Semidefinite Programming and Approximation Algorithms

Michel Goemans

Joint work with David Williamson

Combinatorial Optimization

- Class of optimization problems in which the solution space is discrete
- Decision problems in which each decision has a finite number of possibilities
- Many applications:
  - ▷ Vehicle routing (package pick-up and delivery,...)
  - ▷ Network design (design of SONETs,...)
  - ▷ VLSI design
  - ▷ Scheduling (flexible manufacturing, ...)
  - ▷ Assignment (airline crew scheduling and fleet assignment, ...)
- Combinatorial explosion
- Prototypical CO problem: the Traveling Salesman Problem (TSP)

Intractability

- Many combinatorial optimization problems are NP-complete (Cook '71 and Karp '72):

  Among the hardest. If an exact polynomial-time algorithm can be found for any of them in polynomial-time then any of them can be solved exactly in polynomial-time

  (Basic NP-complete problem: satisfiability)

- Lower expectations:
  - Either develop exact algorithms which are not guaranteed to run in polynomial-time
  - or develop polynomial-time algorithms which are not guaranteed to give an optimum solution
  - or neither...

Relaxations

- Fundamental concept in discrete optimization
- Instead of optimizing over hard feasible space, optimize over larger easy (convex) space
- If solution found belongs to original space, we are done. Optimum solution found.
- If not, can tighten relaxation, or at least have bound on optimum value.
- Classically: Linear programming relaxations

Approximation Algorithms

Polynomial-time algorithms guaranteed to deliver a solution within a certain factor of the optimum value

- Concept introduced in '70s
- For many problems, only elementary approximation algorithms known
- Recent major developments:
  - ▷ New techniques:
    - ∗ randomized rounding (Raghavan-Thompson, ...)
    - ∗ primal-dual (based on linear programming)
    - ∗ semidefinite programming
    - ∗ ...
  - ▷ Strong negative results through probabilistic checkable proofs (PCP)

    ([Feige, Goldwasser, Lovász, Safra, Szegedy], [Arora, Safra], [Arora, Lund, Motwani, Sudan, Szeg edy], [Bellare, Goldreich, Sudan], [Håstad], ...)

Example: Max E3-LIN2

- Given a system of linear equations mod 2 with 3 variables per equation

$$
\begin{aligned}
x_1 + x_3 + x_4 &\equiv 0 \pmod 2\\
x_1 + x_2 + x_5 &\equiv 1 \pmod 2\\
x_1 + x_3 + x_5 &\equiv 1 \pmod 2\\
x_1 + x_2 + x_4 &\equiv 1 \pmod 2
\end{aligned}
$$

- Easy to test if we can satisfy all equations simultaneously (Gaussian elimination)
- Either $x = 0$ or $x = 1$ satisfies at least half of the equations

  $\Longrightarrow$ trivial 2-approximation algorithm
- Håstad '99:

  For any ϵ, no $2 - \epsilon$-approximation algorithm exists unless P=NP !

Positive Semidefinite Matrices

- Symmetric matrix $A \in S_n$ is positive semidefinite ($A \succeq 0$) if $\forall x : x^T A x \geq 0$

  Equivalently,

  ▷ $\lambda_i(A) \geq 0$ for all i

  ▷ $\exists V \in R^{m \times n} : A = V^T V$ (Cholesky decomposition)

  ▷ $\exists v_i \in R^m : A_{ij} = v_i^T v_j$ (Gram matrix)

  ▷ $\exists \lambda_i \geq 0, \exists v_i$: $A = \sum_i \lambda_i v_i v_i^T$

Cone of PSD Matrices

Let $K = \{X : X \succeq 0\}$. Then

- K is a convex cone:

  $A, B \succeq 0$ and $0 \leq \lambda \leq 1$ imply that $\lambda A + (1 - \lambda) B \succeq 0$

- K is closed

- Extreme rays = rank-1 matrices

- [Fejer's theorem] K is self-polar:
  $K^* := \{X : \langle X, Y \rangle \geq 0$ for all $Y \succeq 0\} = K$

  Proof:

  $K^* \subseteq K$ since $y^T X y = \langle X, yy^T \rangle \geq 0$

  $K \subseteq K^*$ since
  $\langle X, Y \rangle = \langle X, \sum_i \lambda_i v_i v_i^T \rangle = \sum_i \lambda_i v_i^T X v_i \geq 0$

- $A \succeq B$ if $A - B \succeq 0$ (Löwner partial order)

Semidefinite Programming (SDP)

- Standard form

$$\text{Inf} \quad \langle C, X \rangle$$

subject to:

$$\langle A_i, X \rangle = b_i \qquad i = 1, \cdots, m$$
$$X \succeq 0$$

Generalizes linear programming

(impose constraints that X is diagonal)

- Infimum not necessarily attained:

$$\text{Inf} \quad x$$

$$\text{subject to:} \quad \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0$$

- Linear Matrix Inequality (LMI) formulation (cfr. control theory)

$$\text{Sup} \quad b^T y$$

subject to:

$$\sum_i y_i A_i \preceq C$$

Many applications of SDP

- Control theory
- Combinatorial optimization
- Probability, statistics
- Coding theory
- Euclidean geometry
- Optimal design
- ...

