Equilibrium Behavior of Stochastic Networks

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Notation

Consider Markov network process $\{X(t) : t \geq 0\}$ with states

$$x = (x_1, \ldots, x_m) = \text{ numbers of units at nodes.}$$

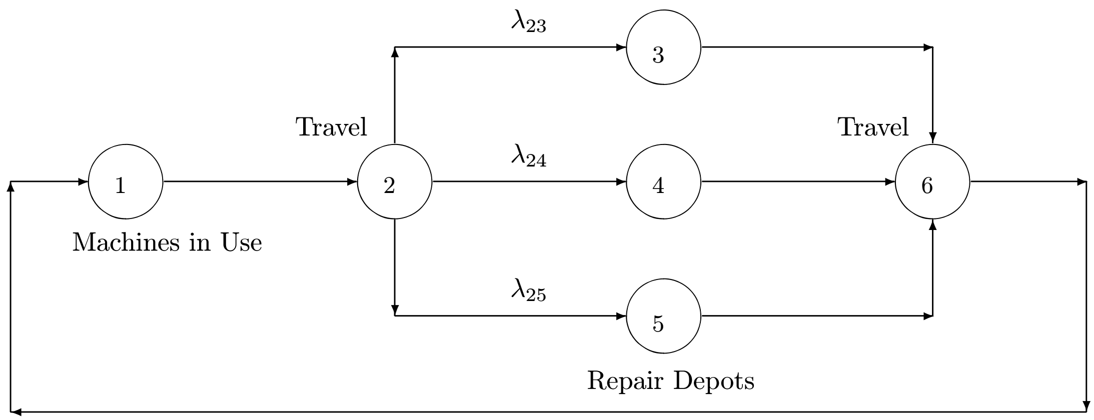
Dynamics are determined by *transition rates*

$$q(x, y) = \lim_{t \downarrow 0} t^{-1} P\{X(t) = y | X(t) = x\}.$$

Stationary distribution π is solution to

$$\pi(x) \sum_y q(x, y) = \sum_y \pi(y) q(y, x).$$

Production –Maintenance Network



Node 1: s_1 Machine Stations

Nodes 2, 6 : Delay Nodes (Infinite Servers)

Nodes 3, 4, 5: Single Servers

μ_j is service rate at node j

$\sum_{j=1}^{6} x_j = \nu$ number of machines in network

Stationary distribution

$$\pi(x) = c\frac{1}{x_2!x_6!}\prod_{n=1}^{x_1}\frac{1}{\min\{n, s_1\}}\prod_{j=1}^{6}(w_j/\mu_j)^{x_j},$$

where

$$w_j = \lambda_{2j}, \quad j = 3, 4, 5, \qquad\qquad w_j = 1 \text{ otherwise.}$$

Closed Jackson Network

Transition rates are

$$q(x, x - e_j + e_k) = \lambda_{jk}\phi_j(x_j).$$

Then

$$\pi(x) = c\Phi(x) \prod_{j=1}^{m} w_j^{x_j},$$

where

$$\Phi(x) = \prod_{j=1}^{m} \prod_{n=1}^{x_j} \phi_j(n)^{-1},$$

and w_j satisfy *traffic equations*

$$w_j \sum_k \lambda_{jk} = \sum_k w_k \lambda_{kj}.$$

Other Variations: Open network with unlimited capacity (or limited capacity)

Performance Parameters

Recall

$$\pi(x) = c \prod_{j=1}^{m} w_j^{x_j} \prod_{r=1}^{x_j} \phi_j(r)^{-1} = c \prod_{j=1}^{m} f_j(x_j), \qquad \sum_j x_j = \nu.$$

Normalization constant

$$c^{-1} = \sum_{\sum_j x_j = \nu} \prod_{j=1}^{m} f_j(x_j) = f_1 \star \ldots \star f_m(\nu).$$

$$f \star g(n) = \sum_{i=0}^{n} f(i) g(n-i), \quad n \geq 0.$$

Marginal Distributions

$$P\{n_1 \text{ in } J_1, \ldots, n_\ell \text{ in } J_\ell\} = c \prod_{i=1}^{\ell} f_{J_i}(n_i), \qquad n_1 + \ldots + n_\ell = \nu,$$

where f_J is convolution of $f_j, j \in J$.

