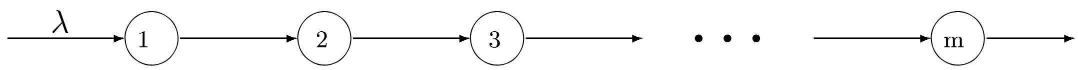
Travel Times and Flows in Stochastic Networks

- Tandem Network (Motivating Example)
- Poisson Functionals of Markov Processes
- Palm Probabilities of Markov Processes
- MUSTA for Jackson Networks
- Travel Times on Overtake-Free Route

Tandem Network



Assumptions:

- Poisson arrivals with rate λ.
- Single-server nodes with exponential services rate μ_j.

Properties of Network in Equilibrium

- Departures at each node form Poisson process with rate λ. (Burke-Reich)
- Moving units see time average (MUSTA) (Palm Probabilities)
- Sojourn time at each node j is exponential with rate $\mu_j - \lambda$.
- Sojourn times at nodes are independent.
- The total number of units Z in network has
$$P\{Z = n\} = \prod_{j=1}^{m} (1 - \rho_j) \sum_{i=1}^{m} \rho_i^{n+m-1} \prod_{\ell \neq i} (\rho_i - \rho_\ell)^{-1},$$
when $\rho_j = \lambda/\mu_j$ are distinct.
- The sojourn time W when $Z \geq b$ has mean
$$EW = P\{Z > b\}/\lambda P\{Z = b\}.$$

Poisson Functionals of Markov Processes

Consider ergodic Markov process $\{X_t : t \in \mathbb{R}\}$ on $\mathbb{E}$ with transition rates $q(x,y)$, stationary distribution π, and transition times

$$\ldots < \tau_{-2} < \tau_{-1} < \tau_0 \leq 0 < \tau_1 < \tau_2 < \ldots$$

A $\mathcal{T}$-*transition* happens at τ_n when

$$(X_{\tau_n -}, X_{\tau_n}) \in \mathcal{T} \subset \mathbb{E}^2.$$

Consider point process

$$N(A) \;=\; \#\text{ of } \mathcal{T}\text{-transitions in time set } A \subset \mathbb{R}.$$

When is N a Poisson process?

By Levy formula, we know

$$EN(A) = \int_A E\alpha(X_t)\,dt,$$

where

$$\alpha(x) = \sum_y q(x,y)1((x,y) \in \mathcal{T}).$$

This is "conditional intensity" of N given X_t

$$E[N(t,t+dt]|X_t] = \alpha(X_t)\,dt.$$

Characterization of Poisson Functionals

Define

$$N_+ \perp X_- \;\equiv \textit{future of } N \textit{ independent of past of } X$$

when

$$\{N(A) : A \subset (t, \infty)\} \perp \{X_s : s \le t\}, \quad t \in \mathbb{R}.$$

Theorem 1 *The N is Poisson process with rate a and $N_+ \perp X_-$ if and only if*

$$\alpha(x) \equiv \sum_y q(x,y) 1((x,y) \in \mathcal{T}) = a, \quad x \in \mathbb{E}. \tag{1}$$

Proof by Watanabe martingale characterization of Poisson processes.

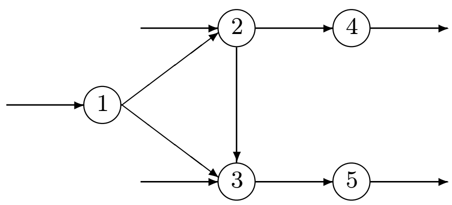
Theorem 2 *Suppose X_t is stationary. The N is Poisson process with rate a and $N_- \perp X_+$ if and only if*

$$\bar{\alpha}(x) = \pi(x)^{-1} \sum_y \pi(y) q(y,x) 1((y,x) \in \mathcal{T}) = a, \quad x \in \mathbb{E}.$$

Proof by time reversibility and preceding theorem.

Results extend to multivariate compound Poisson processes.

Examples of Poisson Functionals



Open Acyclic Jackson Network

Consider

$$N_{jk}(a,b]) \;=\; \#\text{ units moving from } j \text{ to } k \text{ in } (a,b].$$

This is Poisson process with rate $w_j \lambda_{jk}$.

Also,

$$N_{12} \perp N_{13}, \qquad N_{23} \perp N_{24}, \qquad N_{40} \perp N_{50},$$

$$N_{12} \not\perp N_{24}, \qquad N_{23} \not\perp N_{50}.$$

Palm Probabilities for M/M/1 System

Suppose X_t is number of units in $M/M/1$ queueing system with arrival rate λ and service rate μ. In equilibrium, the "conditioinal" probability that an arrival at time 0 sees x units in the system is

$$\begin{aligned} P_N\{X_0 = x+1\} &= \lim_{s\uparrow 0} P\{X_0 = x+1 | X_0 = X_s + 1\} \\ &= \pi(x). \end{aligned}$$

Poisson arrivals see time averages (PASTA).