Control Theory

- [Lyapunov]: Are equivalent
  - ▷ Trajectories of $\dot{x}(t) = Ax(t) \to 0$ (stable)
  - ▷ $Re(\lambda_i(A)) < 0$
  - ▷ $\exists B = LAL^{-1} : B + B^T \succ 0$
  - ▷ $\exists P \succ 0 : A^T P + PA \prec 0$

    Linear Matrix Inequality (LMI)

$$
\begin{aligned}
(\frac{d}{dt} x^T P x &= \dot{x}^T P x + x^T P \dot{x} \\
&= x^T (A^T P + PA) x < 0)
\end{aligned}
$$

- Under uncertainty: Linear Differential Inclusion

$$\dot{x} = A(t, \epsilon) x \text{ where } A(t, \epsilon) \in \Omega$$

  Quadratic stability: $\exists$ Lyapunov funct. $x^T P x \to 0$ on all trajectories

  $\Longrightarrow \exists P \succ 0 : A_i^T P + PA_i \prec 0$ for all $A_i \in \Omega$

- See [Boyd, ElGhaoui,Feron, Balakrishnan '94]

Duality

- Duality over cones:

$$
\begin{array}{rll|c|rll}
z = \text{Inf} & c^T x & & \geq & w = \text{Sup} & b^T y \\
s.t. & a_i^T x = b_i & & & s.t. & (c - \sum_i y_i a_i) \in K^* \\
& x \in K & & & &
\end{array}
$$

$(z - w = c^T x - b^T y = (c - \sum_i y_i a_i)^T x \geq 0)$

- Theorem: Under "regularity" assumption, $z = w$.
- SDP strong duality: If strictly feasible solutions to primal and dual then

$$
\begin{array}{rl|c|rl}
\text{Min} & \langle C, X \rangle & = & \text{Max} & b^T y \\
s.t. & \langle A_i, X \rangle = b_i & & s.t. & \sum_i y_i A_i \preceq C \\
& X \succeq 0 & & &
\end{array}
$$

- Weaker condition: Strictly feasible solution to dual (resp. primal) implies no duality gap and primal (resp. dual) solution set is non-empty and bounded

Bad Behavior

Any combination of the following:

- Primal value not attained
- Dual value not attained
- Non-zero duality gap

Example:

- Primal:

$$\begin{aligned} \text{Inf} \quad & 10X_{11} \\ \text{subject to:} \quad & X_{22} = 0 \\ & X_{11} + X_{23} + X_{32} = 1 \\ & X \succeq 0 \end{aligned}$$

  Any feasible solution has $X_{11} = 1$

  Primal value of 10 is attained

- Dual: Sup y_1 subject to:

$$y_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + y_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \preceq \begin{pmatrix} 10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

  Any dual feasible solution has $y_1 \leq 0$

  Dual value of 0 is attained

Why Slater Type Condition?

Classical argument from convex analysis

- Suppose there exists strictly feasible solution to dual:
  $\exists y : S = C - \sum_i y_i A_i \succ 0$
  Let $S = U \Lambda U^T$ with Λ diagonal and $\succ 0$
- Consider the set P_v of primal feasible solutions X of value $\leq v$
  $\implies \langle X, S \rangle = \langle C, X \rangle - b^T y \leq v - b^T y$
- $S \succeq \lambda_{min}(S) I$ where $\lambda_{min} > 0$

$$\implies \langle X, S \rangle \geq \lambda_{min}(S) \langle X, I \rangle$$

$$\implies \langle X, I \rangle \leq \frac{1}{\lambda_{min}(S)} (v - b^T y)$$

  $\implies P_v$ is bounded (since such X's have bounded diagonal elements)

- Consider a sequence of primal feasible X_i such that $\langle C, X_i \rangle$ tends to infimum

  Boundedness implies that there exists subsequence of X_i's tending to primal optimum solution X

Complementary Slackness

- X and $S = C - \sum_i y_i A_i$ are primal optimum solution and dual optimum slacks if
  - ▷ feasible and
  - ▷ $\langle X, S \rangle = 0$
- For $X, S \succeq 0$, $\langle X, S \rangle = 0$ implies $XS = SX = 0$

  Proof:

$$
\begin{array}{rcl}
\langle X, S \rangle & = & \langle X, \sum_i \lambda_i v_i v_i^T \rangle = 0 \\
& \Downarrow & \\
\langle X, v_i v_i^T \rangle & = & v_i^T X v_i = 0 \\
& \Downarrow & \\
& v_i & \text{is an eigenvector of } X : X v_i = 0 \\
& \Downarrow & \\
XS & = & \sum_i \lambda_i X v_i v_i^T = 0
\end{array}
$$

Exact Feasibility Problem

- Does there exist $X \succeq 0$ such that $\langle X, A_i \rangle = b_i$ for all $i = 1, \cdots, m$?