Throughputs

Arc $j \to k$:

$$\begin{aligned} \rho_{jk} &\equiv E(\#\text{ movements } j \to k \text{ in } [0,1]) = \sum_x \pi(x) q(x, T_{jk}x) \\ &= c_\nu c_{\nu-1}^{-1} w_j \lambda_{jk}. \end{aligned}$$

Sector $J \to K$: $\rho_{JK} = \sum_{j \in J} \sum_{k \in K} \rho_{jk}$.

Node or Sector: $\lambda_j = \sum_{j \neq k} \rho_{jk}, \qquad \lambda_J = \rho_{JJ^c}$.

Sojourn Times via Little Law

Consider average sojourn (waiting) time

$$W_J = \lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} W_i(J) \quad \text{w.p.1},$$

where $W_i(J)$ is sojourn time of ith arrival.

Theorem 1 *The average waiting time W_J exits and $L_J = \lambda_J W_J$, where L_J is average number of units in J.*

Little Law for Markovian System

Suppose number of units in system is

$$X_t = h(Y_t),$$

where Y_t is ergodic Markov process.
Assume existence of arrival rate

$$\lambda = \lim_{t\to\infty} t^{-1} N(t).$$

Key idea

Assume $X_t = 0$ is possible. At these times

$$\int_0^t X_s\, ds = \sum_{n=1}^{N(t)} W_n.$$

Furthermore

$$t^{-1} \int_0^t X_s\, ds = (t^{-1}N(t))N(t)^{-1} \sum_{n=1}^{N(t)} W_n + o(1).$$

Therefore, as $t \to \infty$

$$L = \lambda W.$$

Computations

Recall

$$\pi(x;n) = c\prod_{j=1}^{m} w_j^{x_j}\prod_{r=1}^{x_j}\phi_j(r)^{-1} = c\prod_{j=1}^{m} f_j(x_j), \quad \sum_j x_j = n.$$

Define

$$\alpha_j = \sum_{i\neq j} w_i\lambda_{ij}, \quad h_j(k) = \alpha_j^{-1} f_j(k) f_j(k-1)^{-1}.$$

Proposition 2 *For each $n = 1, 2, \ldots, \nu$ and j,*

$$\begin{aligned}
W_j(n) &= \sum_{k=1}^{n} k h_j(k)\pi_j(k-1; n-1),\\
\lambda_j(n) &= n\alpha_j / \sum_{J'\in\mathcal{M}} \alpha_i W_i(n),\\
L_j(n) &= \lambda_j(n) W_j(n),\\
\pi_j(k;n) &= \lambda_j(n) h_j(k)\pi_j(k-1;n-1), \quad 1 \le k \le n,\\
\pi_j(0;n) &= 1 - \sum_{k=1}^{n}\pi_j(k;n),
\end{aligned}$$

where $L_j(0) = 0$ and $\pi_j(0;0) = 1$.

Another Approach: Monte Carlo Simulation of Metropolis Markov Chain

Whittle Network Process (Closed or Open)

Suppose transition rates of X_t are

$$q(x, x - e_j + e_k) = \lambda_{jk}\phi_j(x).$$

Assume ϕ_j are Φ-*balanced*, where Φ is a positive function such that

$$\Phi(x)\phi_j(x) = \Phi(T_{jk}x)\phi_k(T_{jk}x).$$

Theorem 3 *For the Whittle process X_t,*

$$\pi(x) = c\Phi(x)\prod_{j=1}^{m} w_j^{x_j},$$

where w_j satisfy traffic equations

$$w_j \sum_k \lambda_{jk} = \sum_k w_k \lambda_{kj}.$$

Treelike Network with Load Balancing

Assume

$$\begin{aligned} q(x, T_{0k}x) &= \lambda_{0k}\phi_0(|x|) \\ q(x, T_{jk}x) &= \lambda_{jk}\phi_j(x_j)\phi_{B_j}(x(B_j)). \end{aligned}$$

