Online Algorithms and Competitive Analysis: An Introduction

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Outline

- Algorithms and approximations
- Online problems and algorithms
- Competitive analysis
- Online vs. real-time
- Pros and cons of competitive analysis
- Beyond competitive analysis

An Example: Machine Scheduling

Offline scenario

Given a fixed number of identical machines and a set of jobs, each is specified by a processing time, we are to assign all jobs to the machines in such a way that the makespan is minimized.

Two online scenarios:

While the total number of jobs to be processed is unknown,

- List: the jobs arrive one by one, only when the current job has been assigned, will the next the job become available for assignment. Any job assignment is irrevocable.
- Time: the jobs arrive over time and decisions are made over time.

Optimization Problems

An (offline) optimization problem (of cost minimization) consists of:

- A set $\{I\}$ of inputs and a cost function C.
- Associated with every input I is a (finite) set $F(I)$ of feasible solutions.
- Associated with each feasible solution $f \in F(I)$ is a positive number $C(I, f)$, representing the cost of the solution f with respect to input I.

Algorithms and Approximation

- Given any input I, an algorithm ALG for an optimization problem computes a feasible solution $\mathrm{ALG}[I] \in F(I)$ with cost $\mathrm{ALG}(I) = C(I, \mathrm{ALG}[I])$
- An optimal algorithm OPT is such that for all input I,

$$\mathrm{OPT}(I) = \min_{f \in F(I)} C(I, f)$$

- An algorithm ALG is an asymptotic ρ-approximation algorithm if there is a constant $\alpha \geq 0$ such that for all inputs I,

$$\mathrm{ALG}(I) \leq \rho \mathrm{OPT}(I) + \alpha$$

Online Problems and Algorithms

An online problem:

- Input is received in an online manner
- Output must be produced online

An online algorithm (not necessarily polynomial) ALG is ρ-competitive if there is a constant α such that for all inputs I,

$$\mathrm{ALG}(I) \leq \rho \mathrm{OPT}(I) + \alpha$$

Competitive ratio: infimum over all qualifying ρ.

Online vs. Real-Time

Tow basic online paradigms:

- Over list
- Over time

Intrinsic nature of problems:

- Offline: linear programming
- Either offline or online: scheduling
- Online: investment planning

Why Competitive Analysis?

- Traditional distributional (average-case) complexity.
- Online complexity falls within the framework of worst-case complexity.
- Performance guarantee of an algorithm is necessary and hence competitive analysis is essential for problems such as financial investment planning.
- Online algorithms have been studied implicitly or explicitly for over 30 years.

Why Competitive Analysis?

- Main accomplishment of classical complexity theory:

  Give structure to classes of problems, models, and algorithms

- Competitive analysis has provided some structure to the study of online computation:

  Abstract models of online computation have provided a framework for a number of general techniques and solutions and other fundamental issues have been identified and studied.

Games and Adversaries

A game:

- The online player runs an online algorithm on an input that is created by the adversary.
- The adversary, based on the knowledge of the algorithm, constructs the worst possible input.

Deterministic case

Adversary = offline player

Limitations of Competitive Analysis

- Too pessimistic:

  The malicious adversary chooses the worst input
- How about distributional analysis?
- A consummate analyst considers and often combines all plausible approaches.

Limitation of Deterministic Algorithms

Example:

For on-line (over list) scheduling two identical machines, there can be no deterministic (3/2)-approximation algorithm.

Randomization

It is an important tool in avoiding “traps” (degeneracy), in reaching tricky “corners” of the domain and in many other situations.

Randomized Algorithms

Oblivious adversary

Based on the knowledge of ALG (in particular, the probability distributions ALG uses), choose an instance in advance.

Competitiveness against an oblivious adversary

For any input I, there is a constant α, such that

$$E[\mathrm{ALG}(I)] \leq \rho \mathrm{OPT}(I) + \alpha$$

where $E[]$ is the mathematical expectation with respect to the random choices made by ALG.

Beyond Competitive Analysis

- Refinement: Extra information
- Extra resources
- Comparative analysis: Compare the online algorithms from given classes of algorithms

Thank you!