  Complexity open
- Exists more complex SDP dual for which strong duality holds [Ramana]

  $\Longrightarrow$ if exact SDP feasibility is in NP then it is in co-NP

Sum of Square Root Problem

- Given integers $a_1, \cdots, a_n$ and integer b, is $\sum_{i=1}^n \sqrt{a_i} > b$?
- Complexity open (in the Turing machine model)
- Reformulation:

$$
\begin{array}{ll}
\text{Sup} & x_1 + \cdots + x_n \\
\text{subject to:} & \begin{pmatrix} 1 & x_1 & & & \\ x_1 & a_1 & & & \\ & & \cdots & & \\ & & & 1 & x_n \\ & & & x_n & a_n \end{pmatrix} \succeq 0
\end{array}
$$

Algorithms to Solve Approximately SDPs

- Ellipsoid algorithm: [Yudin & Nemirovskii], [Shor], [Grötschel, Lovász, Schrijver]
- Interior-point algorithms: [Nesterov-Nemirovskii, Alizadeh, Todd, Alizadeh-Haeberly-Overton, Helmberg-Rendl-Vanderbei-Wolkowicz, Monteiro, Klerk-Roos-Terlaky, Nesterov-Todd-Ye, Sturm, Anstreicher, Sheng-Potra, Kojima-Shida-Shindoh, and many more]

see http://www.zib.de/helmberg/semidef.html

Features/properties of Interior-Point Techniques for SDP

- Assumes strictly feasible solutions (strong duality)

  Given sufficiently centered initial strictly feasible solutions, can reduce duality gap by factor ϵ in $O(\sqrt{n}\log(1/\epsilon))$ iterations
- several strategies: path following, potential reduction, ...
- Barrier: $F(X) = -\ln\det X$
  - ▷ Convex
  - ▷ [Nesterov-Nemirovskii] n-self-concordant function

Features (cont'd)

- (Primal) central path: for any $\mu \geq 0$, unique minimum to

$$\begin{array}{lll} \text{Min} & \langle C, X\rangle - \mu \ln \det X & \\ \text{subject to:} & \langle A_i, X\rangle = b_i & i = 1, \cdots, m \\ & X \succeq 0 & \end{array}$$

similar for dual central path

Central path equations:

$$\langle A_i, X\rangle = b_i$$

$$\sum_i y_i A_i + S = C$$

$$XS = \mu I$$

- $XS - \mu I$ in general not symmetric $\Longrightarrow$ need (and flexibility) for symmetrization

(examples: Newton direction for $\frac{1}{2}\bar{X}\bar{S} + \frac{1}{2}\bar{S}\bar{X} = \mu I$ or for $\bar{X}^{1/2}\bar{S}\bar{X}^{1/2} = \mu I$, or $\bar{S} - \mu\bar{X}^{-1}$, or ...)

The Maximum Cut Problem

- MAXCUT: Given a weighted graph $G = (V, E)$, find cut $\delta(S)$ (edges between S and $V - S$) maximizing

$$w(\delta(S)) = \sum_{e \in \delta(S)} w_e$$

- α-approximation algorithm: produces in poly time a cut of value at least αOPT
  - ▷ [Håstad '97] NP-hard to approximate MAXCUT within 0.94117 when $w \geq 0$
  - ▷ [Sahni & Gonzalez '76]: 0.5-approximation algorithm
  - ▷ [G. & Williamson '94]: 0.878-approximation algorithm using SDP

Semidefinite Relaxation for MAX CUT

- Quadratic integer program: Let $y_i = 1$ if $i \in S$ and -1 otherwise.

  Edge $(i, j) \in \delta(S)$ if $\frac{1-y_i \cdot y_j}{2} = 1$; otherwise equal to 0.

$$(Q) \qquad \begin{array}{lll} \text{Maximize} & \displaystyle\sum_{(i,j)\in E} w_{ij} \frac{1 - y_i \cdot y_j}{2} & \\ \text{subject to:} & y_i \in \{-1, 1\} & \forall i \end{array}$$

- Let $Y = [y_i y_j]$

  Observe that

  ▷ $Y_{ii} = 1$ for all i

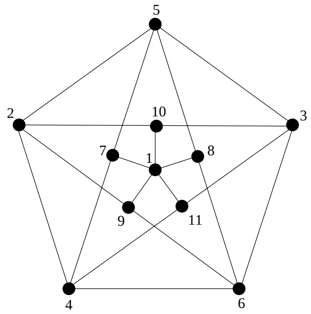
  ▷ $Y \succeq 0$

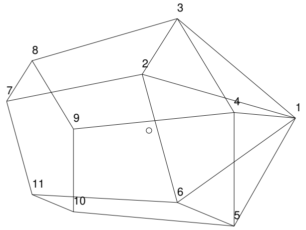
  $\Rightarrow$ Semidefinite Programming relaxation:

$$(SDP) \qquad \begin{array}{lll} \text{Maximize} & \displaystyle\sum_{(i,j)\in E} w_{ij} \frac{1 - Y_{ij}}{2} & \\ \text{subject to:} & Y_{ii} = 1 & \forall i \\ & Y \succeq 0 & \end{array}$$