The ϕ_j is “local service intensity”, and $\phi_{B_j}(x(B_j))$ is a “load-balancing intensity” for branch B_j containing j. Then

$$\pi(x) = \prod_{j=1}^{m} w_j^{x_j} \prod_{i=1}^{|x|} \phi_0(i-1) \prod_{n=1}^{x_j} \phi_j(n)^{-1} \prod_{n'=1}^{x(B_j)} \phi_{B_j}(n')^{-1}, \quad x \in E.$$

Multiclass Whittle Processes

Consider Markov network process X_t with states

$$\begin{aligned} x &= (x_{\alpha j} : \alpha j \in M,\, j \neq 0) \in \mathbb{E} \\ x_{\alpha j} &= \text{number of } \alpha\text{-units at node } j \\ x_j \equiv \sum_\alpha x_{\alpha j} &= \text{number of units at } j. \end{aligned}$$

The transition rates are

$$q(x, T_{\alpha j,\beta k}x) = \lambda_{\alpha j,\beta k}\phi_{\alpha j}(x).$$

Network may be closed, open, or closed-open combination.

The $\phi_{\alpha j}$ are Φ-*balanced*:

$$\Phi(x)\phi_{\alpha j}(x) = \Phi(T_{\alpha j,\beta k}x)\phi_{\beta k}(T_{\alpha j,\beta k}x).$$

Theorem 4 *For the multiclass Whittle network process,*

$$\pi(x) = c\Phi(x) \prod_{\alpha j \in M} w_{\alpha j}^{x_{\alpha j}}, \quad x \in \mathbb{E},$$

where

$$w_{\alpha j} \sum_{\beta k \in M} \lambda_{\alpha j,\beta k} = \sum_{\beta k \in M} w_{\beta k}\lambda_{\beta k,\alpha j}, \quad \alpha j \in M.$$

Service Rates Proportional to Local Populations

$$\phi_{\alpha j}(x) = \frac{x_{\alpha j}}{x_j}\phi_j(x_1, \ldots, x_m).$$

Then

$$\Phi(x) = \tilde{\Phi}(x_1, \ldots, x_m) \prod_{j=1}^m x_j! \prod_\alpha \frac{1}{x_{\alpha j}!}.$$

Kelly Networks: Route-Dependent Services

Consider an open network where units arrive by a Poisson process with rate $\lambda(r)$ and traverse route $r = (r_1, \ldots, r_\ell)$. Arrivals on different routes are independent. A unit traversing route r at stage s (at node r_x) is a *rs-unit*.

The Markov network process X_t has states

$$\begin{aligned} x &= (x_{rs} : rs \in M) \\ x_{rs} &= \text{number of } rs\text{-units in the network at node } r_s. \end{aligned}$$

The transition rates are

$$\begin{aligned} q(x, x + e_{r1}) &= \lambda(r) \\ q(x, x - e_{rs} + e_{r(s+1)}) &= \phi_{rs}(x). \end{aligned}$$

Assume ϕ_{rs} are Φ-balanced.

Corollary 5 *For the Kelly network process,*

$$\pi(x) = c\Phi(x) \prod_r \lambda(r)^{x_r},$$

where x_r is number of units on route r.

Proof. This is multiclass Whittle network with traffic equations

$$w_{r1} = \lambda(r), \quad w_{rs} = w_{r(s-1)}, \quad s = 2, \ldots, \ell.$$

The solution is $w_{rs} = \lambda(r)$.

Birth-Death Processes

Consider Markov jump process X_t on $\mathbb{E}$ with rates $q(x,y)$. It is *reversible* with respect to a positive π if

$$\pi(x)q(x,y) = \pi(y)q(y,x), \quad x,y \in \mathbb{E}.$$

This *detailed balance* implies the total balance

$$\pi(x)\sum_y q(x,y) = \sum_y \pi(y)q(y,x), \quad x \in \mathbb{E}.$$

Hence π is invariant measure (or stationary distribution).

