Approximation Algorithms for Survivable Network Design via Iterative Rounding

Parts I & II of talks at Lunteren 2002

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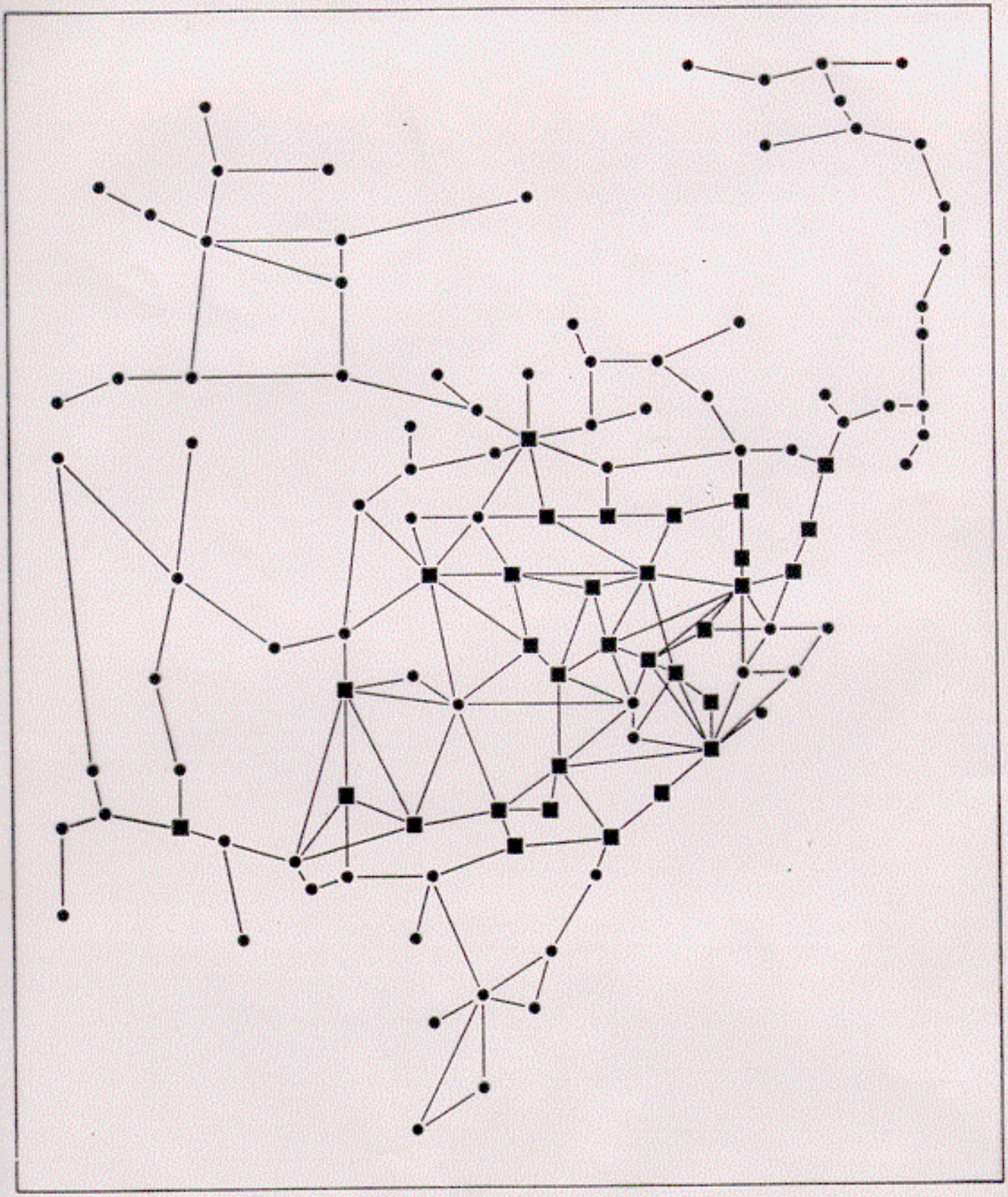
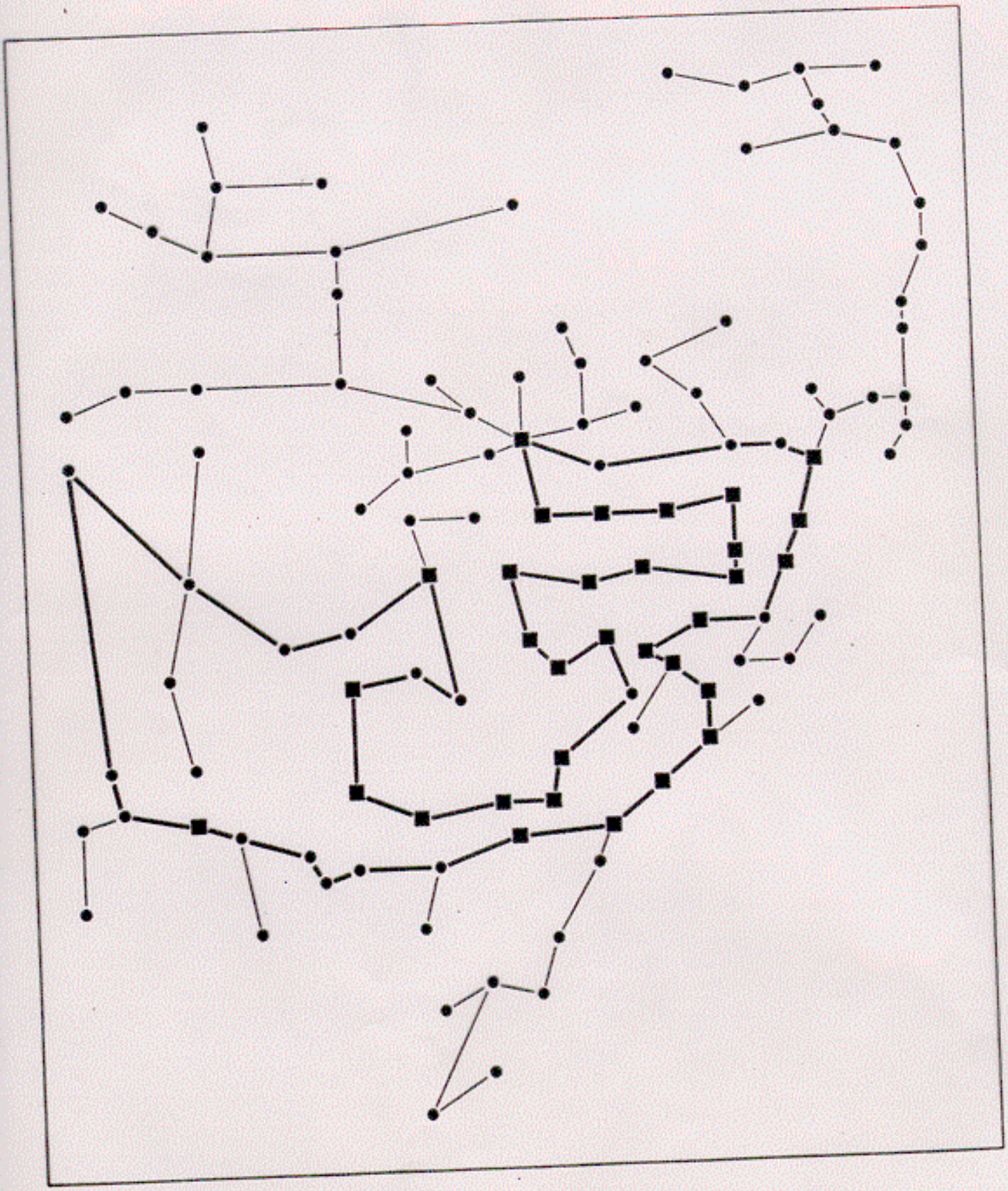


