Lecture II

The Ordered Set Mirror Descent Algorithm and Its Use in Medical Imaging

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Extended Abstract

We develop and implement a new algorithm for solving an extremely large-scale convex optimization problem arising in Nuclear Medicine—that of reconstructing 3D medical images from data acquired by Positron Emission Tomography (PET) scanners.

PET is an imaging technique which helps in the diagnosis of diseases and allows a better understanding of the functioning of the human body. A PET scan involves the use of a small amount of radioactive material which has the property of emitting positrons. The image reconstruction problem is to determine the unknown distribution of the radioactive tracer within the body, given a list of detected events (projections). A mathematical model of the problem, based on the Maximum Likelihood principle, is posed as a problem of minimizing a convex function of several million variables over the standard simplex. This large-scale problem should be solved quickly in order to enable medical diagnosis in a realistic time.

Currently, the 3D PET image reconstruction problem is solved mainly by using analytical methods, based on inverse transform, which suffer from several drawbacks. In recent years, the study of iterative (statistical) reconstruction methods has received much attention. However, to the best of our knowledge, there is only one iterative algorithm, namely, OSEM (Ordered Subsets Expectation Maximization), capable of solving the 3D tomography reconstruction problems of realistic size. In this research, we adapt Convex Optimization methodology to solve the real-world optimization problem arising from PET.

The specific characteristics of the reconstruction problem rule out most advanced optimization methods, and as a result, we focus on gradient-type methods. Specifically,

we develop an accelerated version of the Mirror Descent (MD) method of Nemirovski and Yudin (1978). The MD method is a non-Euclidean version of the usual Subgradient Descent (SD) method and can be adjusted to the "geometry" of the problem to be solved. In order to motivate our choice of MD method for solving the PET image reconstruction problem, we prove optimality of the MD method for large-scale convex minimization over the standard simplex. In particular, we state that for such problems, the efficiency estimate (an upper bound on the error in the objective function value) of the MD method can be better than the efficiency estimate of the SD method by a factor of order of $(n/\ln n)^{1/2}$, n being the dimension of the problem, which for large-scale problems can make a huge difference.

Our acceleration of the MD method is based on the *Incremental Gradient* idea, also known as the *Ordered Subsets* methodology, which was proved to be extremely useful in a variety of applications. In this methodology, the data vector is divided into sub-vectors ("ordered sets"); the algorithm then processes the data incrementally, one subset of the data at a time, and cycles through all subsets, to complete one full iteration. The main theoretical result of the research is the proof of convergence and efficiency estimate for the Ordered Subsets Mirror Descent (OSMD) method. In comparison, OSEM does not have a "solid" theoretical justification, and it is known that for certain data or certain choices of the subsets it may not converge.

In order to test the practical behavior of the OSMD method, we perform several tests on simulated and real clinical data. Our main conclusion from these experiments is that OSMD is a reliable and efficient algorithm for PET reconstruction. MD achieves better results than the Subgradient Descent method, and OSMD indeed accelerates the performance as compared to the MD method. In addition, the quality of our reconstructions compares favourably to the one given by the commercially-used analytic algorithms. As compared to the "golden standard" of the new generation of 3D imaging algorithms—the OSEM algorithm, OSMD, is highly competitive both in image quality and computational effort.

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