Robust Optimization — Methodology and Applications

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Abstract

Robust Optimization (RO) is a modeling methodology, combined with computational tools, to process optimization problems in which the data is uncertain and is only known to belong to some uncertainty set. The paper surveys the main results of RO as applied to uncertain linear, conic quadratic and semidefinite programming. For these cases, computationally tractable robust counterparts of uncertain problems are explicitly obtained, or good approximations of these counterparts are proposed, making RO a useful tool for real-world applications. We discuss some of these applications, specifically: antenna design, truss topology design and stability analysis/synthesis in uncertain dynamic systems. We also describe a case study of 90 LPs from the NETLIB collection. The study reveals that the feasibility properties of the usual solutions of real world LPs can be severely affected by small perturbations of the data and that the RO methodology can be successfully used to overcome this phenomenon.

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1 Introduction

A generic mathematical programming problem is of the form

$$\min_{x_0 \in \mathbf{R}, x \in \mathbf{R}^n} \{x_0 : f_0(x,\zeta) - x_0 \le 0,\ f_i(x,\zeta) \le\ i = 1,...,m\} \qquad (\mathrm{P}[\xi])$$

where x is the design vector, the functions f_0 (the objective function) and $f_1, ..., f_m$ are *structural elements* of the problem, and ζ is a *data* specifying a particular problem instance.

For real-world optimization problems, the "decision environment" is often characterized by the following facts:

F.1. The data is uncertain/inexact;

F.2. The optimal solution, even if computed very accurately, may be difficult to implement accurately;

F.3. The constraints *must* remain feasible for *all* meaningful realizations of the data;

F.4. Problems are *large-scale* (n or/and m are large);

F.5. "Bad" optimal solutions (those which become severely infeasible in the face of even relatively small changes in the nominal data) are *not* uncommon.

F.1 (and in fact **F.2** as well) imply that in many cases we deal with *uncertain optimization problems* – families of the usual ("certain") optimization problems

$$\{(\mathrm{P}[\zeta]) \mid \zeta \in \mathcal{U}\}, \qquad (1)$$

where $\mathcal{U}$ is come "uncertainty set" in the space of the data.

Fact **F.3** implies that a meaningful candidate solution (x_0, x) of an uncertain problem (1) is required to satisfy the semi-infinite system of constraints

$$f_0(x,\zeta) \le x_0,\ f_i(x,\zeta) \le 0\ i = 1,...,m \quad \forall(\zeta \in \mathcal{U}). \tag{2}$$

Fact **F.4** on the other hand imposes a severe requirement of being able to process efficiently the semi-infinite system of constraints (2) for large-scale problems.

Robust Optimization originating from [2, 3, 4, 9, 10, 11] is a modeling methodology, combined with a suite of computational tools, which is aimed at accomplishing the above requirements. The urgency of having such a methodology stems from the fact **F.5**, whose validity is well illustrated in the cited papers (and in the examples to follow).

In the Robust Optimization methodology, one associates with an uncertain problem (1) its *robust counterpart*, which is a usual (semi-infinite) optimization program

$$\min_{x_0,x} \{x_0 : f_0(x,\zeta) \le x_0,\ f_i(x,\zeta) \le 0,\ i = 1,...,m \quad \forall(\zeta \in \mathcal{U})\}; \tag{3}$$

fesible/optimal solutions of the robust counterpart are called *robust feasible/robust optimal* solutions of the uncertain problem (1).

The major challenges associated with the Robust Optimization methodology are:

C.1. When and how can we reformulate (3) as a "computationally tractable" optimization problem, or at least approximate (3) by a tractable problem.

C.2. How to specify reasonable uncertainty sets $\mathcal{U}$ in specific applications.

In what follows, we overview the results and the potential of the Robust Optimization methodology in the most interesting cases of uncertain Linear, Conic Quadratic and Semidefinite Programming. For these cases, the instances of our uncertain optimization problems will be in the *conic form*

$$\min_{x_0\in\mathbf{R},x\in\mathbf{R}^n} \left\{x_0 : c^T x \le x_0,\ A_i x + b_i \in \mathbf{K}_i,\ i = 1,...,m, \right\}, \tag{C}$$

where for every i $\mathbf{K}_i$ is

– either a non-negative orthant $\mathbf{R}^{m_i}_+$ (linear constraints),

– or the Lorentz cone $\mathbf{L}^{m_i} = \left\{ y \in \mathbf{R}^{m_i} \mid y_{m_i} \ge \sqrt{\sum\limits_{j=1}^{m_i-1} y_j^2} \right\}$ (conic quadratic constraints),

– or a semidefinite cone $\mathbf{S}^{m_i}_+$ – the cone of positive semidefinite matrices in the space $\mathbf{S}^{m_i}$ of $m_i \times m_i$ symmetric matrices (Linear Matrix Inequality constraints).

The class of problems which can be modeled in the form of (C) is extremely wide (see, e.g., [8, 7]). It is also clear what is the *structure* and what is the *data* in (C) – the former is the design dimension n, the number of "conic constraints" m and the list of the cones $\mathbf{K}_1$,..., $\mathbf{K}_m$, while the latter is the collection of matrices and vectors c, $\{A_i, b_i\}_{i=1}^m$ of appropriate sizes. Thus, an uncertain problem (C) is a collection

$$\left\{ \min_{x_0\in\mathbf{R},x\in\mathbf{R}^n} \left\{ x_0 : \underbrace{x_0 - c^T x}_{A_0 x + b_0} \in \mathbf{K}_0 \equiv \mathbf{R}_+,\ A_i x + b_i \in \mathbf{K}_i,\ i = 1,...,m \right\} | (c, \{A_i, b_i\}_{i=1}^m) \in \mathcal{U} \right\} \tag{4}$$

of instances (C) of a common structure $(n, m, \mathbf{K}_1, ..., \mathbf{K}_m)$ and data $(c, \{A_i, b_i\}_{i=1}^m)$ varying in a given set $\mathcal{U}$. Note that the robust counterpart is a "constraint-wise" notion: in the robust counterpart of (4), every one of the uncertain constraints $A_i x + b_i \in \mathbf{K}_i$, $i = 0, ..., m$, of (4) is replaced with its robust counterpart

$$A_i x + b_i \in \mathbf{K}_i \quad \forall (A_i, b_i) \in \mathcal{U}_i,$$

where $\mathcal{U}_i$ is the projection of the uncertainty set $\mathcal{U}$ on the space of the data of i-th constraint. This observation allows us in the sequel to focus on the "pure cases" (all $\mathbf{K}_i$'s are orthants, or all of them are Lorentz cones, or all are the semidefinite cones) only; although the results can be readily applied to the "mixed cases" as well.

In the rest of this paper, we consider in turn uncertain Linear, Conic Quadratic and Semidefinite programming problems; our major interest is in the relevant version of **C.1** in the case when the uncertainty set is an intersection of ellipsoids (which seems to be a general enough model from the viewpoint of **C.2**). We shall illustrate our general considerations by several examples, mainly coming from engineering. We believe that these examples will convince the reader that in many applications taking into account from the very beginning data uncertainty is a necessity which cannot be ignored, and that the Robust Optimization methodology does allow to deal with this necessity successfully.