Figure 10. Original graph of the LATADL problem.

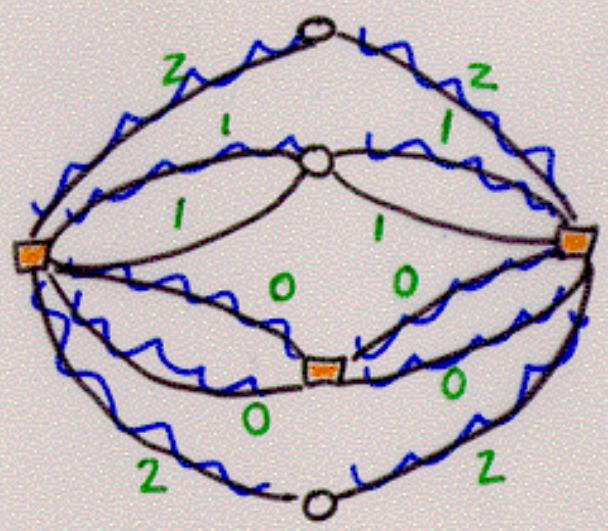


Solution of the LATADI problem.

Vertex Connectivity

vertex disjoint paths

$G=(V, E)$

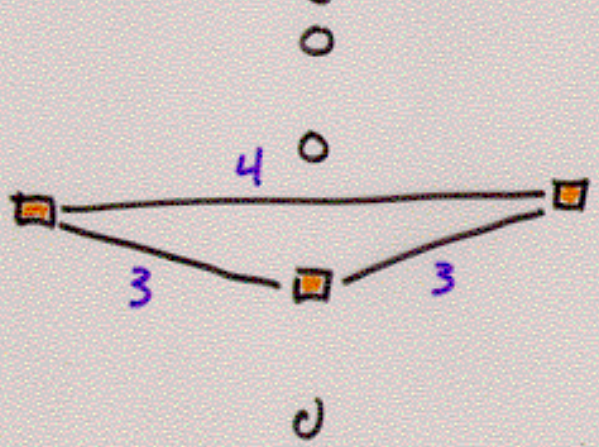


$c(e)$ cost of e

r_{ij} Connectivity required

Goal:

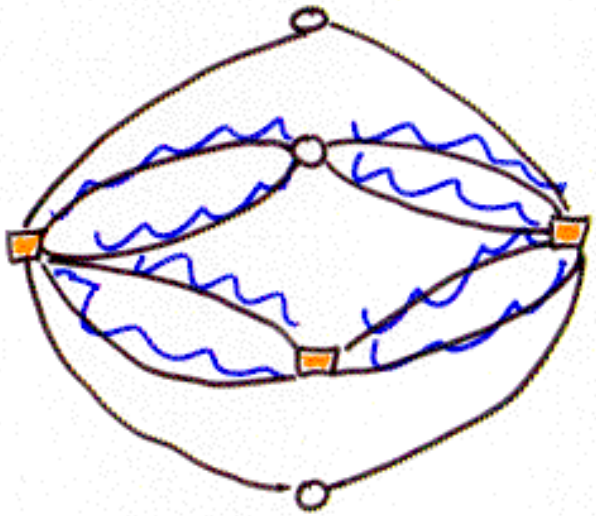
Find min cost subgraph satisfying required connectivity



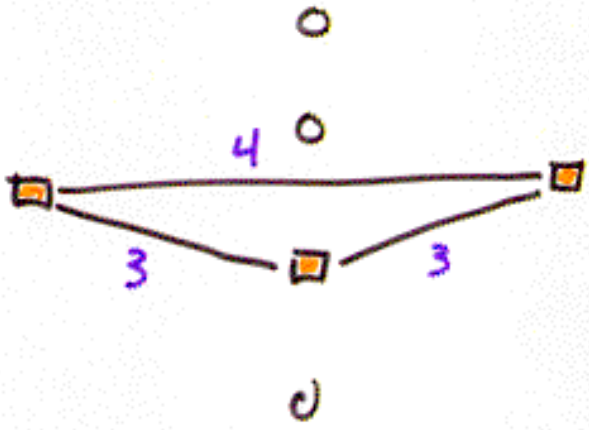
Edge Connectivity

edge disjoint paths

$G = (V, E)$



r_{ij} Connectivity required



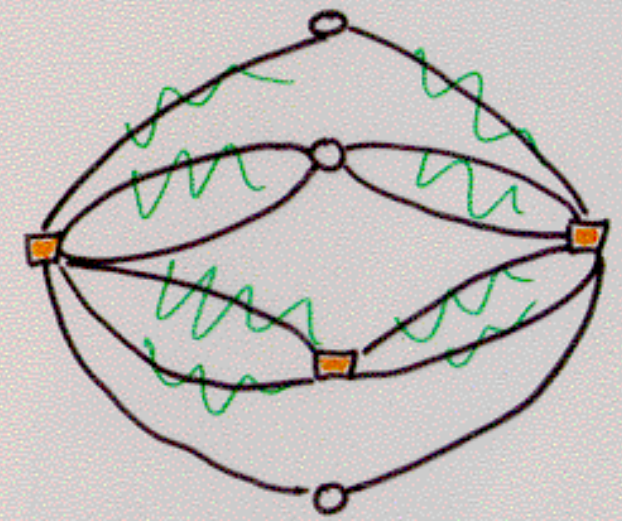
Element Connectivity

element disjoint paths

$V = R \cup Q$

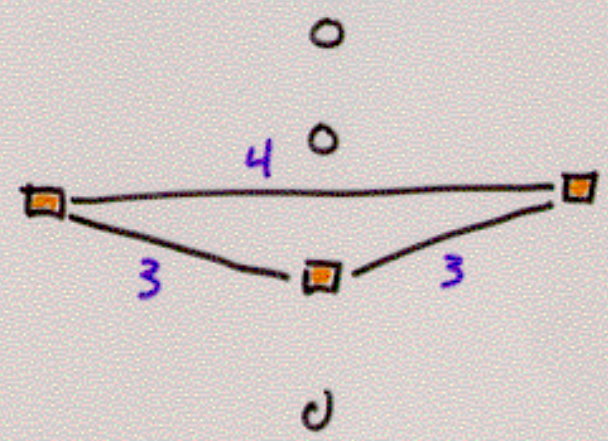
■ terminals o steiner nodes

elements := $E \cup Q$



- Edge connectivity special case: $R = V$, $Q = \emptyset$
- Allows For node Failures

r_{ij} Connectivity required



Algorithm $\mathcal{A}$ is a ρ-approximation algorithm if, for any instance, the cost of the solution obtained by $\mathcal{A}$ is $\leq \rho \cdot OPT$

(Note: $\rho \geq 1$)

Problem $\mathcal{P}$ is max SNP hard if there exists a $\delta > 0$ such that for all $\delta > \varepsilon > 0$, there is no polynomial time $\varepsilon+1$ approximation algorithm for $\mathcal{P}$, unless $P = NP$.