GW Randomized Approximation Algorithm

1. Solve SDP relaxation $\Rightarrow Y$
2. Cholesky decomposition $\Rightarrow v_i \in R^n$ for $i \in V$
3. Let r be sampled uniformly from unit sphere in R^n
4. $S = \{i : r^T v_i \geq 0\}$





Theorem:

$$E[w(\delta(S))] \geq 0.87856 SDP \geq 0.87856 OPT$$

when $w \geq 0$ where $SDP =$ value of relaxation

Analysis

$$Pr[(i,j) \in \delta(S)] = \frac{\text{angle between } v_i \text{ and } v_j}{\pi} = \frac{\arccos(y_{ij})}{\pi}$$

$\Longrightarrow E[w(\delta(S)] = \sum_{(i,j)\in E} w_{ij} \frac{\arccos(y_{ij})}{\pi}$

compare with $SDP = \sum_{(i,j)\in E} w_{ij} \frac{1-y_{ij}}{2}$

$\Longrightarrow E[w(\delta(S))] \geq 0.878 SDP$ since $\frac{\arccos(x)}{\pi} \geq 0.878 \frac{1-x}{2}$

New formulation for MAXCUT:

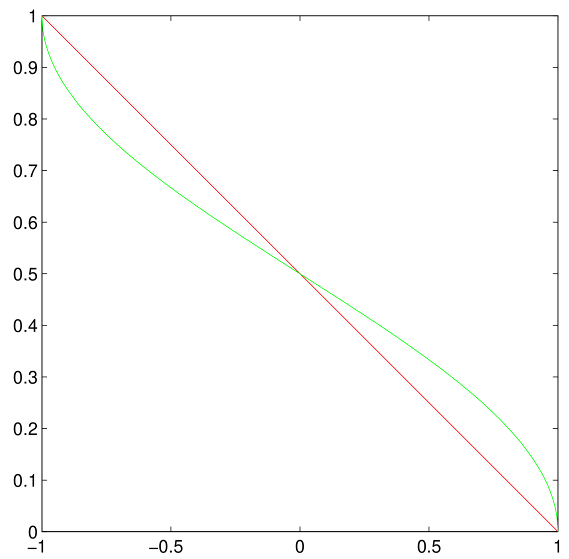
$$OPT = \text{Max} \quad \frac{1}{2} \sum_{(i,j)\in E} w_{ij} \frac{\arccos(y_{ij})}{\pi}$$

subject to: $\quad y_{ii} = 1 \qquad \forall i$

$\qquad Y = (y_{ij}) \succeq 0$

Slide 23

Ratio of 0.87856



Slide 24

More General Analysis based on [Nesterov '97]

- Assume $w : L \in K$ (convex cone)
- Then guarantee of α if can show

$$\left[\frac{2}{\pi}\arcsin(Y) - \alpha Y\right] \in K^* \text{ for all } Y : y_{ii} = 1, Y \succeq 0$$

- E.g. $K = \{L \succeq 0\}$

  $\Longrightarrow \alpha = \frac{2}{\pi}$ [Nesterov '97]

  since $\arcsin(Y) \succeq Y$ for $Y \succeq 0$:

  ▷ $\arcsin(x) = x + \frac{1}{2\cdot 3}x^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}x^5 + \cdots$

  ▷ Schur product: $A \circ B = [A_{ij}B_{ij}]$

  $A, B \succeq 0 \Longrightarrow A \circ B \succeq 0$

  In particular, $Y \succeq 0 \Longrightarrow [Y]^2, [Y]^3, \cdots \succeq 0$

  ▷ $\Longrightarrow \arcsin(Y) - Y \succeq 0$

Tightness

What is the **worst-case** ratio OPT/SDP?

- For 5-cycle: ratio $= 0.88445$
- Random hyperplane technique no better than 0.87856 [Karloff]
- If $rank(Y) \leq 2$, can show $OPT/SDP \geq 0.88445$
- [Feige and Schechtman '00]: There exist graphs for which $OPT/SDP \sim 0.87856$
- What happens when adding triangle inequalities?

  ($Y_{ij} + Y_{jk} + Y_{ik} \geq -1$ for all i, j, k)

  [Feige and Schechtman '00]: There exists graphs for which the ratio is 0.891

Approximating

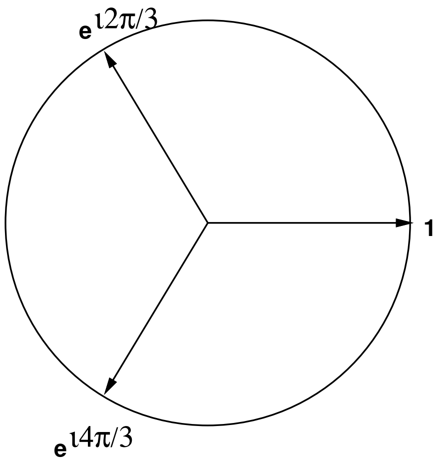
Ternary Optimization Problems via Complex Semidefinite Programming

Michel Goemans

Joint work with David Williamson

Ternary optimization

- For binary decisions, represent the two choices either by 0, 1 or by $\pm\mathbf{1}$ (square roots of unity)
- Idea: Use the cube roots of unity to represent three choices!