2 Robust Linear Programming

2.1 Robust counterpart of uncertain LP

In [3] we showed that the robust counterpart

$$\min_{t,x}\left\{t : t \geq c^T x, Ax \geq b \quad \forall (c, A, B) \in \mathcal{U}\right\} \tag{5}$$

of an uncertain LP

$$\left\{\min_x \left\{c^T x : Ax \geq b\right\} \middle| (c, A, b) \in \mathcal{U} \subset \mathbf{R}^n \times \mathbf{R}^{m\times n} \times \mathbf{R}^m\right\} \tag{6}$$

is equivalent to an explicit computationally tractable problem, provided that the uncertainty set $\mathcal{U}$ itself is "computationally tractable". To simplify our presentation, we restrict ourselves to the "most typical" types of "computationally tractable" sets, and here is the corresponding result:

Theorem 2.1 [[3]] *Assume that the uncertainty set $\mathcal{U}$ in* (6) *is given as the affine image of a bounded set $\mathcal{Z} = \{\zeta\} \subset \mathbf{R}^N$, and $\mathcal{Z}$ is given*
either

(i) *by a system of linear inequality constraints*

$$P\zeta \leq p$$

or

(ii) *by a system of Conic Quadratic inequalities*

$$\| P_i\zeta - p_i \|_2 \leq q_i^T \zeta - r_i, \; i = 1, ..., M,$$

or

(iii) *by a system of Linear Matrix Inequalities*

$$P_0 + \sum_{i=1}^{\dim \zeta} \zeta_i P_i \succeq 0.$$

In the cases (ii), (iii) *assume also that the system of constraints defining $\mathcal{U}$ is strictly feasible. Then the robust counterpart* (5) *of the uncertain LP* (6) *is equivalent to*

– *a Linear Programming problem in case* (i),
– *a Conic Quadratic problem in case* (ii),
– *a Semidefinite program in case* (iii).

In all cases, the data of the resulting robust counterpart problem are readily given by m, n and the data specifying the uncertainty set. Moreover, the sizes of the resulting problem are polynomial in the size of the data specifying the uncertainty set. □

Proof. A point $y = (t, x) \in \mathbf{R}^{n+1}$ is a feasible solution of (5) if and only if y solves a semi-infinite system of inequalities of the form

$$\begin{array}{c} (C_i) \quad [B_i\zeta + \beta_i]^T y + [c_i^T \zeta + d_i] \geq 0 \quad \forall \zeta \in \mathcal{Z}, \\ i = 0, ..., m \end{array} \tag{7}$$

The fact that y solves (C_i) means that the optimal value in the problem

$$\min_{\zeta} \left\{ [B_i\zeta + \beta_i]^T y + [c_i^T \zeta + d_i] : \zeta \in \mathcal{Z} \right\} \tag{$P_i[y]$}$$

is nonnegative. Now, by assumption $\mathcal{Z}$ is representable as

$$\mathcal{Z} = \{\zeta \mid P\zeta - p \in \mathbf{K}\}$$

where $\mathbf{K}$ is either a nonnegative orthant, or a direct product of the second-order cones, of the semidefinite cone. Consequently, $(P_i[y])$ is the conic problem

$$\min_{\zeta} \left\{ [B_i^T y + c_i]^T \zeta + [\beta_i^T y + d_i] : P\zeta - p \in \mathbf{K} \right\}$$

Note that this conic problem is bounded (since $\mathcal{Z}$ is bounded). Applying either the LP Duality Theorem or the Conic Duality Theorem [12], Chapter 4, depending on whether $\mathbf{K}$ is or is not polyhedral, we see that the optimal value in $(P_i[x])$ is equal to the one in the (solvable) dual problem

$$\max_{\xi} \left\{ p^T \xi + [\beta_i^T y + d_i] : \xi \in \mathbf{K}, P^T \xi = B_i^T y + c_i \right\} \tag{$D_i[y]$}$$

Thus, y solves (C_i) if and only if there exists $\xi \in \mathbf{K}$ such that $P^T \xi = B_i^T y + c_i$ and $p^T \xi + [\beta_i^T y + d_i] \geq 0$. We conclude that (5) is equivalent to the problem

$$\min_{y=(t,x), \{\xi^i\}_{i=0}^m} \left\{ t : \begin{array}{c} \xi^i \in \mathbf{K},\ i = 0, ..., m, \\ P^T \xi^i = B_i^T y + c_i,\ i = 0, ..., m, \\ p^T \xi^i + [\beta_i^T y + d_i] \geq 0,\ i = 0, ..., m. \end{array} \right\}$$ ■

2.2 An example: robust antenna design

To illustrate the practical potential of Robust LP, consider a particular application – the *antenna design* problem.

A monochromatic electro-magnetic antenna is characterized by its *diagram*, which is a complex-valued function $D(\delta)$ of a 3D-direction δ; the sensitivity of the antenna w.r.t. a flat wave incoming along a direction δ is proportional to $|D(\delta)|^2$.

An important property of the diagram is that the diagram of an antenna array – a complex antenna consisting of elements with diagrams $D_1, ..., D_N$ – is just the sum of the diagrams of the elements. When amplification of the outputs of the elements is allowed, the diagram of the resulting complex antenna becomes $\sum_{j=1}^{N} x_j D_j(\cdot)$, where x_j are (in general, complex-valued) "weights" of the elements.

A typical antenna design problem is:

(AD) *Given antenna elements with diagrams* $D_1, ..., D_N$, *find (complex-valued) weights* $x_1, ..., x_N$ *such that the diagram* $D(\cdot) = \sum_{j=1}^{N} x_j D_j(\cdot)$ *of the resulting antenna array satisfies given design specifications* (i.e., $|D(\cdot)|$ is as close as possible to a given target function).

As a simple example, consider the following problem:
Circular Antenna. Let the antenna array consist of rings, centered at the origin, in the XY-plane. The diagram of such a ring is real-valued and depends solely on the altitude angle θ (the angle between a direction and the XY-plane): specifically,

$$D_\kappa(\theta) = \frac{1}{2}\int_0^{2\pi} \cos\left(2\pi\kappa\cos(\theta)\cos(\phi)\right) d\phi$$

where κ is the ratio of the ring radius and the wavelength. Assume there are 40 rings in the array, with $\kappa_j = j/10$, $j = 1, ..., 40$. Our goal is to choose *real* weights x_j such that the diagram

$$D(\theta) = \sum_{j=1}^{40} x_j D_{\kappa_j}(\theta)$$

is nearly uniform in the "angle of interest" $77^o \le \theta \le 90^o$, specifically,

$$77^o \le \theta \le 90^o \Rightarrow 0.9 \le \sum_{j=1}^{40} x_j D_{\kappa_j}(\theta) \le 1;$$

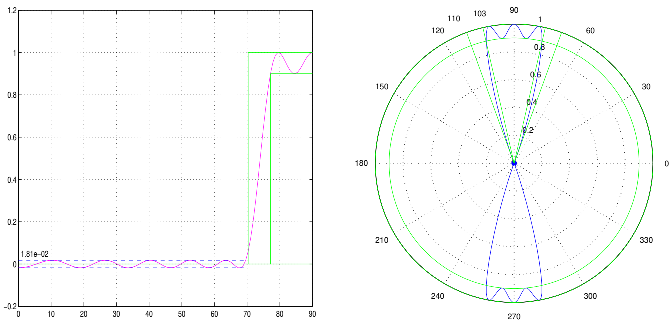
under this restriction, we want to minimize the "sidelobe attenuation level" $\max\limits_{0 \le \theta \le 70^o} |D(\theta)|$.

With discretization in θ, the problem can be modeled as a simple LP

$$\min_{\tau, x_1, ..., x_{40}} \left\{ \tau : \begin{array}{c} 0.9 \le \sum_{j=1}^{40} x_j D_{\kappa_j}(\theta) \le 1,\ \theta \in \Theta_{\rm cns} \\ -\tau \le \sum_{j=1}^{40} x_j D_{\kappa_j}(\theta) \le \tau,\ \theta \in \Theta_{\rm obj} \end{array} \right\} \tag{8}$$

where $\Theta_{\rm cns}$ and $\Theta_{\rm obj}$ are finite grids on the segments $[77^o, 90^o]$ and $[0^o, 70^o]$, respectively.

