Lecture 1 : fair division of indivisible units

Division according to claims

- $N = \{i\}$ agents

- $\mathbb{N} = \{0, 1, 2, ...\}; t \in \mathbb{N}$: resources; $x_i \in \mathbb{N}$: agent i's claim

- $t \leq \sum_N x_i = x_N$: rationing, urn emptying, scheduling

- $t < \sum_N x_i$: surplus sharing, urn filling

- (N, t, x) : rationing problem

- $Y = (Y_i)_{i \in N}$: random variable $s.t.$ $Y_i(\varpi) \in \mathbb{R}$, $0 \leq Y_i \leq x_i$, $\sum_N Y_i = t$

- $(N, t, x) \rightarrow r(N, t, x) = Y$: rationing method

- duality: $r \to r^*$: $r^*(t,x) = x - r(x_N - t, x)$

- distribution of the first unit:$\rho_i(x) = proba\{r(1,x) = e^i\}$,

- of the first unit of tax: $\tau_i(x) = proba\{r(x_N - 1, x) = x - e^i\}$

Examples of rationing methods

- priority $prio^\sigma$: $\sigma =$ fixed ordering of N, $(\pi^\sigma)^* = \pi^{\sigma^*}$

- random priority rp : $rp(t,x) = prio^\sigma(t,x)$ where σ is uniformly distributed over all orderings of N, $rp^* = rp$

- proportional pro : $\rho_i(x) = \tau_i(x) = \frac{x_i}{x_N}$, iterate; $pro^* = pro$

- $m(x) = \{i \in N \mid x_i > 0\}, M(x) = \{i \in N \mid x_i = \max_j x_j\}$;

- fair queuing fq : $\rho(x)$ uniform on $m(x)$; $\tau(x)$ uniform on $M(x)$; iterate on τ

- fair queuing* fq^* : $\rho(x)$ uniform on $M(x)$, $\tau(x)$ uniform on $m(x)$; iterate on ρ

- equal chances ec : $\rho(x)$ uniform on $m(x)$, iterate

- equal chances* ec^* : $\tau(x)$ uniform on $m(x)$, iterate mild properties:

- Demand Monotonicity DM : $x' = x + e^i \Longrightarrow$ $r_i(t, x')$ stochastically dominates $r_i(t, x)$

- Demand Monotonicity* DM^* : $x' = x + e^i \Longrightarrow$ $x'_i - r_i(t, x')$ stochastically dominates $x_i - r_i(t, x)$

basic equity property

- Equal Treatment ExAnte $ETEA$: $x_i = x_j \Longrightarrow$ $r_i(t, x)$ and $r_j(t, x)$ identically distributed

All methods above meet DM, DM^*

All except priority meet ETE

It is always interesting to drop the ETE property axioms with much bite

Two dual markovian properties: $UC^* = LC$

- Upper Composition UC : $t < t' \leq x_N \Longrightarrow r(t,x) = r(t, r(t',x))$; equivalently: $r(t,x)$ obtains by iteration of $\tau(x)$

- Lower Composition LC : $0 \leq t' < t \leq x_N \Longrightarrow r(t,x) = r(t',x) + r(t-t', x - r(t',x))$; equivalently: $r(t,x)$ obtains by iteration of $\rho(x)$

  examples

- priority, proportional: LC <u>and</u> UC

- fair queuing, equal chances*: UC, <u>not</u> LC

- fair queuing*, equal chances: LC, <u>not</u> UC

- random priority: <u>neither</u> LC <u>nor</u> UC

A variable population property

- Consistency CSY : fix (N, t, x) and write $Y_k = r_k(N, t, x)$;then $Y_i = r_i(N \backslash j, t - Y_j, x_{-j})$

Note: $CSY = CSY^*$

Examples

- priority, proportional, fair queuing, fair queuing*: YES

- random priority, equal chances, equal chances*: NO

An incentive-compatibility property

- Strategyproofness SP : fix i, x_{-i} and x_i, x_i'; then $r_i(t, (x_i, x_{-i}))$ stochastically dominates $proj_{[0,x_i]} r_i(t, (x_i', x_{-i}))$

Note: $SP \Longrightarrow DM$ (take $x_i' \leq x_i$)