Classical Birth-Death Process. With birth rates λ_x and death rates μ_x has

$$q(x,y) = \lambda_x 1(y = x+1) + \mu_x 1(y = x-1), \quad x,y \in \mathbb{R}_+. \tag{1}$$

This is reversible w.r.t

$$\pi(x) = \prod_{k=1}^{x} \lambda_{k-1}/\mu_k. \tag{2}$$

Batch Birth-Death Process. Batch sizes $\le b$.

$$q(x,y) = \prod_{k=x}^{y-1} \lambda_k 1(y - x \le b) + \prod_{k=y+1}^{x} \mu_k 1(x - y \le b).$$

This is reversible w.r.t (2).
Key idea

$$q(x,y) = \prod_{k=1}^{n} \overline{q}(x_k, x_{k+1}),$$

where $\overline{q}$ is defined by (1) and $x = x_1, x_2, \ldots, x_n = y$ is path from x to y.

Space-Time Poisson Process

A *Poisson process* N on $\mathbb{E}$ with finite intensity measure w is a random element of $(\mathbb{M}, \mathcal{M})$ such that:
• The number of points $N(A)$ in any $A \in \mathcal{E}$ has a Poisson distribution with mean $w(A)$.
• N has independent increments.

The Poisson *distribution of* N is

$$\begin{aligned}
\pi_w(C) &\equiv P\{N \in C\} \\
&= 1(0 \in C) + e^{-w(\mathbb{E})} \sum_{n=1}^{\infty} \int_{\mathbb{E}^n} \frac{1}{n!} w(dx_1) \cdots w(dx_n) 1(\sum_{k=1}^{n} \delta_{x_k} \in C).
\end{aligned}$$

Space-Time Poisson process is Poisson process M on $\mathbb{R}_+ \times \mathbb{E}$. It is time homogeneous with *spatial intensity* λ if

$$E[M((a,b] \times A)] = (b-a)\lambda(A).$$

It is *Space-Time Customer Arrival Process* if

$$M((a,b] \times A) \ = \text{number of arrivals in } A \text{ during } (a,b].$$

Arrivals into disjoint $A_1, \ldots, A_n$ form independent homogeneous Poisson processes with rates $\lambda(A_i)$, for $1 \leq i \leq n$.

More Birth-Death Processes

Multi-variate Batch Birth-Death Process

.

$$q(\mathbf{x}, d\mathbf{y}) = [e^{-u(\mathbf{x},\mathbf{y}]} 1(\mathbf{x} < \mathbf{y}) + e^{-v(\mathbf{y},\mathbf{x}]} 1(\mathbf{y} < \mathbf{x})]\psi(d\mathbf{y}),$$

where u, v and ψ are measures on $\mathbb{R}_+^m$. This is reversible w.r.t

$$\pi(d\mathbf{x}) = e^{-[u(0,\mathbf{x}]-v(0,\mathbf{x}]]}\psi(d\mathbf{x}).$$

Simple Spatial Birth-Death Process

.

Units arrive into system by time homogeneous space-time Poisson process with spatial intensity λ. Each unit spends exponential time with rate $\gamma(x)$ then exits.

Consider

$$X_t(A) \;=\; \text{number of units in } A \text{ at time } t.$$

The X_t is reversible w.r.t. the Poisson distribution π_w with intensity

$$w(dx) = \gamma(x)^{-1}\lambda(0, dx).$$

When is it ergodic?

$$\lim_{t\to\infty} \sup_C |P\{X_t \in C\} - \pi(C)| = 0.$$

The X_t is ergodic if and only if $w(\mathbb{E}) < 1/2$.

Space-Time Poisson-Markov Process

• Space-time Poisson arrivals with expectation $(b-a)\lambda(0,A)$.