Max E2-LIN3

- Given a set of equations/inequations modulo 3 ($x_i - x_j \equiv c \pmod 3$ or $\not\equiv c \pmod 3$), find a largest (weight) subset that can be simultaneously satisfied:

$$\begin{aligned} x_1 - x_3 &\equiv 0 \pmod 3 \\ x_2 - x_3 &\equiv 2 \pmod 3 \\ x_1 - x_2 &\not\equiv 1 \pmod 3 \end{aligned}$$

- NP-hard
- [Andersson, Engebretson, and Håstad '99]

  Cannot be approximated within $17/18 + \epsilon \sim 0.94445$

  Approximation algorithm with guarantee $\frac{1}{3} + 10^{-8}$ (for more general case)
- Case of all inequations of the form $x_i - x_j \not\equiv 0$ is equivalent to MAX 3-CUT

  Frieze-Jerrum '94: 0.83601-approximation algorithm
- Both Frieze and Jerrum '94 and Andersson et al. '99 use (real) semidefinite programming (with twists)

(Complex) Positive Semidefinite Matrices

- For complex matrix $M = (m_{ij})$, its *conjugate transpose* $M^* = (\bar{m}_{ji})$
- A square M is *Hermitian* if $M = M^*$. Say $M \in H_n$.
- For vectors $u, v \in C^n$, use inner product $\langle u, v\rangle = v^* u = \sum_k \bar{v}_k u_k$.
- For matrices $A, B \in H_n$, (Frobenius) inner product $\langle A, B\rangle = Tr(B^* A)$
- A $n \times n$ Hermitian matrix M is positive semidefinite $(M \succeq 0)$ iff $x^* M x \geq 0$ for all $x \in C^n$.

  Equivalently:

  ▷ $\exists V \in C^{m\times n}$ (or in $R^{m\times n}$ for real matrices): $A = V^* V$ (Cholesky decomposition)

  ▷ $\lambda_i(A) \geq 0$ for all i

Complex SDP

- A complex semidefinite program:

$$
\begin{aligned}
\text{Max} \quad & \langle C, Z\rangle & \\
& \langle A_k, Z\rangle \geq b_k & \forall k \\
& Z = (z_{ij}), Z \in H_n \text{ psd}, &
\end{aligned}
$$

where $C = (c_{ij})$ and $A_k = (a_{ijk})$ are Hermitian. Once again, polytime solvable.

- Special case of self-scaled cones (see e.g. Nesterov and Todd) or symmetric cones (see e.g. Schmieta and Alizadeh)

- Equivalent to complex vector program:

$$
\begin{aligned}
\text{Max} \quad & \sum_{i,j} c_{ij} \langle v_i, v_j\rangle & \\
& \sum_{ij} a_{ijk} \langle v_i, v_j\rangle \geq b_k & \forall k \\
& v_i \in C^n &
\end{aligned}
$$

- Also equivalent to real semidefinite programming using linear transformation $\mathcal{T}$ that maps matrices in H_n to matrices in S_{2n}:

$$
\mathcal{T}(Z) = \begin{bmatrix} \text{Re } Z & -\text{Im } Z \\ \text{Im } Z & \text{Re } Z \end{bmatrix}.
$$

Model of MAX E2-LIN3

- Say that $x_u = k$ if $y_u = e^{i2\pi k/3}$
- $\langle y_u, y_v\rangle = \langle e^{i2\pi x_u/3}, e^{i2\pi x_v/3}\rangle = e^{i2\pi(x_u - x_v)/3}$

Thus $x_u - x_v \equiv c \pmod 3$ is satisfied iff $\langle y_u, y_v\rangle = \bar{y}_v y_u = e^{i2\pi c/3}$

- For $y_u, y_v \in \{0, e^{i2\pi/3}, e^{i4\pi/3}\}$, we have $\langle y_u, y_v\rangle \in \{0, e^{i2\pi/3}, e^{i4\pi/3}\}$

Hence, contribution of $x_u - x_v \equiv c \pmod 3$ to the objective function is

$$\frac{1}{3} + \frac{1}{3} e^{-i2\pi c/3} \langle y_u, y_v\rangle + \frac{1}{3} e^{i2\pi c/3} \langle y_v, y_u\rangle$$

- Similarly for $x_u - x_v \not\equiv c \pmod 3$

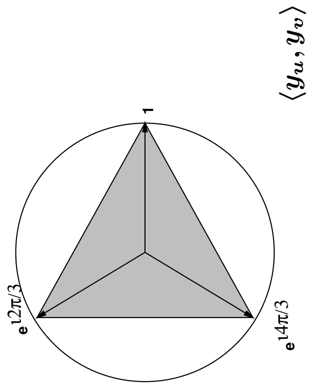
$$\frac{2}{3} - \frac{1}{3} e^{-i2\pi c/3} \langle y_u, y_v\rangle - \frac{1}{3} e^{i2\pi c/3} \langle y_v, y_u\rangle$$