The P_N is the Palm probability of the point process N of arrivals to the system, where there is an arrival at time 0 with probability 1.

Furthermore, if W is the sojourn time of a unit arriving at time 0, then

$$\begin{aligned} P_N\{W \le t\} &= \lim_{s\uparrow 0} P\{W \le t | X_0 = X_s + 1\} \\ &= 1 - e^{-(\mu-\lambda)t}. \end{aligned}$$

Is there a general formula for Palm probabilities of stationary Markov processes?

Palm Probabilities for Markov Processes

Consider stationary Markov process $\{X_t : t \in \mathbb{R}\}$ on $\mathbb{E}$ with transition rates $q(x, y)$, stationary distribution π, and transition times

$$\ldots < \tau_{-2} < \tau_{-1} < \tau_0 \leq 0 < \tau_1 < \tau_2 < \ldots$$

Suppose $\mathcal{T}$ is a subset of sample paths of X.
A $\mathcal{T}$*-transition* occurs at τ_n if

$$S_{\tau_n} X = \{X_{t+\tau_n} : t \in \mathbb{R}\} \in \mathcal{T}.$$

Consider point process

$$N(A) \ = \ \text{number of } \mathcal{T}\text{-transitions of } X \text{ in time set } A \subset \mathbb{R}.$$

The N is a stationary point process (its distribution is invariant under time shifts), and its *intensity* is

$$\begin{aligned} \lambda_{\mathcal{T}} &= EN(0,1] \\ &= \sum_x \pi(x) \sum_{y \neq x} q(x,y) P\{S_{\tau_1} X \in \mathcal{T} | X_{\tau_0} = x, X_{\tau_1} = y\}. \end{aligned}$$

The *Palm probability* P_N of the stationary Markov process X given that a $\mathcal{T}$-transition occurs at time 0 is

$$P_N\{X \in \mathcal{T}'\} = \lambda_{\mathcal{T}'}/\lambda_{\mathcal{T}}, \quad \mathcal{T}' \subset \mathcal{T}.$$

Clearly $P_N\{X \in \mathcal{T}\} = 1$: a $\mathcal{T}$-transition "occurs at 0".

MUSTA for Jackson Processes

Suppose X_t is a stationary Jackson network process. Consider a $\mathcal{T}$-transition in which one unit moves from node j to node k. Let $\tilde{X}_t$ denote the vector of the unmoved units at the transition.
If the network is open with unlimited capacity, then

$$P_N\{\tilde{X}_0 = x\} = \pi(x).$$

If the network is closed with ν units (or open with capacity ν), then

$$P_N\{\tilde{X}_0 = x\} = \pi_{\nu-1}(x),$$

the distribution of the network with $\nu - 1$ units.

Travel Times on Overtake-Free Routes in a Jackson Network

Suppose X_t is stationary Jackson network process. Consider an *overtake–free* route $r = (r_1, \ldots, r_\ell)$:

- Each node j is single server with rate μ_j and FIFO discipline.
- Routing is such that a unit cannot overtake another one.

Let P_{r_1} denote the Palm probability of a network transition in which a unit enters node r_1 at time 0 and traverses the route r.

Let $W_{r_1}, \ldots, W_{r_\ell}$ denote sojourn times at nodes on the route for that unit.

Theorem 3 (Open Network, Unlimited)
The $W_{r_1}, \ldots, W_{r_\ell}$ *"under the Palm probability* P_{r_1}*" are independent exponential random variables with rates* $\mu_{r_1} - w_{r_1}, \ldots, \mu_{r_\ell} - w_{r_\ell}$.

Travel Times on Overtake-Free Routes in a Jackson Network

Theorem 4 (Closed Network with ν Units)
For $t_1, \ldots, t_\ell$ in $\mathbb{R}$,

$$P_{r_1}\{W_{r_1} \le t_1, \ldots, W_{r_\ell} \le t_\ell\} = \sum_{x \in \mathbb{E}'} \pi_{\nu-1}(x) \prod_{i=1}^{\ell} F(t_i | \mu_{r_i}, x_{r_i} + 1).$$

Here $F(t|\mu, n)$ is the Erlang distribution with parameters μ and n. Its density is

$$\frac{dF(t|\mu, n)}{dt} = \mu(\mu t)^{n-1} e^{-\mu t}/(n-1)!, \quad t \ge 0.$$

Same theorem applies to a ν-capacity open network.