Minimum Cost Connectivity

NP-hard, Max SNP-hard: Steiner tree is special case

$k :=$ max connectivity required

Connectivity problem	Approximation Algorithm: Primal-Dual	Comb.	Iterative Rounding
Edge	$\log k$ GGPSTW		2 Jain 98
Element	$\log k$ JMVW		2 FJW 01
Vertex	—	—	—
uniform k		$O(\log k)$ $^{k}_{CVV}$ ^{KN}	$O(\sqrt{n}/\varepsilon)$ CVV 02
unif + c a metric		~2 KR	
$r_{ij} \in \{0,1,2\}$	3 RW		2 F 01

Iterative Rounding for Edge Connectivity

(1) Formulate residual IP: $x_e = 1$ if take $e \in E$
0 else

$$\min \sum_e c(e)\, x_e$$

$$\forall \emptyset \neq S \subset V \qquad x(S) := x(\delta_{E+F}(S)) \geq f(S) - |\delta_F(S)|$$

$$x_e \in \{0,1\}$$

$$f(S) := \max_{\substack{i \in S \\ j \notin S}} r_{ij}$$

$$\delta_E(S) := \{(i,j) \in E \mid i \in S,\ j \notin S\}$$

$$x(H) := \sum_{e \in H} x_e$$

(2) Solve residual LP relaxation

(3) Fix to 1 all e with $x_e \geq 1/2$

$$F := \{e \mid x_e \text{ fixed to } 1\}$$

(4) If some connectivity requirement unmet, go to (2)

Analysis of Iterative Rounding for Edge Conn.

x current solution to Edge-LP with set F

F' new complete set of Fixed edges

Lemma: x restricted to $E-F'$ is feasible for Edge-LP with fixed set F'.

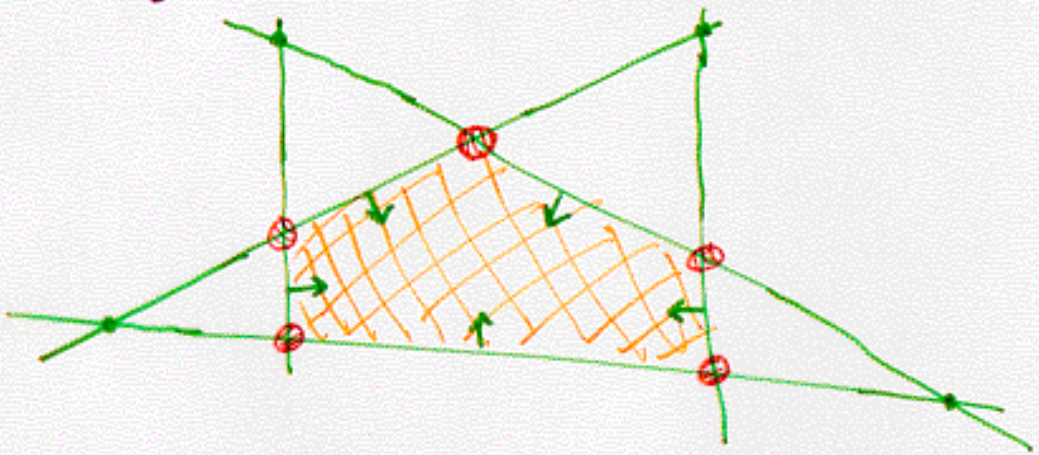
Theorem: For any **basic** solution x to Edge-LP, there is an $e \in E-F$ with $x_e \geq 1/2$.

Lemma + Theorem $\Rightarrow$ 2-approx for edge connectivity

Basic Feasible Solutions to an LP

- the intersection of $|E-F|$ linearly independent equalities defining inequalities of the LP.

- a <u>feasible</u> solution to the LP that cannot be represented as the convex combination of two distinct feasible solutions.

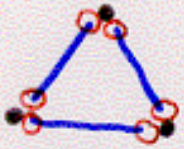


Main Ideas in Proof of Theorem

① For all basic x, there is a laminar basis $\mathcal{B}$ of tight inequalities at x.