Solving (8), one arrives at the *nominal design* with a nice diagram as follows:

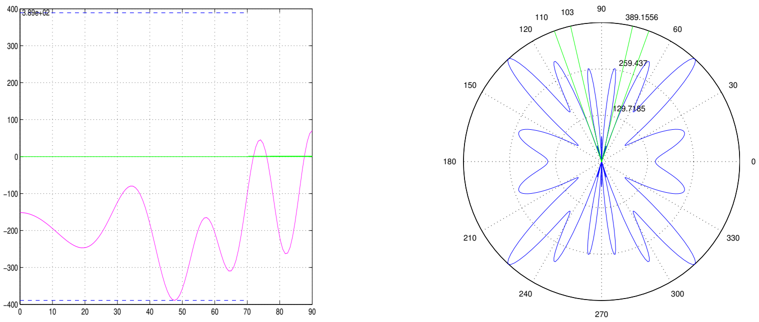


Dream: Nominal design, no implementation errors
Sidelobe attenuation level 0.018

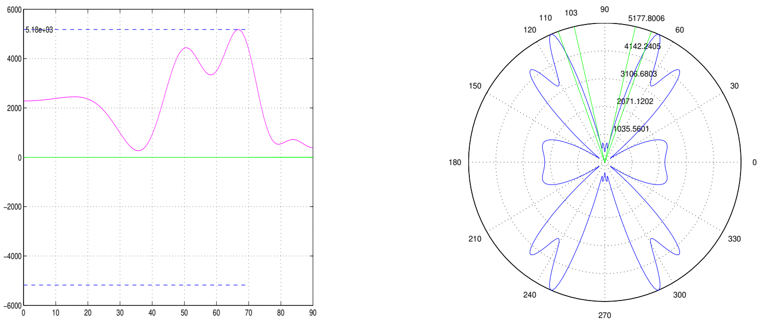
In reality, the optimal weights x_j correspond to characteristics of certain physical devices and as such cannot be implemented exactly. Thus, it is important to know what happens if the actual weights are affected by "implementation errors"

$$x_j \mapsto (1+\xi_j)x_j, \quad \xi_j \sim \text{Uniform}[-\epsilon, \epsilon] \text{ are independent.} \tag{9}$$

It turns out that even quite small implementation errors have a disastrous effect on the designed antenna:



Reality: Nominal design, implementation errors $x_j \mapsto (1+\xi_j)x_j$
$[\xi_j \sim \text{Uniform}[-0.001, 0.001]]$
100-diagram sample: Sidelobe level $\in [97, 536]$



Reality: Nominal design, implementation errors $x_j \mapsto (1+\xi_j)x_j$
$[\xi_j \sim \text{Uniform}[-0.02, 0.02]]$
100-diagram sample: Sidelobe level $\in [2469, 11552]$

We see that from the practical viewpoint the nominal design is meaningless – its "optimality" is completely destroyed by fairly small implementation errors!

2.2.1 From nominal to a robust design: Interval model of uncertainty

In order to "immune" the design against implementation errors, one can use the Robust Counterpart methodology. Indeed, the influence of multiplicative errors

$$x_j \mapsto (1+\xi_j)x_j, \quad |\xi_j| \leq \epsilon_j$$

on a solution of an LP

$$\min_x \left\{ c^T x : Ax \geq b \right\}$$

is as if there were no implementation errors, but the matrix $\begin{bmatrix} c^T \\ A \end{bmatrix}$ was known up to multiplication from the right by a diagonal matrix D with the diagonal entries D_{jj} varying in the segments $[1-\epsilon_j, 1+\epsilon_j]$. As far as the robust counterpart is concerned, the resulting uncertainty is (equivalent to) a particular case of *interval uncertainty*, where every entry in the data of an LP, independently of other entries, runs through a given interval. Note that the robust counterpart of an uncertain LP with interval uncertainty

$$\left\{ \min_x \left\{ c^T x : Ax \geq b \right\} \mid \begin{array}{c} |c_j - c_j^{\rm n}| \leq \delta c_j,\, j = 1, ..., n \\ |A_{ij} - A_{ij}^{\rm n}| \leq \delta A_{ij},\, i = 1, ..., m, j = 1, ..., n \\ |b_i - b_i^{\rm n}| \leq \delta b_i,\, i = 1, ..., m \end{array} \right\}$$

is clearly equivalent to the LP

$$\min_{x,y,t} \left\{ \sum_j [c_j^{\rm n} x_j + \delta c_j y_j] : \begin{array}{c} \sum_j [A_{ij}^{\rm n} x_j + \delta A_{ij} y_j] \leq b_i - \delta b_i,\, i = 1, ..., m, \\ -y_j \leq x_j \leq y_j,\, j = 1, ..., n \end{array} \right\}$$

It is worthy of mentioning that uncertain LPs with interval uncertainty and their robust counterparts were considered by A.L. Soyster as early as in 1973.

As applied to our Circular Antenna problem affected by implementation errors, the outlined approach leads to the Interval robust counterpart

$$\min_{\tau,x,y} \left\{ \tau : \begin{array}{c} 0.9 \leq \sum\limits_{j=1}^{40} \left[x_j D_{\kappa_j}(\delta) - \epsilon |D_{\kappa_j}(\delta)| y_j \right],\ \theta \in \Theta_{\rm cns} \\ \sum\limits_{j=1}^{40} \left[x_j D_{\kappa_j}(\delta) + \epsilon |D_{\kappa_j}(\delta)| y_j \right] \leq 1,\ \theta \in \Theta_{\rm cns} \\ -\tau \leq \sum\limits_{j=1}^{40} \left[x_j D_{\kappa_j}(\delta) - \epsilon |D_{\kappa_j}(\delta)| y_j \right],\ \theta \in \Theta_{\rm obj} \\ \sum\limits_{j=1}^{40} \left[x_j D_{\kappa_j}(\delta) + \epsilon |D_{\kappa_j}(\delta)| y_j \right] \leq \tau,\ \theta \in \Theta_{\rm obj} \\ -y_j \leq x_j \leq y_j,\ j = 1, ..., 40 \end{array} \right\},$$

where ϵ is the level of implementation errors we intend to withstand.

2.2.2 From nominal to a robust design: Ellipsoidal model of uncertainty

The "worst-case-oriented" interval model of uncertainty looks "too conservative". Indeed, when the perturbations in coefficients of an uncertain linear inequality are of stochastic nature, it is "highly unprobable" that they will *simultaneously* take the "most dangerous" values. This is the case, e.g., in our Antenna Design problem, where we have all reasons to assume that the implementation errors are as in (9).

Let us look what happens with a linear inequality

$$a_0 + \sum_{j=1}^{n} a_j x_j \leq 0 \tag{10}$$

at a given candidate solution x, when the coefficients of the constraint are affected by random perturbations, so that the vector $a = (a_0, a_1, ..., a_n)^T$ is random. In this case, the left hand side of (10) is a random variable ζ_x with the mean and the standard deviation given by

$$\begin{array}{c} \mathbf{E}\{\zeta_x\} = (1, x^T)^T a^{\rm n}, \quad a^{\rm n} = \mathbf{E}\{a\}, \\ \mathbf{S}tD\{\zeta_x\} \equiv \mathbf{E}^{1/2}\{[\zeta_x - \mathbf{E}\{\zeta_x\}]^2\} = \sqrt{(1, x^T) V (1, x^T)^T}, \quad V = \mathbf{E}\left\{ (a - a^{\rm n})(a - a^{\rm n})^T \right\}. \end{array} \tag{11}$$

Now let us choose a "safety parameter" Ω and ignore the "rare event" $\zeta_x > \mathbf{E}\{\zeta_x\} + \Omega \mathbf{S}tD\{\zeta_x\}$, but, at the same time, let us take upon ourselves full responsibility to satisfy (10) in all other events. With this approach, a "safe" version of our inequality becomes

$$a_0^{\mathrm{n}} + \sum_{j=1}^{n} a_j^{\mathrm{n}} x_j + \Omega\sqrt{(1, x^T)V(1, x^T)^T} \le 0; \tag{12}$$

if this inequality is satisfied at a given x, then the "true" inequality (10) is *not* satisfied at x with probability $p(x, \Omega)$ which approaches 0 as Ω grows. (For example, in the case of normally distributed a, one has $p(x, \Omega) \le \exp\{-\Omega^2/2\}$, so that when choosing $\Omega = 5.24$, we are sure that if x satisfies (12), then the probability that x violates (10) is less than 1.e-6.)