Examples

- priority, random priority, fair queuing, equal chances: YES

- proportional, fair queuing*, equal chances*: NO

Characterization results

- $UC + LC + ETE \Longleftrightarrow UC+$ self dual $\Longleftrightarrow$ proportional

- the $UC + LC$ family consists of interesting variants of the proportional method

- $UC + LC + CSY \Longleftrightarrow$ priority composition of proportional methods (US bankruptcy law)

Equal Treatment Ex Post $ETEP$: $x_i = x_j \Longrightarrow \mid Y_i(w) - Y_j(w) \mid \leq 1$

- $UC + DM^* + ETEA + ETEP \Longleftrightarrow$ fair queuing

- $LC + DM + ETEA + ETEP \Longleftrightarrow$ fair queuing*

- Standard of loss: an ordering $\succsim$ (complete, transitive) of $N \times \mathbb{N}$ such that $x'_i \geq x_i \Longrightarrow (i, x'_i) \succsim (i, x_i)$

- Standard of loss method: an UC method such that $(i, x_i) \succ (j, x_j) \Longrightarrow \tau_j(x) = 0$

If the standard is a strict ordering, this defines a *deterministic* method

- $UC + CSY + DM^* +$ deterministic $\Longleftrightarrow$ standard of loss

- $UC + CSY + DM^* \Longleftrightarrow$ probabilistic standard of loss

- example: $UC + CSY + DM^* + ETEA \Longleftrightarrow$ "mixtures" of fair queuing and proportional

- the dual notion of standards of gain allows a dual description of the family $LC + CSY + DM$

- $LC + SP \iff$ fixed chances; in particular $LC + SP + ETEA \iff$ equal chances

- $UC + SP + ETEA \iff$ fair queuing

- $UC + SP \iff$ "quasi deterministic" standard of loss methods

- $CSY + SP \iff$??

- $SP+$ self-dual $\iff$ random priority (*conjecture)*

Lecture 2 :Cost and benefit sharing

- agents N, agent i demands $x_i = 0, 1, 2, ...$

- cost function $C : N^N \longrightarrow R_+$, non decreasing, $C(0) = 0$

- cost sharing method: $\varphi : (N, C, x) \longrightarrow y = \varphi(N, C, x) \in \mathbb{R}_+^N$, $\sum_i y_i = C(x)$

- special case $x = (1, ..., 1)$ is classical cooperative game framework.

- Additivity axiom: $\varphi(C^1 + C^2, x) = \varphi(C^1, x) + \varphi(C^2, x)$

- shared flow on $[0, x]$: $f(z - e^i, z)$ a unit flow on $[0, x]$ from 0 to x ; $s_j(z - e^i, z)$ is agent i's share, $s_i \geq 0, \sum_i s_i = 1$

- associate to (f, s) an additive method :$\varphi(C, x) = \sum_{z \in]0,x]} \sum_{i: z_i > 0} \partial_i C(z) \cdot f(z - e^i, z) \cdot s(z - e^i, z)$

- *Theorem:* every additive cost sharing method $\varphi(x)$ is represented (in at least one way) by a shared flow on $[0, x]$

Examples of additive cost sharing methods

- fixed shares: $\varphi(C,x) = C(x) \cdot s$, where s is a fixed vector of shares.

- simple proportional: $\varphi_i(C,x) = C(x) \cdot \frac{x_i}{\sum_j x_j}$

- incremental: fix an ordering of N say $\{1,2,..\}$ $\varphi_1(C,x) = C(x_1,0), \varphi_2(C,x) = C(x_1,x_2,0) - C(x_1,0), \varphi_3(C,x) = C(x_1,x_2,x_3,0) - C(x_1,x_2,0), ...$

- cross-subsidizing serial: say $N = \{1,2,3\}$ and $x_1 \leq x_2 \leq x_3$, then

  $\varphi_1(C,x) = \frac{1}{3}C(x_1,x_1,x_1); \varphi_2(C,x) = \varphi_1(C,x) + \frac{1}{2}(C(x_1,x_2,x_2) - C(x_1,x_1,x_1)); \varphi_3(C,x) = \varphi_2(C,x) + C(x) - C(x_1,x_2,x_2)$

Responsibility in idiosyncratic demand: normative and incentive justification.