Relaxation

Relax $y_u = e^{i2\pi k/3}$ to

- $y_u \in C^n$ for all u,
- $\langle y_u, y_u \rangle = 1$ for all u,
- $\alpha \langle y_u, y_v \rangle + \bar{\alpha} \langle y_v, y_u \rangle \geq -1$ for all u, v and $\alpha \in \{1, e^{i2\pi/3}, e^{i4\pi/3}\}$

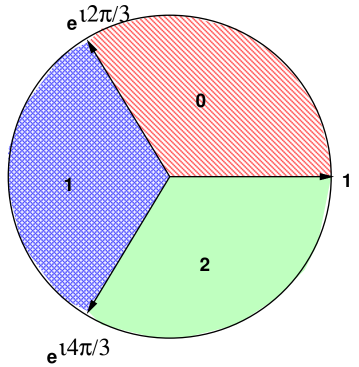
$\Rightarrow$ complex vector program



Algorithm

- Solve complex vector program,
- Choose random vector r from *complex* normal distribution (i.e. each $Re(r_k)$ and $Im(r_k)$ are i.i.d. Gaussian)
- Let

$$x_u = \begin{cases} 0 & \text{if } \arg\langle r, y_u \rangle \in [0, 2\pi/3) \\ 1 & \text{if } \arg\langle r, y_u \rangle \in [2\pi/3, 4\pi/3) \\ 2 & \text{if } \arg\langle r, y_u \rangle \in [4\pi/3, 2\pi). \end{cases}$$



- Theorem: (Randomized) algorithm gives performance guarantee of .79373.
- Improves upon guarantee of $\frac{1}{3}(1 + 10^{-8})$ of Andersson, Engebretson, Håstad '99.

A few steps of the analysis

- $X_k = 1$ if kth (in)equation is satisfied and 0 otherwise.

  $\Rightarrow E\left[\sum_k X_k\right] = \sum_k E[X_k] = \sum_k Pr($equation k is satisfied$)$

  $\Rightarrow$ can focus on

  ▷ single equation, say $x_u - x_v \equiv c$ (can even assume $c = 0$),

  ▷ and single inequation, sqy $x_u - x_v \not\equiv c$

- Because of spherical symmetry of distribution of r, can perform rotation and can assume that $y_u, y_v \in C^2$

  (probability depends only on $\langle y_u, y_v \rangle$ and not on actual position)

- Can further assume that $y_u = e_1$

- then tedious calculations...

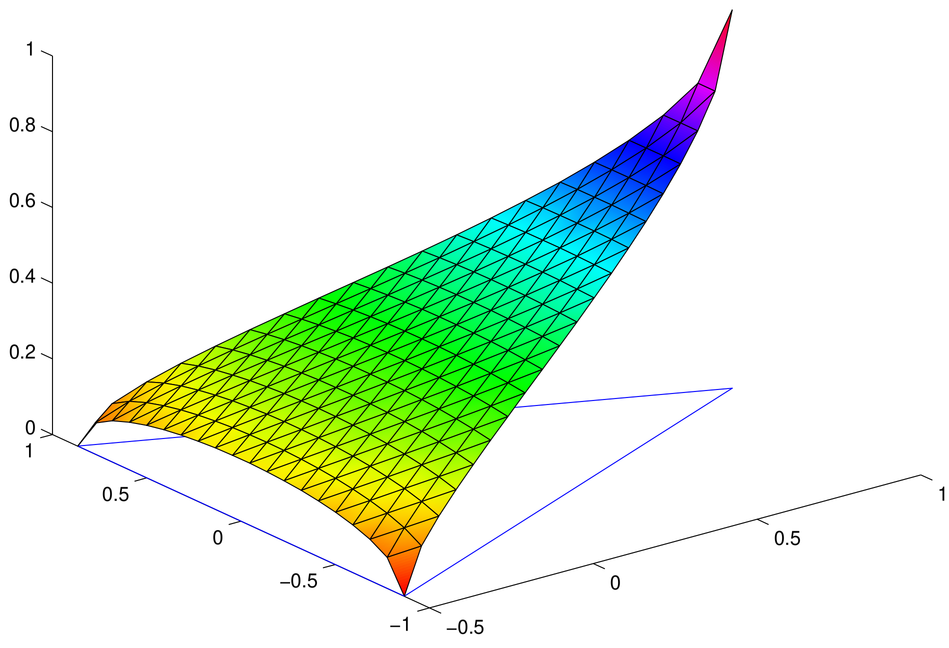
Probability that a single equation be satisfied