It remains to note that (12) is *exactly* the robust counterpart of the uncertain inequality (10) corresponding to choosing the uncertainty set as the ellipsoid

$$\mathcal{U} = \left\{a^{\mathrm{n}} + da \mid da = \Omega V^{1/2}\zeta, \zeta^T\zeta \le 1\right\}.$$

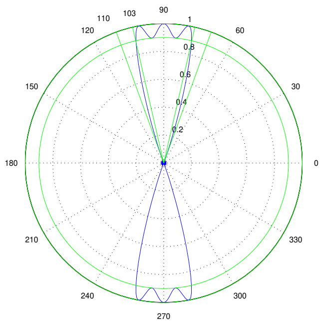
We see that an ellipsoidal uncertainty is a natural way to model, in an "ϵ-reliable fashion", random perturbations of the data in uncertain LPs; usually this approach is significantly less conservative than the worst-case-oriented interval one (interval uncertainty).

As applied to our Circular Antenna problem affected by implementation errors (9), the approach leads to the Ellipsoidal robust counterpart

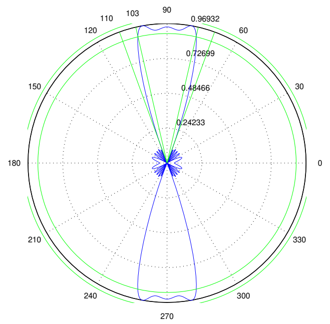
$$\min_{\tau,x,y}\left\{\tau : \begin{array}{c} 0.9 + \Omega\epsilon\sqrt{\sum\limits_{j=1}^{40} x_j^2 D_{\kappa_j}^2(\theta)} \le \sum\limits_{j=1}^{40} x_j D_{\kappa_j}(\delta),\ \theta \in \Theta_{\mathrm{cns}} \\ \sum\limits_{j=1}^{40} x_j D_{\kappa_j}(\delta) + \Omega\epsilon\sqrt{\sum\limits_{j=1}^{40} x_j^2 D_{\kappa_j}^2(\theta)} \le 1,\ \theta \in \Theta_{\mathrm{cns}} \\ -\tau + \Omega\epsilon\sqrt{\sum\limits_{j=1}^{40} x_j^2 D_{\kappa_j}^2(\theta)} \le \sum\limits_{j=1}^{40} x_j D_{\kappa_j}(\delta),\ \theta \in \Theta_{\mathrm{obj}} \\ \sum\limits_{j=1}^{40} x_j D_{\kappa_j}(\delta) + \Omega\epsilon\sqrt{\sum\limits_{j=1}^{40} x_j^2 D_{\kappa_j}^2(\theta)} \le \tau,\ \theta \in \Theta_{\mathrm{obj}} \end{array}\right\}.$$

2.2.3 Comparing the designs

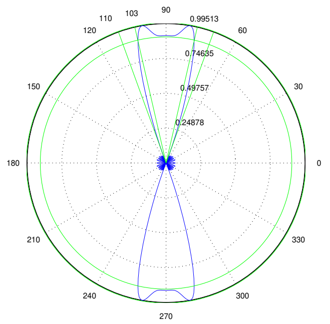
Below we summarize a comparison of the designs corresponding to the Nominal problem and its Interval and Ellipsoidal Robust counterparts for our Circular Antenna example.



Nominal, no errors



Interval Robust, 2% errors



Ellipsoidal Robust, 2% errors

Design	ϵ in (9)					
	$\epsilon = 0$		$\epsilon = 0.001$		$\epsilon = 0.02$	
	Sidelobe level	Constr. viol.	Sidelobe level[1]	Constr. viol.[2]	Sidelobe level[1]	Constr. viol.[2]
Nom	0.018	0	97 536	1338	2469 11552	23756
Ell_Rob	0.065	0	0.0653 0.0657	0	0.0657 0.0712	0.009
Int_Rob	0.110	0	0.1095 0.1099	0	0.1077 0.1151	0

[1] minimum and maximum in a 100-diagram sample
[2] maximum in a 100-diagram sample

Note that although the Interval and the Ellipsoidal Robust designs are built for the implementation error $\epsilon = 0.001$, both designs are capable to withstand even 20 times larger implementation errors. We see also that in terms of the objective the Ellipsoidal Robust design is 50% better than the Interval Robust one, although both designs possess the same capability to withstand random implementation errors.

2.3 `NETLIB` case study

Consider a real-world LP program `PILOT4` from the `NETLIB` library (1,000 variables, 410 constraints). The constraint # 372 is:

$$\begin{array}{r}
[a^{\rm n}]^T x \equiv -15.79081x_{826} - 8.598819x_{827} - 1.88789x_{828} - 1.362417x_{829} - 1.526049x_{830} \\
-0.031883x_{849} - 28.725555x_{850} - 10.792065x_{851} - 0.19004x_{852} - 2.757176x_{853} \\
-12.290832x_{854} + 717.562256x_{855} - 0.057865x_{856} - 3.785417x_{857} - 78.30661x_{858} \\
-122.163055x_{859} - 6.46609x_{860} - 0.48371x_{861} - 0.615264x_{862} - 1.353783x_{863} \\
-84.644257x_{864} - 122.459045x_{865} - 43.15593x_{866} - 1.712592x_{870} - 0.401597x_{871} \\
+x_{880} - 0.946049x_{898} - 0.946049x_{916} \\
\geq b \equiv 23.387405.
\end{array} \qquad (13)$$

Most of the coefficients are "ugly reals" like -8.598819; we have all reasons to believe that these coefficients are characteristics of certain technological processes/devices and as such cannot be known with high accuracy; thus, we can treat the ugly coefficients as uncertain and coinciding with the "true" coefficients with an accuracy of, say, 3-4 digits, not more. An exception is the coefficient 1 at x_{880} which perhaps represents the structure of the problem and is therefore certain. With these natural assumptions, we can ask what happens with the constraint at the optimal solution $x^{\rm n}$, as reported by `CPLEX`, when we perturb the uncertain coefficients within, say, 0.01% margin. The answer is as follows:

• The worst, over all 0.01%-perturbations of uncertain coefficients, violation of the constraint at $x^{\rm n}$ is as large as 450% of the right hand side;

• With independent random multiplicative perturbations distributed uniformly , distributed uniformly on of uncertain coefficients, the constraint at $x^{\rm n}$ is violated by at most 150% of the right hand side with probability 0.18.

We see that the usual solution to a real-world LP can be highly unreliable – it can become heavily infeasible as a result of fairly small perturbations of the uncertain data. To realize how frequent is this unpleasant phenomenon and how to struggle with it, we have carried out a case study as follows (for full details, see [5]).

• We have looked through the list of NETLIB LPs and for every one of them treated as *certain* the coefficients of equality constraints and "simple" (representable as fractions with denominators ≤ 100) coefficients of the inequality constraints; all other coefficients were treated as *uncertain*.

• At the "analysis" stage of our case study, we looked what happens with the feasibility properties of the nominal solution (as reported by CPLEX 6.2) when the uncertain coefficients are subject to small (0.01%) random perturbations.

It turned out that with these data perturbations, in 13 (of totally 90) problems the nominal solution violates some of the constraints by more than 50% of the right hand side!