- Dummy axiom: for any i ,$\{\partial_i C(z) = 0$ for all $z \in \mathbb{N}^N\} \Longrightarrow \varphi_i(C, x) = 0$

- Non Dummy axiom: for any i ,$\{C(z) = c(z_i)$ for all $z \in \mathbb{N}^N\} \Longrightarrow \varphi_i(C, x) = c(z_i)$

- Lemma: $\{$Additivity $+$ Dummy$\}\Longleftrightarrow\{$Additivity $+$ Non Dummy$\}$

The Dummy axiom eliminates the fixed shares, simple proportional, and cross subsidizing serial methods.

Representation of Additive + Dummy methods

- a unit flow f from 0 to x defines a simple shared flow by $s(z-e^i,z)=e^i$.

- we associate to flow f the probabilistic rationing method where $proba\{r(t,x)=z\}$ is the in-flow at z.

- we associate to flow f a cost sharing method $\varphi_i(C,x)=\sum_{z\in]0,x],z_i>0}\partial_i C(z)\cdot f(z-e^i,z)$. Dummy holds true.

- *Theorem :* every additive c.s. method $\varphi(x)$ meeting Dummy is represented *uniquely* by a unit flow f.

Thus the Additive and Dummy methods are in one-to-one correspondence with the subset of rationing methods constructed by flows.

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Examples

- priority rationing $\longleftrightarrow$ incremental cost sharing

- random priority rationing $\longleftrightarrow$ Shapley-Shubik cost sharing : uniform average of all incremental methods $\Leftrightarrow$ Shapley value of the stand alone game $v(S) = C(x(S), 0(N \setminus S))$

- proportional rationing $\longleftrightarrow$ Aumann-Shapley cost sharing : uniform average over all paths from 0 to x.

Note that *Aumann Shapley* and *simple proportional* coincide when outputs are perfect substitute: $C(x) = c(\sum_i x_i)$.

- fair queuing $\longleftrightarrow$ subsidy-free serial cost sharing

The cost sharing methods corresponding to fair queuing*, equal chances rationing, equal chances*, are easy to define but have not emerged in the literature.

Characterization results: not as developed as one may want.

- Ordinality axiom: for all $i \in N, z_i \in \mathbb{N}$,$\{\partial_i C(z_i, z_{-i}) =$ 0 for all $z_{-i}\} \Longrightarrow \{$we can *merge* z_i and $z_i - 1\}$

Theorem: Additivity $+$ Dummy $+$ Ordinality $\Longleftrightarrow$ $\{$convex combinations of incremental methods$\}$.

Corollary: add Equat Treatment of Equals to pick the Shapley-Shubik method.

The Aumann Shapley method is characterized by Additivity, Dummy, and 2 additional properties:

- Simple proportional for perfect substitutes

- The corresponding rationing method meets Upper Composition

Limits of the additive approach

- Demand Monotonicity axiom: for all i , C and x, $\frac{\partial\varphi_i}{\partial x_i}(C,x) \geq 0$

Note that the Aumann-Shapley method fails Demand Monotonicity !

- *Theorem:* if an additive method meets Dummy and Demand Monotonicity, it can't be simple proportional for perfect substitute outputs.

Strengthening Demand Monotonicity

- Group Demand Monotonicity: for all i, S, C and x, $i \in S \Longrightarrow \frac{\partial \varphi_S}{\partial x_i}(C, x) \geq 0$

- Solidarity : for all i, j, k all distinct, all C, x, $\frac{\partial \varphi_j}{\partial x_i}(C, x) \cdot \frac{\partial \varphi_k}{\partial x_i}(C, x) \geq 0$

Theorem: Additivity + Dummy + $\{$ Group Demand Monotonicity or Solidarity$\} = \emptyset$

Generalized proportional methods :

$\varphi(C,x) = \frac{f_i(x_i)}{\sum f_j(x_j)} \cdot C(x)$, where f_i is nondecreasing and positive.

- *Theorem:* The generalized proportional methods meet Group Demand Monotonicity and Solidarity. They are *characterized* by the combination $\{$Additivity $+$ Demand Monotonicity $+$ Solidarity$\}$.