Lecture 1

Robust Optimization — Methodology and Applications

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Extended Abstract

A generic mathematical programming problem is of the form

$$\min_{x_0 \in \mathbf{R}, x \in \mathbf{R}^n} \{x_0 : f_0(x,\zeta) - x_0 \le 0,\ f_i(x,\zeta) \le\ i = 1, ..., m\} \qquad (\mathrm{P}[\xi])$$

where x is the design vector, the functions f_0 (the objective function) and $f_1, ..., f_m$ are *structural elements* of the problem, and ζ is a *data* specifying a particular problem instance.

For real-world optimization problems, the "decision environment" is often characterized by the following facts:

F.1. The data is uncertain/inexact;

F.2. The optimal solution, even if computed very accurately, may be difficult to implement accurately;

F.3. The constraints *must* remain feasible for *all* meaningful realizations of the data;

F.4. Problems are *large-scale* (n or/and m are large);

F.5. "Bad" optimal solutions (those which become severely infeasible in the face of even relatively small changes in the nominal data) are *not* uncommon.

F.1 (and in fact **F.2** as well) imply that in many cases we deal with *uncertain optimization problems* – families of the usual ("certain") optimization problems

$$\{(\mathrm{P}[\zeta]) \mid \zeta \in \mathcal{U}\}, \qquad (1)$$

where $\mathcal{U}$ is an "uncertainty set" in the space of the data.

Fact **F.3** implies that a meaningful candidate solution (x_0, x) of an uncertain problem (1) is required to satisfy the semi-infinite system of constraints

$$f_0(x,\zeta) \le x_0,\ f_i(x,\zeta) \le 0\ i = 1, ..., m \quad \forall(\zeta \in \mathcal{U}). \qquad (2)$$

Fact **F.4** on the other hand imposes a severe requirement of being able to process efficiently the semi-infinite system of constraints (2) for large-scale problems.

Robust Optimization is a modeling methodology, combined with a suite of computational tools, which is aimed at accomplishing the above requirements. The urgency of having such a methodology stems from fact **F.5**, whose validity will be illustrated via several engineering design problems.

In the Robust Optimization methodology, one associates with an uncertain problem (1) its *robust counterpart*, which is a usual (semi-infinite) optimization program

$$\min_{x_0,x} \{x_0 : f_0(x,\zeta) \le x_0,\ f_i(x,\zeta) \le 0,\ i = 1,...,m \quad \forall(\zeta \in \mathcal{U})\}\,; \tag{3}$$

fesible/optimal solutions of the robust counterpart are called *robust feasible/robust optimal* solutions of the uncertain problem (1).

The major challenges associated with the Robust Optimization methodology are:

C.1. When and how can we reformulate (3) as a "computationally tractable" optimization problem, or at least approximate (3) by a tractable problem.

C.2. How to specify reasonable uncertainty sets $\mathcal{U}$ in specific applications.

In the lecture, we overview the results and the potential of the Robust Optimization methodology in the most interesting cases of uncertain Linear, Conic Quadratic and Semidefinite Programming. For these cases, the instances of our uncertain optimization problems will be in the *conic form*

$$\min_{x_0\in\mathbf{R},x\in\mathbf{R}^n} \left\{x_0 : c^T x \le x_0,\ A_i x + b_i \in \mathbf{K}_i,\ i = 1,...,m,\right\}, \tag{C}$$

where for every i $\mathbf{K}_i$ is

– either a non-negative orthant $\mathbf{R}^{m_i}_+$ (linear constraints),

– or a Lorentz cone $\mathbf{L}^{m_i} = \left\{y \in \mathbf{R}^{m_i} \mid y_{m_i} \ge \sqrt{\sum\limits_{j=1}^{m_i-1} y_j^2}\right\}$ (conic quadratic constraints),

– or a semidefinite cone $\mathbf{S}^{m_i}_+$ – the cone of positive semidefinite matrices in the space $\mathbf{S}^{m_i}$ of $m_i \times m_i$ symmetric matrices (Linear Matrix Inequality constraints).

The class of problems which can be modeled in the form of (C) is extremely wide. It is also clear what is the *structure* and what is the *data* in (C) – the former is the design dimension n, the number of "conic constraints" m and the list of the cones $\mathbf{K}_1$,..., $\mathbf{K}_m$, while the latter is the collection of matrices and vectors c, $\{A_i, b_i\}_{i=1}^m$ of appropriate sizes. Thus, an uncertain problem (C) is a collection

$$\left\{\min_{x_0\in\mathbf{R},x\in\mathbf{R}^n} \left\{x_0 : \underbrace{x_0 - c^T x}_{A_0 x + b_0} \in \mathbf{K}_0 \equiv \mathbf{R}_+,\ A_i x + b_i \in \mathbf{K}_i,\ i = 1,...,m\right\} |(c, \{A_i, b_i\}_{i=1}^m) \in \mathcal{U}\right\} \tag{4}$$

of instances (C) of a common structure $(n, m, \mathbf{K}_1, ..., \mathbf{K}_m)$ and data $(c, \{A_i, b_i\}_{i=1}^m)$ varying in a given set $\mathcal{U}$. In the robust counterpart of (4), every one of the uncertain constraints $A_i x + b_i \in \mathbf{K}_i$, $i = 0, ..., m$, of (4) is replaced with its robust counterpart

$$A_i x + b_i \in \mathbf{K}_i \quad \forall (A_i, b_i) \in \mathcal{U}_i,$$

where $\mathcal{U}_i$ is the projection of the uncertainty set $\mathcal{U}$ on the space of the data of i-th constraint.

In the lecture, we consider in turn uncertain Linear, Conic Quadratic and Semidefinite programming problems; our major interest is in the relevant version of **C.1** in the case when the uncertainty set is an intersection of ellipsoids (which seems to be a general enough model from the viewpoint of **C.2**). We shall illustrate our general considerations by several examples, mainly coming from engineering; these examples will show that in many applications taking into account from the very beginning data uncertainty is a necessity which cannot be ignored, and that the Robust Optimization methodology offers an efficient way to deal with this necessity successfully.

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