• At the "synthesis" stage, we used the Interval and the Ellipsoidal robust counterparts to get "uncertainty-immuned" solutions to the "bad" NETLIB LPs given by the Analysis stage. It turned out that both approaches yield fairly close results, and that *in terms of optimality, "immunization" against ϵ-uncertainty is basically costless, provided that ϵ is not too large.* For example, passing from the nominal solutions of NETLIB problems to the robust ones, immuned against 0.1% uncertainty, we never lost more than 1% in the value of the objective.

We believe that the outlined case study demonstrates the high potential of the Robust Optimization methodology in processing correctly real-world LPs.

3 Robust Quadratic Programming

Consider an uncertain convex quadratically constrained problem

$$\left\{\min_x \left\{c^T x : x^T A_i x \leq 2b_i^T x + c_i,\, i = 1, ..., m\right\} | \{A_i, b_i, c_i\}_{i=1}^m \in \mathcal{U}\right\} \tag{14}$$

(w.l.o.g., we may treat the objective as certain, moving, if necessary, the original objective to the list of constraints). Here, in contrast to the case of LP, uncertainty sets even of fairly simple geometry (e.g., a box) can yield NP-hard (and thus computationally intractable) robust counterparts [4]. In these cases, the Robust Optimization methodology recommends to use *approximate* robust counterpart instead of the true one.

3.1 Approximate robust counterpart of an uncertain problem

Consider an uncertain optimization problem

$$\mathcal{P} = \left\{\min_x \left\{c^T x : F(x, \zeta) \leq 0\right\} | \zeta \in \mathcal{U}\right\} \tag{15}$$

(as above, we loose nothing when assuming the objective to be linear and certain). Typically, the uncertainty set $\mathcal{U}$ is given as

$$\mathcal{U} = \zeta^{\mathrm{n}} + \mathcal{V}, \tag{16}$$

where ζ^{n} is the *nominal data*, and $\mathcal{V}$ is a *perturbation set*. From now on, we postulate this structure and assume that $\mathcal{V}$ is a convex compact set containing the origin.

In the case of (16), our uncertain problem $\mathcal{P}$ can be treated as a member of the parametric family of uncertain problems

$$\mathcal{P}_\rho = \left\{\min_x \left\{c^T x : F(x, \zeta) \leq 0\right\} | \zeta \in \mathcal{U}_\rho = \zeta^{\mathrm{n}} + \rho\mathcal{V}\right\}, \tag{17}$$

where $\rho \geq 0$ is the "level of uncertainty" (which is equal to 1 for the original problem). Let

$$\mathcal{X}_\rho = \{x \mid F(x, \zeta) \leq 0 \ \forall \zeta \in \mathcal{U}_\rho\} \tag{18}$$

be the robust feasible set of $\mathcal{P}_\rho$; these sets clearly shrink as ρ grows.

We call an optimization problem

$$\min_{x,u}\left\{c^T x : G(x,u) \le 0\right\} \tag{19}$$

an *approximate robust counterpart* of $\mathcal{P}$, if the projection $\mathcal{Y}$ of the feasible set of (19) on the x-plane is contained in $\mathcal{X}_1$, and we say that the *level of conservativeness* of an approximate robust counterpart (19) *does not exceed* α, if $\mathcal{X}_\alpha \subset \mathcal{Y}$. In other words, (19) is an approximate RC of $\mathcal{P}$ with the level of conservativeness $\le \alpha$, if

1. Whenever a given x can be extended to a feasible solution of (19), x is robust feasible for $\mathcal{P}$; an approximate RC is "more conservative" than the true RC;
2. Whenever x *cannot* be extended to a feasible solution of (19), it may or may not be robust feasible for $\mathcal{P}$, but it certainly looses robust feasibility when the level of perturbations is increased by factor α.

Taking into account that the level of perturbations is usually something which is known "up to a factor of order of 1 only", an approximate robust counterpart with an $O(1)$ level of conservativeness can be treated as an appropriate alternative for the true RC.

3.2 Approximate robust counterparts of uncertain convex quadratic problems with ellipsoidal uncertainty

Assume that the uncertainty set $\mathcal{U}$ in (14) is *ellipsoidal*:

$$\mathcal{U} = \left\{ \left\{ (c_i, A_i, b_i) = (c_i^{\mathrm{n}}, A_i^{\mathrm{n}}, b_i^{\mathrm{n}}) + \sum_{\ell=1}^{L} \zeta_\ell (c_i^\ell, A_i^\ell, b_i^\ell) \right\}_{i=1}^{m} \middle| \zeta^T Q_j \zeta \le 1,\ j = 1, ..., k \right\}, \tag{20}$$

where $Q_j \succeq 0$, $\sum\limits_{j=1}^{k} Q_j \succ 0$. Note that this is a fairly general model of uncertainty (including, as a particular case, the interval uncertainty).

Given an uncertain convex quadratically constrained problem (14) with the ellipsoidal uncertainty (20), let us associate with it the semidefinite program

$$\begin{array}{l} \text{minimize } c^T x \\ \text{subject to} \\ \left(\begin{array}{c|ccc|c} 2x^T b_i^{\mathrm{n}} + c_i^{\mathrm{n}} - \sum\limits_{j=1}^{k} \lambda_{ij} & \frac{c_i^1}{2} + x^T b_i^1 & \cdots & \frac{c_i^L}{2} + x^T b_i^L & [A_i^{\mathrm{n}} x]^T \\ \hline \frac{c_i^1}{2} + x^T b_i^1 & & & & [A_i^1 x]^T \\ \vdots & & \sum\limits_{j=1}^{k} \lambda_{ij} Q_i & & \vdots \\ \frac{c_i^L}{2} + x^T b_i^L & & & & [A_i^L x]^T \\ \hline A_i^{\mathrm{n}} x & A_i^1 x & \cdots & A_i^L x & I \end{array}\right) \succeq 0,\ i = 1, ..., m, \\ \lambda_{ij} \ge 0,\ i = 1, ..., m,\ j = 1, ..., k \end{array} \tag{21}$$

in variables x, λ_{ij}.

Theorem 3.1 [[6], Theorem 3.1] *Problem* (21) *is an approximate robust counterpart of the uncertain convex quadratically constrained problem* (14) *with ellipsoidal uncertainty* (20), *and the level of conservativeness* Ω *of this approximation can be bounded as follows:*

(i) *In the case of a general-type ellipsoidal uncertainty* (20), *one has*

$$\Omega \leq \sqrt{31 + 2\ln\left(\sum_{j=1}^{k} \mathrm{Rank}(Q_j)\right)}. \tag{22}$$

Note that the right hand side in (22) *is* < 8, *provided that* $\sum\limits_{j=1}^{k} \mathrm{Rank}(Q_j) \leq 15{,}000{,}000$.