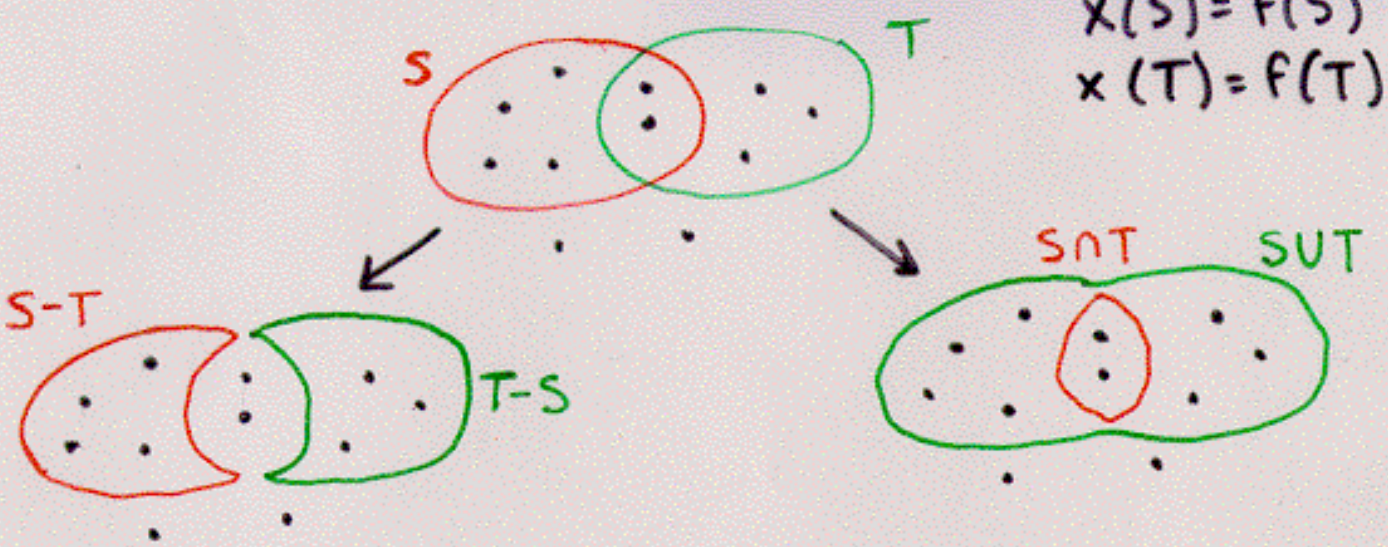
② By contradiction: if all e have $x_e < \frac{1}{2}$ then there are more than $2|\mathcal{B}|$ endpoints of edges in support of x.

(contradicts $|\mathcal{B}| = |E - F|$)



6 endpoints ∘

Obtaining a Laminar Basis



$x(S) = f(S)$
$x(T) = f(T)$

$$f(S) + f(T) \leq \max\{ f(S-T) + f(T-S),\ f(S \cup T) + f(S \cap T) \}$$

"weakly supermodular"

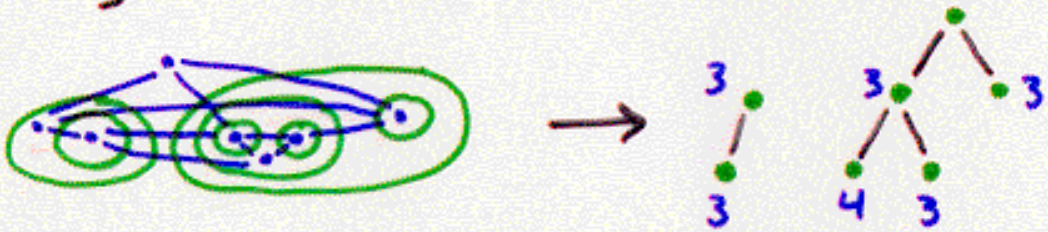
$$x(S) + x(T) \geq \max\{ x(S-T) + x(T-S),\ x(S \cup T) + x(T \cap S) \}$$

"submodular"

$\Rightarrow$ either $S-T$ & $T-S$ or $S \cap T$ & $S \cup T$ are both <u>tight</u> and can <u>replace</u> S or T in basis.

If $x < \frac{1}{2}$ then $\exists > 2|B|$ endpoints of support(x)

Using inclusion tree of sets in laminar basis,



charge endpoints to sets in bottom-up order.

Redistribute so that each node gets at least 2,
each root gets at least 3.

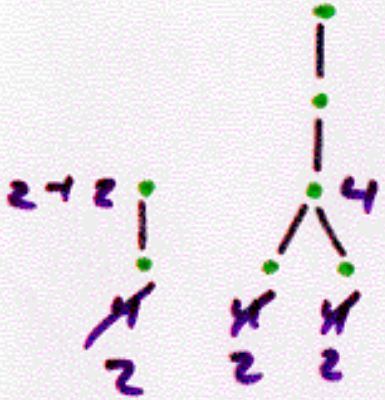
⇒ either $S-T$ & $T-S$ or $S \cap T$ & $S \cup T$ are both tight
and can replace S or T in basis.

