with the following LP:

$$\max_{\mathbf{z}} \sum_{r} w_r z_r, \quad \text{s.t.} \quad \sum_{r \ni k} z_r \leq C_k,\ k \in r,\ z_r \leq \lambda_r,\ r \in \mathcal{R};$$

or, the following NLP:

$$\max_{\mathbf{z}} \sum_{r} w_r \mathsf{E}[min\{z_r, N_r\}] \quad \text{s.t.} \quad \sum_{r \ni k} z_r \leq C_k,\ k \in r,\ r \in \mathcal{R}.$$

- Recall our model:

$$\max_{\mathbf{y}} \sum_{r \in \mathcal{R}} w_r h(\lambda_r, y_r) \quad \text{s.t.} \quad y_r + \sum_{s \neq r, s \ni k} h(\lambda_s, y_s) \leq C_k, \quad k \in r,\ r \in \mathcal{R}.$$

- Note $h(\lambda_r, y_r) = \mathsf{E}[N(\lambda_r) \wedge y_r] \le \lambda_r \wedge y_r$. Hence,

$$y_r + \sum_{s \neq r, s \ni k} h(\lambda_s, y_s) \le \sum_{r \ni k} y_r \le C_k;$$

i.e., the constraints, $\sum_{r \ni k} z_r \le C_k$, in the existing (NLP) model is too conservative.

- Existing models use the dual variables (shadow prices) to come up with a "nested capacity allocation" scheme. We simply apply the solution y_r as booking limit to each route r, and then just follow plain FCFS.

- Our model leads to a richer set of dual varibles; in particular, explicitly characterizes:
  - the capacity implication of accepting an order (π_r),
  - the penalty in terms of impacting other routes.($\pi_{\bar{r}}$),

  - the “shadow price” on demand (ξ_{rn}).

- Numerical studies demonstrate that our route-based booking limit control outperforms the best nested capacity allocation, in particular when the demand factor (mean demand over link capacity) is high.

The Pricing Problem

- Next, let w_r be decision variables too. The arrival rate λ_r is determined by the price w_r through a decreasing function $\lambda_r = g_r(w_r)$, with $w_r \in [\underline{w}_r, \overline{w}_r]$. The pricing problem is:

$$\begin{aligned} \max_{\mathbf{y},\mathbf{w}} \quad & \sum_{r \in \mathcal{R}} w_r h(g_r(w_r), y_r) \\ \text{s.t.} \quad & y_r + \sum_{s \neq r, s \ni k} h(g_s(w_s), y_s) \leq C_k, ; \\ & w_r \in [\underline{w}_r, \overline{w}_r], \quad k \in r,\ r \in \mathcal{R}. \end{aligned} \tag{2}$$

- Since $w_r = g_r^{-1}(\lambda_r)$, the problem can be rewritten as:

$$\begin{aligned} \max_{\mathbf{y},\lambda} \quad & \sum_{r\in\mathcal{R}} w_r(\lambda_r)h(\lambda_r, y_r) \\ \text{s.t.} \quad & y_r + \sum_{s\neq r, s\ni k} h(\lambda_s, y_s) \leq C_k, ; \\ & \lambda_r \in [g_r(\overline{w}_r), g_r(\underline{w}_r)], \quad k \in r,\ r\in\mathcal{R}. \end{aligned}$$

- Let η_{rk} still be the Lagrangian multipliers. We have another set of optimality equations:

$$w_r' h(\lambda_r, y_r) + w_r \frac{\partial h(\lambda_r, y_r)}{\partial \lambda_r} - \Big(\sum_{s\neq r}\sum_{k\in s\cap r} \eta_{sk}\Big)\frac{\partial h(\lambda_r, y_r)}{\partial \lambda_r} = 0.$$

That is,

$$\frac{\partial h(\lambda_r, y_r)}{\partial \lambda_r} = \frac{-w_r' h}{w_r - \pi_{\bar{r}}} = P_{\lambda_r}(\lfloor y_r \rfloor).$$

- Combining the two set of optimality equations, we have

$$w_r - \pi_r - \pi_{\bar{r}} = -w_r' h$$

- That means at optimality, orders on route r should be priced such that the expected incremental revenue due to changes in order arrivals is equal to the net profit (revenue minus the shadow prices).

Asymptotic Optimality

- If the prices are optimally set, the booking limits become ineffective; one can simply follow FCFS.

- For asymptotic optimality, we scale the original system: the link capacity to nC, and the arrival rate for each route r to $n\lambda_r$.

- Suppose J^* is the expected revenue under the optimal control policy, and J^*_{bl} is the the optimal objective value from the booking-limt control problem. Then,

$$\lim_{n\to\infty} \frac{J^*_{bl}(n)}{n} = \lim_{n\to\infty} \frac{J^*(n)}{n}.$$

- A similar asymptotic optimality holds for the pricing problem.

Concluding Remarks

- The fixed-point approximation provides a very accurate estimate of the accepted orders on each route, given the booking limits.

- Booking-limit control, a nonlinear integer programming problem, can be solved by linear programming, and the control policy is asymptotically optimal.

- Revenue optimization via pricing is also asymptotically optimal. At optimal pricing, there's no need to exercise booking control, FCFS suffices.

- Both booking control and pricing solutions are easy to interpret and simple to implement, and outperform the best solutions from other existing approaches.