(ii) *In the case of box uncertainty (i.e.,* $\zeta^T Q_j \zeta = \zeta_j^2$, $1 \leq j \leq k = L \equiv \dim \zeta$*),*

$$\Omega \leq \frac{\pi}{2}.$$

(iii) *In the case of simple ellipsoidal uncertainty (*$k = 1$ *in* (20)*),*

$$\Omega = 1,$$

i.e., problem (21) *is equivalent to the robust counterpart of* (14), (20). □

3.3 Approximate robust counterparts of uncertain conic quadratic problems with ellipsoidal uncertainty

Consider an uncertain conic quadratic program

$$\left\{\min_x \left\{c^T x : \| A_i x + b_i \|_2 \leq \alpha_i^T x + \beta_i,\ i = 1, ..., m\right\} \mid \{(A_i, b_i, \alpha_i, \beta_i)\}_{i=1}^m \in \mathcal{U}\right\} \tag{23}$$

affected by *side-wise* uncertainty:

$$\mathcal{U} = \left\{ \{A_i, b_i, \alpha_i, \beta_i\}_{i=1}^m \mid \begin{array}{l} \{A_i, b_i\}_{i=1}^m \in \mathcal{U}^{\mathrm{left}}, \\ \{\alpha_i, \beta_i\}_{i=1}^m \in \mathcal{U}^{\mathrm{right}} \end{array} \right\} \tag{24}$$

and assume that

1. The left-hand-side uncertainty set is ellipsoidal:

$$\mathcal{U}^{\mathrm{left}} = \left\{ \left\{ (A_i, b_i) = (A_i^{\mathrm{n}}, b_i^{\mathrm{n}}) + \sum_{\ell=1}^{L} \zeta_\ell (A_i^\ell, b_i^\ell) \right\}_{i=1}^m \mid \zeta^T Q_j \zeta \leq 1,\ j = 1, ..., k \right\} \tag{25}$$

   where $Q_j \succeq 0$, $j = 1, ..., k$, and $\sum\limits_j Q_j \succ 0$;

2. The right-hand-side uncertainty set $\mathcal{U}^{\mathrm{right}}$ is bounded and *semidefinite-representable:*

$$\begin{array}{c} \mathcal{U}^{\mathrm{right}} = \left\{ \left\{ (\alpha_i, \beta_i) = (\alpha_i^{\mathrm{n}}, \beta_i^{\mathrm{n}}) + \sum\limits_{r=1}^{R} \eta_r (\alpha_i^r, \beta_i^r) \right\}_{i=1}^m \mid \eta \in \mathcal{V} \right\}, \\ \mathcal{V} = \{\eta \mid \exists u :\ P(\eta) + Q(u) - R \succeq 0\}, \end{array} \tag{26}$$

   where $P(\eta) : \mathbf{R}^R \to \mathbf{S}^N$, $Q(u) : \mathbf{R}^p \to \mathbf{S}^N$ are linear mappings taking values in the space $\mathbf{S}^N$ of symmetric $N \times N$ matrices and $R \in \mathbf{S}^N$.

Let us associate with (23) – (26) the semidefinite program

$$\text{minimize } c^T x$$

subject to

$$\left(\begin{array}{c|c|c} \tau_i - \sum\limits_{j=1}^{k} \lambda_{ij} & & [A_i^{\rm n} x + b_i^{\rm n}]^T \\ \hline & \sum\limits_{j=1}^{k} \lambda_{ij} Q_i & \begin{array}{c} [A_i^1 x + b_i^1]^T \\ \vdots \\ [A_i^L x + b_i^L]^T \end{array} \\ \hline A_i^{\rm n} x + b_i^{\rm n} & A_i^1 x + b_i^1 \quad \cdots \quad A_i^L x + b_i^L & \tau_i I \end{array}\right) \succeq 0, \; i = 1, ..., m,$$

$$\begin{array}{c} \lambda_{ij} \geq 0, \; i = 1, ..., m, \; j = 1, ..., k, \\ \tau_i \leq x^T \alpha_i^{\rm n} + \beta_i^{\rm n} + \mathrm{Tr}(R V_i), \; i = 1, ..., m, \\ P^*(V_i) = \begin{pmatrix} x^T \alpha_i^1 + \beta_i^1 \\ \vdots \\ x^T \alpha_i^R + \beta_i^R \end{pmatrix}, \; i = 1, ..., m, \\ Q^*(V_i) = 0, \; i = 1, ..., m, \\ V_i \succeq 0, \; i = 1, ..., m \end{array} \tag{27}$$

in variables $x, \lambda_{ij}, \tau_i, V_i$; here for a linear mapping $S(y) = \sum\limits_{i=1}^{q} y_i S_i : \mathbf{R}^q \to \mathbf{S}^N$, $S^*(Y) = \begin{pmatrix} \mathrm{Tr}(Y S_1) \\ \vdots \\ \mathrm{Tr}(Y S_q) \end{pmatrix} : \mathbf{S}^N \to \mathbf{R}^q$ is the conjugate mapping.

Theorem 3.2 [[6], Theorem 4.1] *Assume that the semidefinite representation of $\mathcal{V}$ in* (26) *is strictly feasible, i.e., there exist $\bar{\eta}$, $\bar{u}$ such that $P(\bar{\eta}) + Q(\bar{u}) - R \succ 0$. Then problem* (27) *is an approximate robust counterpart of the uncertain conic quadratic problem* (23) *with uncertainty* (24) – (26)*, and the level of conservativeness Ω of this approximation can be bounded as follows:*

(i) *In the case of a general-type ellipsoidal uncertainty* (25) *in the left hand side data, one has*

$$\Omega \leq \sqrt{31 + 2 \ln \left(\sum_{j=1}^{k} \mathrm{Rank}(Q_j) \right)}. \tag{28}$$

(ii) *In the case of box uncertainty in the left hand side data ($\zeta^T Q_j \zeta = \zeta_j^2$, $1 \leq j \leq k = L \equiv \dim \zeta$),*

$$\Omega \leq \frac{\pi}{2}.$$

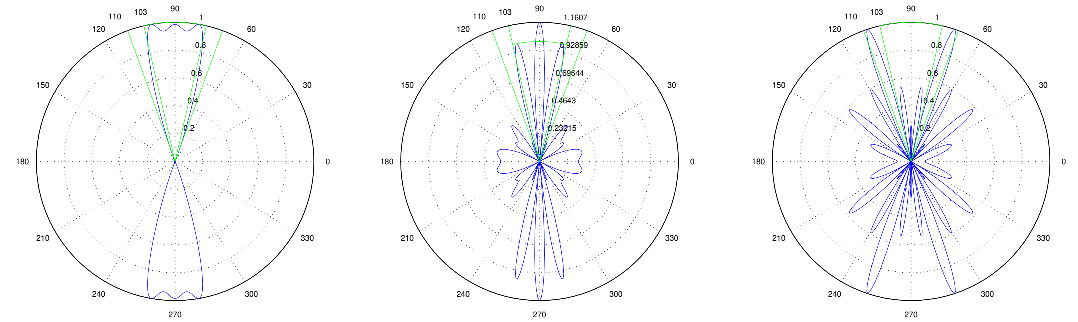
(iii) *In the case of simple ellipsoidal uncertainty in the left hand side data ($k = 1$ in* (25)*), $\Omega = 1$, so that problem* (27) *is equivalent to the robust counterpart of* (23) – (26). □

3.4 Example: Least Squares Antenna Design

To illustrate the potential of the Robust Optimization methodology as applied to conic quadratic problems, consider the Circular Antenna problem from Section 2.2 and assume that now our goal is to minimize the (discretized) L_2-distance from the synthesied diagram $\sum\limits_{j=1}^{40} x_j D_{\kappa_j}(\cdot)$ to the "ideal" diagram $D_*(\cdot)$ which is equal to 1 in the range $77^o \leq \theta \leq 90^o$ and is equal to 0 in the range $0^o \leq \theta \leq 70^o$. The associated problem is just the Least Squares problem

$$
\begin{array}{c}
\min_{\tau,x}\left\{\tau : \underbrace{\sqrt{\dfrac{\sum\limits_{\theta\in\Theta_{\rm cns}} D_x^2(\theta) + \sum\limits_{\theta\in\Theta_{\rm obj}} (D_x(\theta)-1)^2}{\mathrm{card}\,(\Theta_{\rm cns}\cup\Theta_{\rm obj})}}}_{\|D_*-D_x\|_2} \leq \tau\right\}, \\
D_x(\theta) = \sum\limits_{j=1}^{40} x_j D_{\kappa_j}(\theta)
\end{array}
\tag{29}
$$

The *Nominal Least Squares design* obtained from the optimal solution to this problem is completely unstable w.r.t. small implementation errors:



Dream	Reality	Reality
no errors	0.1% errors	2% errors
$\parallel D_* - D \parallel_2 = 0.014$	$\parallel D_* - D \parallel_2 \in [0.17, 0.89]$	$\parallel D_* - D \parallel_2 \in [2.9, 19.6]$