Easier claim,

If $x < \frac{1}{3}$ then $\exists > 2|B|$ endpoints of support(x)

yields 3-approximation

- Redistribute endpoints in inclusion tree of laminar basis so that each node gets ≥ 2 & each root gets ≥ 4



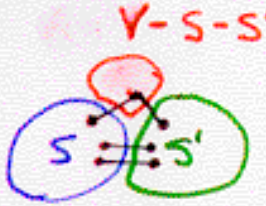
by induction on size of subtree:

- ✓ leaf
- ✓ parent with 2^+ children
- ✓ parent with 1 child & 2 endpoints
- ✗ parent with 1 child & ≤ 1 endpoint

$0 < x_e = |f(B) - f(A)| \in \mathbb{Z}$

Extending Jain's Algorithm to Vertex Connectivity : IP formulation

Stoer '92



Element

~~Vertex~~ disjoint paths From S to S' use either

- edge from S to S'
- ~~vertex~~ in V-S-S'
  steiner node

Menger

$f(S,S')$ max requirement from S to S'

$\delta_F(S,S')$ set of edges in F with one endpoint in each of S and S'

$x_F(S,S')$ LP value of edges in $\delta_F(S,S') = \sum_{\substack{i \in S \\ j \in S' \\ (i,j) \in F}} x(i,j)$

minimize $c^T x$

$$x_E(S,S') \geq f(S,S') - |V-S-S'| \qquad V-S-S' \subseteq Q$$

$$x \in \{0,1\}^E$$

When does this have properties similar to Edge-LP?

Extending Jain's Algorithm to Vertex Connectivity : IP formulation

Stoer '92

Vertex disjoint paths from S to S' use either

- edge from S to S'
- vertex in $V-S-S'$

Menger

$f(S,S')$ max requirement from S to S'

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minimize $c^T x$

$$x_{E-F}(S,S') \geq f(S,S') - |V-S-S'| - |\delta_F(S,S')|$$

(for iterative problem)

$$x \in \{0,1\}^{E-F}$$

When does this have properties similar to Edge-LP?

Main Results

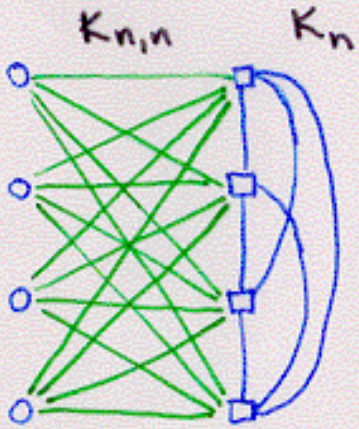
$$k = \max_{i,j} r_{ij}$$

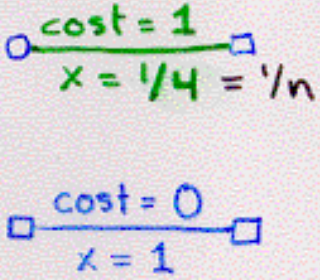
☹ There is an instance of Vertex Conn. with an optimal basic solution to Vertex-LP with $\max_{e \in E-F} x(e) = \frac{1}{k}$

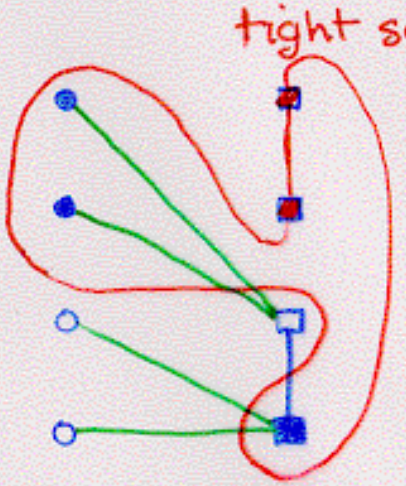
☺ If $r_{ij} \in \{0,1,2\}$, then every basic feasible solution to Vertex-LP has an edge $e \in E-F$ with $x(e) \geq 1/2$.

☺ For any $F \subset E$ and any basic solution x to the residual Element-LP, there is an edge $e \in E-F$ with $x(e) \geq 1/2$.