• Units move independently according to Markov kernel $\lambda(x,dy)$ until they exit network. Assume $\lambda(x,A)$ is irreducible with stationary distribution w on $\overline{\mathbb{E}}$:

$$\int_A w(dx)\lambda(x,\overline{\mathbb{E}}) = \int_{\overline{\mathbb{E}}} w(dy)\lambda(y,A), \quad A \in \overline{\mathbb{E}}.$$

Then $X_t(A) =$ number of units in A, is measure-valued Markov process with transition rate kernel

$$q(\mu,\mathcal{C}) = \sum_{x\in\mathbb{E}} \int_{\mathbb{E}} \lambda(x,dy) 1(\mu - \delta_x + \delta_y \in \mathcal{C}), \quad \mu \in \mathbb{M},\ \mathcal{C} \in \mathcal{M}.$$

Theorem 6 *The* **space-time Poisson-Markov process** X_t *has a Poisson stationary distribution with intensity* w*. The process is ergodic if and only if* $w(\mathbb{E}) < 1/2$*.*

Reversible Spatial Queueing Process

Consider system as above with

- Space-time Poisson arrivals with expectation $(b-a)\lambda(0, A)$

- Units routed by Markov kernel $\lambda(x, A)$.
- $X_t(A)$ = number of customers (units) in A.

Now, movements of units depend on congestion as in a queueing network. The transition rates are

$$q(\mu, C) = \sum_{x\in\mathbb{E}} \int_{\mathbb{E}} r(\mu, T_{xy}\mu)\lambda(x, dy)1(T_{xy}\mu \in C)M.$$

where

$$T_{xy}\mu \equiv \mu - \delta_x + \delta_y \in \mathbb{M},$$

(a unit at x moves to the location y).

Here $r(\mu, T_{xy}\mu)$ is a *departure-attraction* rate of departure from x and be attracted to y.

For instance

$$r(\mu, T_{xy}\mu) = \frac{u(\mu - \delta_x)}{u(\mu)} \frac{v(\mu + \delta_y)}{v(\mu)}.$$

Reversible Spatial Queueing Process

Theorem 7 *Suppose*
- $\lambda(x, A)$ *is reversible w.r.t.* w.
- *There is positive function* f *such that* $f(\mu)r(\mu, \eta)$ *is symmetric.*

Then X_t *is reversible w.r.t.*

$$\pi(d\mu) = f(\mu)\pi_w(d\mu),$$

where π_w *is the Poisson distribution with intensity* w.

Key idea

$$q(\mu, d\eta) = r(\mu, \eta)\overline{q}(\mu, d\eta),$$

where $\overline{q}(\mu, d\eta)$ is Space-Time Poisson-Markov process with Poisson stationary distribution π_w.

For example, $f(\mu) = u(\mu)v(\mu)$ when

$$r(\mu, T_{xy}\mu) = \frac{u(\mu - \delta_x)}{u(\mu)}\frac{v(\mu + \delta_y)}{v(\mu)}.$$

Jackson Network with Occasional Clearing

Consider Jackson network, where all units from a node may occassionally be deleted.

Transitions	**Rates**	
$x \to x - e_j + e_k$	$\lambda_{jk}\phi_j(x_j)$	
$x \to x - x_j e_j$	$\lambda_{\infty j}\phi_j(x_j)$	(Clearing)

Then

$$\pi(x) = c\Phi(x) \prod_{j=1}^{m} w_j^{x_j},$$

where $0 < w_j < 1$ satisfy *traffic equations*

$$w_j[\sum_k \lambda_{jk} + \lambda_{\infty j}(1 - w_j)^{-1}] = \sum_k w_k \lambda_{kj}.$$

This equation is

$$w_j \tilde{\mu}_j = \tilde{\lambda}_j.$$

- $\tilde{\mu}_j$ is departure rate when node is not empty.
- $\tilde{\lambda}_j$ is effective arrival rate.
- $\tilde{\lambda}_j/\tilde{\mu}_j$ is traffic intensity.

Basic Types of Stochastic Network Processes

- Jackson Networks– Whittle Networks
- Multiclass Jackson and Whittle Networks
  Kelly and BCMP Networks
- Quasi-reversible Networks (Product-form distributions)
- Reversible Networks (Batch movements, dependent services, blocking)
- Networks with String Transitions (Batch movements, multiple activities at transition)
- Spatial Queueing Systems (Poisson-Markov, Birth-Death)
- Brownian Motion Networks (Diffusion approximations for heavy traffic, fluid models)