Theorem: If $\langle y_u, y_v \rangle = re^{-i\theta}$ then the probability that the equation $x_u - x_v \equiv c \pmod 3$ is satisfied is

$$
\begin{aligned}
P &= \frac{1}{3} + \frac{3}{8\pi^2}\left[2\arccos^2\left(-r\cos\left(\frac{2\pi c}{3} + \theta\right)\right)\right. \\
&\quad - \arccos^2\left(-r\cos\left(\frac{2\pi(c+1)}{3} + \theta\right)\right) \\
&\quad \left. - \arccos^2\left(-r\cos\left(\frac{2\pi(c-1)}{3} + \theta\right)\right)\right].
\end{aligned}
$$

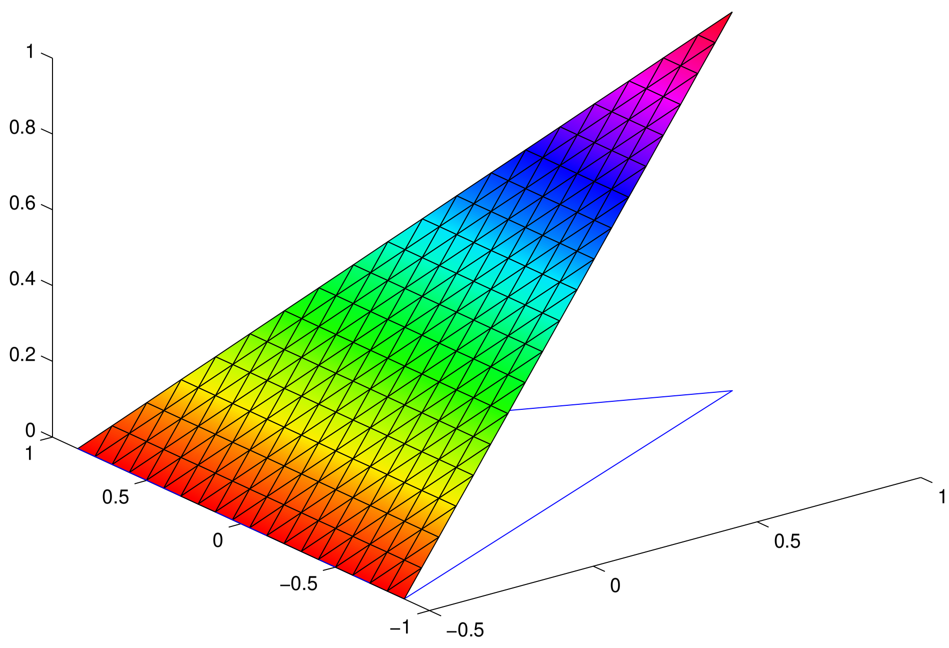
(1-above for $x_i - x_j \not\equiv c \pmod 3$)

Slide 37



Probability that $x_u - x_v \equiv 0 \pmod 3$ is satisfied

Slide 38



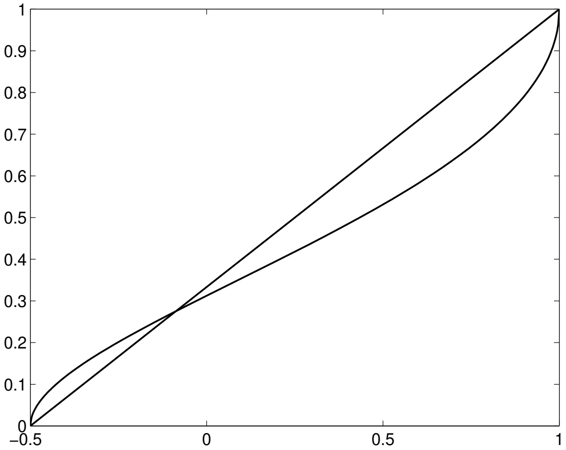
Contribution of $x_u - x_v \equiv 0 \pmod 3$ to objective function

For $x_i - x_j \equiv 0 \pmod 3$

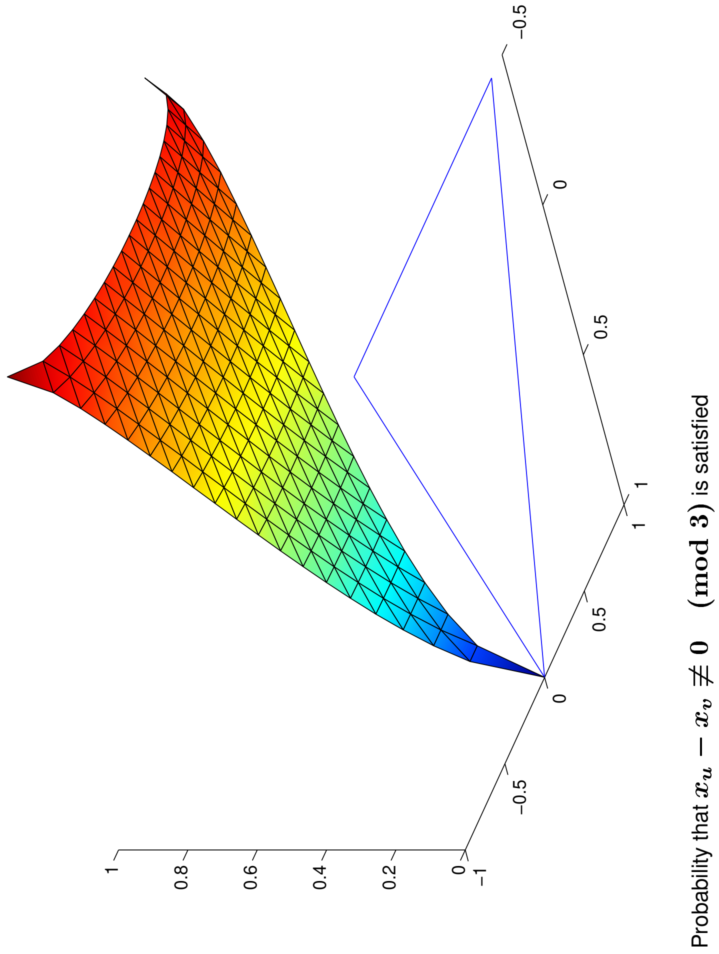
- Probability concave for objective constant