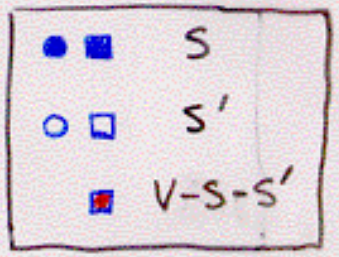
Basic Solution for Vertex LP with small maximum $x(e)$





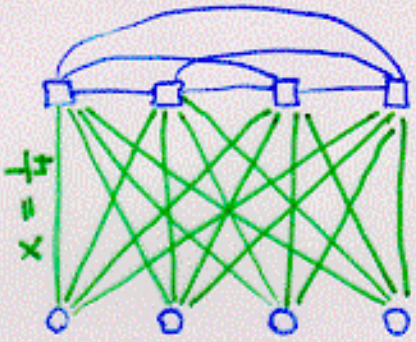


$r_i = 1$ $\quad r_j = 4 = n$ $\quad r_{ij} = \min\{r_i, r_j\}$



Basis of tight sets is highly crossing,

Largest fraction is $\frac{1}{\max r_{ij}} = \frac{1}{n}$



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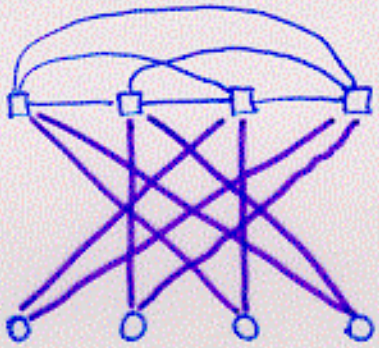

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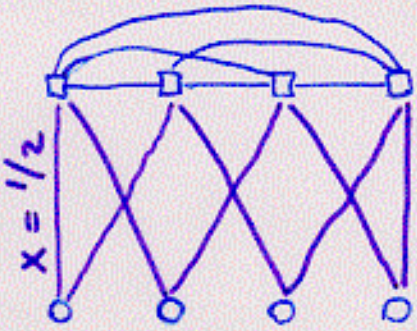

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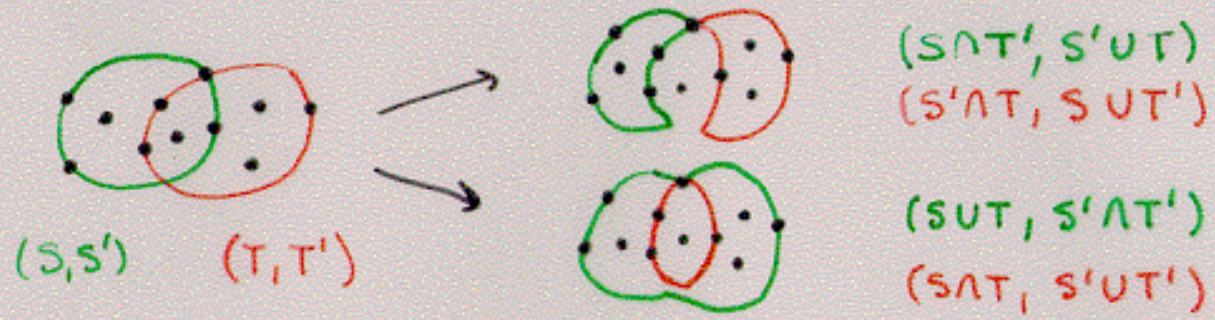
Extending Jain's Algorithm: Uncrossing Set Pairs



two-submodular

$$f(S,S') + f(T,T') \geq^{①} f(S\cap T', S'\cup T) + f(S'\cap T, S\cup T')$$
$$\geq^{②} f(S\cup T, S'\cap T') + f(S\cap T, S'\cup T')$$

↑

two-supermodular $\leq$

weakly two supermodular

$$f(S,S') + f(T,T') \leq \max\{①, ②\}$$

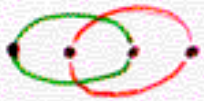
Lemma

- $|\delta_F(S,S')|$ and $x_F(S,S')$ are two-submodular
- (weakly two-supermod) − (two-submod) = weakly two-supermod
- can uncross if f is weakly two-supermodular

Which vtx connectivity problems have weakly two-supermodular RHS ?

✗ Example ⇒ not general vertex connectivity

✗ not k-uniform connectivity:



intersection & differences are empty

✗ not $\{0,1,2\}$ connectivity:

$r_{ij} = \min\{r_i, r_j\}$

• = demand 2 vertex

$f(S \cap T', S' \cup T) = f(S \cup T, S' \cap T') = 0$

✓ general element connectivity R = terminals Q = steiner nodes

$$f^{el}(S,S') = \max_{\substack{i \in S \cap R \\ j \in R \setminus S = R \cap S'}} r_{ij} \qquad S \subset V, \quad V - S - S' \subseteq Q$$