Nominal Least Squares design: dream and reality
Data over a 100-diagram sample

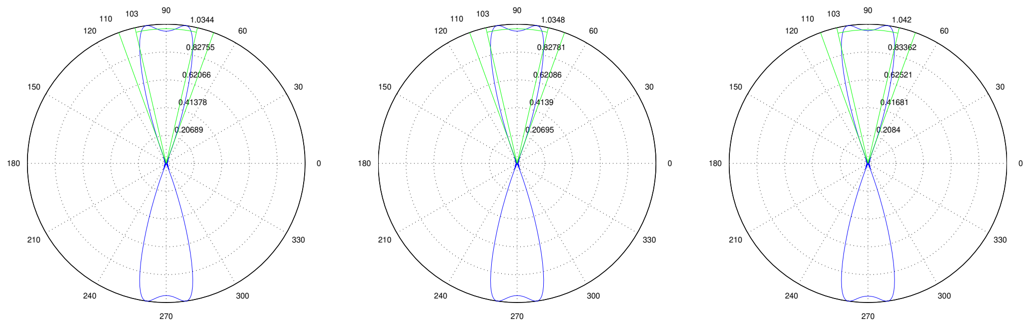
In order to take into account implementation errors (9), we should treat (29) as an uncertain conic quadratic problem

$$\left\{\min_{\tau,x}\{\tau : \parallel Ax - b \parallel_2 \leq \tau\}\,|A\in\mathcal{U}\right\}$$

with the uncertainty set of the form

$$\mathcal{U} = \{A = A^{\rm n} + A^{\rm n}\,\mathrm{Diag}(\delta)\mid \parallel \delta \parallel_\infty \leq \epsilon\};$$

in the results to follow, we use $\epsilon = 0.02$. The corresponding approximate RC (27) yields the Robust design as follows:



Dream	Reality	Reality
no errors	0.1% errors	2% errors
$\parallel D_* - D \parallel_2 = 0.025$	$\parallel D_* - D \parallel_2 \approx 0.025$	$\parallel D_* - D \parallel_2 \approx 0.025$

Robust Least Squares design: dream and reality
Data over a 100-diagram sample

Other examples of applications of Robust Optimization in quadratic and conic quadratic programming can be found in [10, 11].

4 Robust Semidefinite Programming

The robust counterpart of an uncertain semidefinite program

$$\left\{ \min_x \left\{ c^T x : A_0 + \sum_{i=1}^n x_i A_i \succeq 0 \right\} |(A_0, ..., A_n) \in \mathcal{U} \right\} \tag{30}$$

is NP hard already in the simple case where the uncertainty set $\mathcal{U}$ is an ellipsoid. Thus, we are forced to look for "moderately conservative" approximate robust counterparts of uncertain SDPs. The strongest result in this direction deals with the case of box uncertainty.

4.1 Approximate robust counterpart of uncertain semidefinite program affected by a box uncertainty

Theorem 4.1 [6] *Assume that the uncertain SDP* (30) *is affected by "box uncertainty", i.e.,*

$$\mathcal{U} = \left\{ (A_0, ..., A_n) = (A_0^{\rm n}, ..., A_n^{\rm n}) + \sum_{\ell=1}^L \zeta_\ell (A_0^\ell, ..., A_n^\ell) | \parallel \zeta \parallel_\infty \leq 1 \right\} \tag{31}$$

Then the semidefinite program

$$\min_{x, X^\ell} \left\{ c^T x : \begin{array}{c} X^\ell \succeq A_\ell[x] \equiv A_0^\ell + \sum\limits_{j=1}^n x_j A_j^\ell, \; \ell = 1, ..., L, \\ X^\ell \succeq -A_\ell[x], \; \ell = 1, ..., L, \\ \sum\limits_{\ell=1}^L X^\ell \preceq A_0^{\rm n} + \sum\limits_{j=1}^n x_j A_j^{\rm n} \end{array} \right\} \tag{32}$$

is an approximate robust counterpart of the uncertain semidefinite program (30) – (31), *and the level of conservativeness* Ω *of this approximation can be bounded as follows. Let*

$$\mu = \max_{1 \leq \ell \leq L} \max_x \mathrm{Rank}(A_\ell[x])$$

(note $\ell \geq 1$ in the max*). Then*

$$\Omega \leq \vartheta(\mu) = \begin{cases} \frac{\pi}{2}, & \mu \leq 2 \\ \frac{\pi\sqrt{k}}{2}, & \mu \geq 3 \end{cases}. \tag{33}$$

□

We see that *the level of conservativeness of the approximate robust counterpart* (32) *depends solely on the ranks of the "basic perturbation matrices" $A_\ell[\cdot]$ and is independent of other sizes of the problem.* This is good news, since in many applications the ranks of the perturbation matrices are $O(1)$. Consider an instructive example.

4.1.1 Example: Lyapunov Stability Synthesis

An LMI region is a set in the complex plane $\mathbf{C}$ representable as

$$\mathcal{C} = \{z \in \mathbf{C} \mid f_{\mathcal{C}}(z) \equiv L + Mz + M^T\bar{z} \prec 0\},$$

where $L = L^T$ and M are real $k \times k$ matrices and $\bar{z}$ is the complex conjugate of z. The simplest examples of LMI regions are:

1. *Open left half-plane*: $f_{\mathcal{C}}(z) = z + \bar{z}$;
2. *Open disk* $\{z \mid |z + q| \leq r\}$, $q \in \mathbf{R}, r > 0$: $f_{\mathcal{C}}(z) = \begin{pmatrix} -r & \bar{z} + q \\ z + q & -r \end{pmatrix}$;
3. *The interior of the sector* $\{z \mid \pi - \theta \leq |\arg(z)| \leq \pi\}$ $(-\pi < \arg(z) \leq \pi,\ 0 < \theta < \frac{\pi}{2})$:

$$f_{\mathcal{C}}(z) = \begin{pmatrix} (z + \bar{z})\sin\theta & -(z - \bar{z})\cos\theta \\ (z - \bar{z})\cos\theta & (z + \bar{z})\sin\theta \end{pmatrix};$$

4. *The stripe* $\{z \mid h_1 < \Re(z) < h_2\}$: $f_{\mathcal{C}}(z) = \begin{pmatrix} 2h_1 - (z + \bar{z}) & 0 \\ 0 & (z + \bar{z}) - 2h_2 \end{pmatrix}$.

It is known that

> (!) *The spectrum $\Sigma(A)$ of a real $n \times n$ matrix A belongs to $\mathcal{C}$ if and only if there exists $Y \in \mathbf{S}^n$, $Y \succ 0$, such that the matrix*
>
> $$\mathcal{M}[Y, A] = \left[L_{pq}Y + M_{pq}AY + M_{qp}YA^T\right]_{1 \leq p,q \leq k}$$
>
> *is negative definite.*
>
> Here and in what follows $[B_{pq}]_{1 \leq p,q \leq k}$ denotes the block matrix with blocks B_{pq}.

We can treat Y from (!) as a certificate of the inclusion $\Sigma(A) \subset \mathcal{C}$, and by homogeneity reasons we can normalize this certificate to satisfy the relations $Y \succeq I$, $\mathcal{M}[Y, A] \preceq -I$. From now on, we speak about normalized certificates only.

In many control applications, we are interested to solve the following problem:

(LSS[ρ]) *Given an LMI region $\mathcal{C}$ and an "uncertain interval matrix"*

$$\mathcal{AB}_\rho = \Big\{(A,B) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \mid |A_{ij} - A^{\rm n}_{ij}| \leq \rho C_{ij},\, |B_{ij} - B^{\rm n}_{i\ell}| \leq D_{i\ell},\ 1 \leq i,j \leq n, 1 \leq \ell \leq m\Big\}$$

find a linear feedback $K \in \mathbf{R}^{n\times m}$ and a matrix Y which certifies that the spectra of all matrices of the form $A + BK$, $(A,B) \in \mathcal{AB}_\rho$, belong to $\mathcal{C}$.