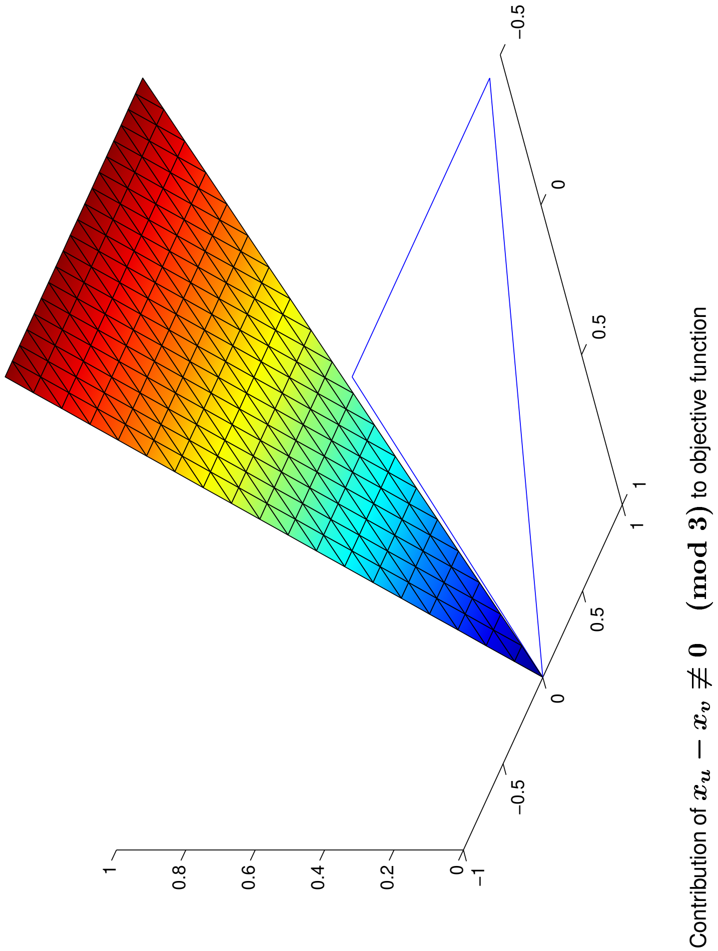
  $\Rightarrow$ ratio worst along sides

- On side:



- Minimum ratio = $\mathbf{0.79373}$ $\Rightarrow$ approximation guarantee



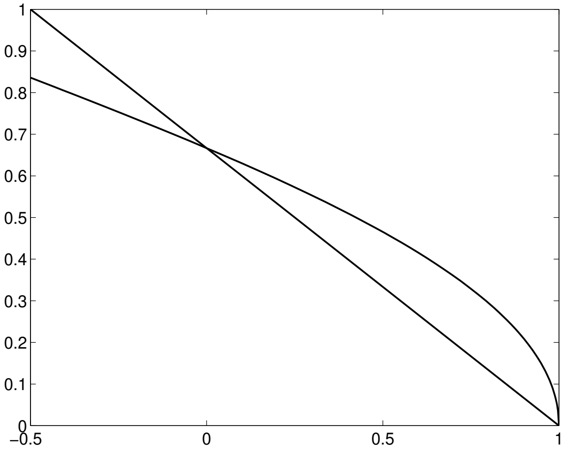


For $x_i - x_j \not\equiv 0 \pmod 3$

- Probability convex for objective constant

  $\Rightarrow$ ratio worst on real axis

- On real axis:



- Concave $\Rightarrow$ minimum for $\langle y_u, y_v \rangle = -0.5$:

$$\text{ratio } = \frac{7}{12} + \frac{3}{4\pi^2} \arccos^2(-1/4) \sim .83601$$

Final remarks

- Complex SDP + algorithm can be shown to be equivalent to Andersson et al.' real SDP relaxation + algorithm
- For MAX 3-CUT, guarantee is:

$$\frac{7}{12} + \frac{3}{4\pi^2} \arccos^2(-1/4) \sim .83601$$

- Work in progress, more extensions being considered