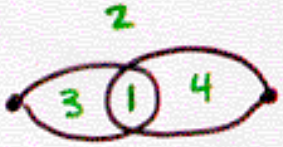
For $\{0,1,2\}$ vertex connectivity

By swapping S with S' and T with T' can assume

$$S \subseteq T \cup T' \quad \text{and} \quad T \subseteq S \cup S'$$

Very weakly two-supermodular

$$f(S,S') + f(T,T') \leq \max \{ f^{1}(S \cap T, S' \cup T') + f^{2}(S \cup T, S' \cap T'),$$
$$f^{3}(S \cap T', S' \cup T) + f^{4}(S' \cap T, S \cup T'),$$
$$f^{1}(S \cap T, S' \cup T'),$$
or perhaps swapping $(S,S') \nleftrightarrow (T,T')$
$$f^{1}(S \cap T, S' \cup T') + f^{4}(S' \cap T, S \cup T') \}$$

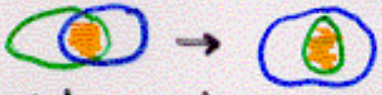


Uncrossing Lemma for $\{0,1,2\}$ Vertex Connectivity

(S,S') and (T,T') tight $\Rightarrow$ (modulo swapping $S \leftrightarrow S'$ or $T \leftrightarrow T'$) one of the following holds:

1) $(S \cap T, S' \cup T')$ and $(S \cup T, S' \cap T')$ tight and

$$\chi_x(S,S') + \chi_x(T,T') = \chi_x(S \cap T, S' \cup T') + \chi_x(S \cup T, S' \cap T')$$



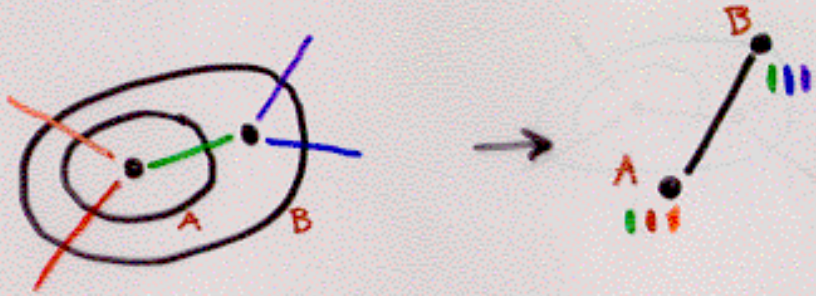
2) $(S \cap T', S' \cup T)$ and $(S' \cap T, S \cup T')$ tight and

$$\chi_x(S,S') + \chi_x(T,T') = \chi_x(S \cap T', S' \cup T) + \chi_x(S' \cap T, S \cup T')$$

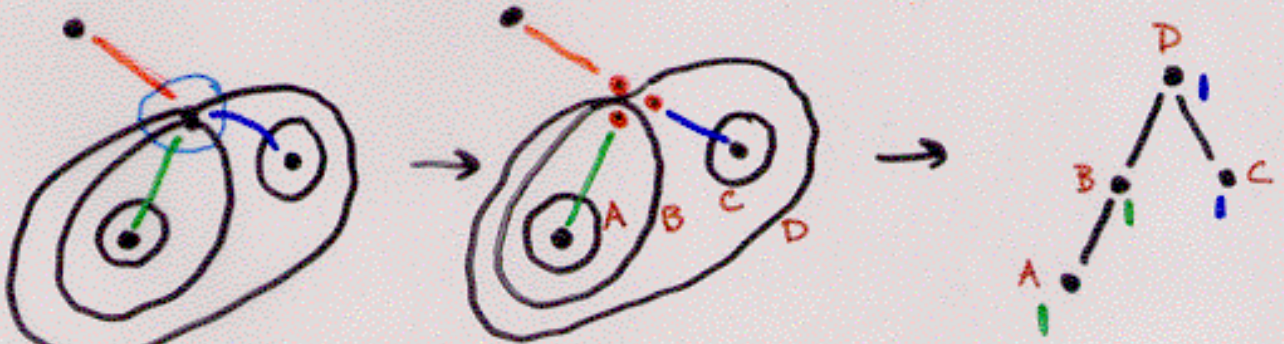
3) $(S \cap T, S' \cup T')$ and $(S' \cap T, S \cup T')$ tight and

$$2\,\chi_x(S,S') + \chi_x(T,T') = \chi_x(S \cap T, S' \cup T') + \chi_x(S' \cap T, S \cup T')$$

For edge connectivity, initial assignment of endpoints to sets is clear:



For element connectivity, endpoints corresponding to the same vertex can be assigned to different sets.



Conclusion

Simple iterative rounding yields

- 2-approx. for $\{0,1,2\}$ vertex connectivity
- 2-approx. for general element connectivity

$\Downarrow$ ZNI

d-approx. for hyperedge connectivity

Also
- $O(\frac{\sqrt{|V|}}{\varepsilon})$ approx for uniform vertex conn. CVV
- 4-approx for directed uniform edge conn. MT

Open

- Integrality gap for vertex connectivity