For example, in the case when $\mathcal{C}$ is the open left half-plane, our question becomes: find K such that all matrices of the form $A+BK$, $(A,B) \in \mathcal{AB}_\rho$, share a common "stability certificate" $Y \succeq I$:

$$(A + BK)Y + Y(A + BK)^T \preceq -I \quad \forall (A,B) \in \mathcal{AB}_\rho.$$

The interpretation is as follows: given an uncertain time-varying controlled linear system

$$\frac{d}{dt}x(t) = A(t)x(t) + B(t)u(t), \quad (A(t), B(t)) \in \mathcal{AB}_\rho \ \forall t,$$

we are looking for a linear feedback

$$u(t) = Kx(t)$$

such that the stability of the resulting uncertain closed loop system

$$\frac{d}{dt}x(t) = (A(t) + B(t)K)x(t) \qquad\qquad (A(t), B(t)) \in \mathcal{AB}_\rho \ \forall t,$$

can be certified by a quadratic Lyapunov function $x^T Y^{-1} x$:

$$\frac{d}{dt}(x^T(t) Y^{-1} x(t)) < 0$$

for all t and all nonzero trajectories $X(\cdot)$ of the closed loop system.

Note that replacing in the preceding story the open left half-plane with the open unit disk, we come to a completely similar problem for a discrete time uncertain controlled system.

Problem (LSS[ρ]) is NP-hard. It turns out, however, that with the aid of Theorem 4.1 we can build a fairly good tractable approximation of the problem. Note that $Y \succeq I$ and K form a solution of (LSS[ρ]) if and only if (Y,K) is a robust feasible solution of the uncertain matrix inequality

$$\{\{\mathcal{M}[Y, A + BK] \preceq -I\} | (A,B) \in \mathcal{AB}_\rho\}$$

with box uncertainty. Passing from the variables Y, K to $Y, Z = KY$, we convert this uncertain matrix inequality into an uncertain *Linear* Matrix Inequality (LMI)

$$\{ \Big[L_{pq}Y + M_{pq}(AY + BZ) + M_{qp}(AY + BZ)^T\Big]_{1\leq p,q\leq k} \preceq -I\} \tag{I[ρ]}$$

in variables Y, Z. Thus, $(Y, K = ZY^{-1})$ solves (LSS[ρ]) if and only if $Y \succeq I$ and (Y,Z) is a robust feasible solution of the uncertain LMI (I[ρ]) with a box uncertainty. In the notation from Theorem 4.1, $x = (Y,Z)$ and the perturbation matrices $A_\ell[x]$, $\ell = 1, ..., L$, are

$$\begin{array}{c} C_{ij}\left[M_{pq}E^{ij}Y + M_{qp}YE^{ji}\right]_{1\leq p,q\leq k},\ i,j = 1,...,n, \\ D_{i\ell}\Big[M_{pq}F^{i\ell}Z + M_{qp}Z^T(F^{i\ell})^T\Big]_{1\leq p,q\leq k},\ i = 1,...,n, \ell = 1,...,m, \end{array} \tag{34}$$

where E^{ij} are the standard $n \times n$ basic matrices ("1 in the cell ij, zeros in other cells") and $F^{i\ell}$ are the standard $n \times m$ basic matrices.

The approximate robust counterpart (32) of the uncertain LMI (I[ρ]) is the system of LMI's

$$
\begin{array}{c}
X^{ij} \succeq \pm C_{ij}\left[M_{pq}E^{ij}Y + M_{qp}YE^{ji}\right]_{1\le p,q\le k},\ i,j=1,...,n,\\
Z^{i\ell} \succeq \pm D_{i\ell}\left[M_{pq}F^{i\ell}Z + M_{qp}Z^T(F^{i\ell})^T\right]_{1\le p,q\le k},\ i=1,...,n,\ell=1,...,m,\\
\rho\left[\sum\limits_{i,j} X^{ij} + \sum\limits_{i,\ell} Z^{i\ell}\right] \preceq -I - \left[L_{pq}Y + M_{pq}(A^{\mathrm{n}}Y + B^{\mathrm{n}}Z) + M_{qp}(A^{\mathrm{n}}Y + B^{\mathrm{n}}Z)^T\right]_{1\le p,q\le k},\\
Y \succeq I
\end{array}
\tag{II[ρ]}
$$

Theorem 4.1 implies the following

Corollary 4.1 *Let μ be the maximum, over Y,Z and the indices i,j,ℓ, of the ranks of matrices* (34). *Then:*

(i) *If the system of LMI's* (II[ρ]) *is solvable, so is the problem* (LSS[ρ]), *and every solution* $(Y,Z,...)$ *of the former system yields a solution* $(Y,K=ZY^{-1})$ *of the latter problem;*

(ii) *If the system of LMI's* (II[ρ]) *is unsolvable, so is the problem* (LSS[$\vartheta(\mu)\rho$]), *cf.* (33). □

The point is that the parameter μ in Corollary 4.1 normally is a small integer; specifically, we always have $\mu \le 2k$, where k is the size of the matrices L,M specifying the LMI region $\mathcal{C}$ in question; for the most important cases where $\mathcal{C}$ is, respectively, the open left half-plane and the open unit disk, we have $\mu=2$, i.e., (II[ρ]) is $\frac{\pi}{2}$-conservative approximation of (LSS[ρ]).

There are many other applications of Theorem 4.1 to systems of LMI's arising in Control and affected by an interval data uncertainty. Usually the structure of such a system ensures that when perturbing a single data entry, the right hand side of every LMI is perturbed by a matrix of a small rank, which is exactly the case considered in Theorem 4.1.

4.2 Approximate robust counterpart of uncertain semidefinite program affected by a ball uncertainty

Theorem 4.2 [9] *Assume that* (30) *is affected by a ball uncertainty:*

$$
\mathcal{U} = \left\{(A_0,...,A_n) = (A_0^{\mathrm{n}},...,A_n^{\mathrm{n}}) + \sum_{\ell=1}^{L}\zeta_\ell(A_0^\ell,...,A_n^\ell) \mid \parallel \zeta \parallel_2 \le 1\right\}
\tag{35}
$$

Then the semidefinite program

$$
\min_{x,F,G}\left\{c^Tx : \begin{array}{c}
\begin{pmatrix}
G & A_1[x] & A_2[x] & \dots & A_L[x]\\
A_1[x] & F & & & \\
A_2[x] & & F & & \\
\vdots & & & \ddots & \\
A_L[x] & & & & F
\end{pmatrix} \succeq 0,\\
F + G \preceq 2\left(A_0^{\mathrm{n}} + \sum\limits_{j=1}^{n} x_j A_j^{\mathrm{n}}\right)
\end{array}\right\}
\tag{36}
$$

(cf. (32)*) is an approximate robust counterpart of the uncertain semidefinite program* (30), (35), *and the level of conservativeness Ω of this approximation does not exceed*

$$
\min\left[\sqrt{M},\sqrt{L}\right],
$$

where M is the row size of the matrices A_i in (30). □

4.3 Uncertain semidefinite programs with tractable robust counterparts

The only known "general" geometry of the uncertainty set $\mathcal{U}$ in (30) which leads to a computationally tractable robust counterpart of (30) is the trivial one, namely, a polytope given as a convex hull of a finite set:

$$\mathcal{U} = \mathrm{Conv}\left\{(A_0^1, A_1^1, ..., A_n^1), ..., (A_0^L, A_1^L, ..., A_n^L)\right\};$$

in this case, the robust counterpart of (30) merely is the semidefinite program

$$\min_x \left\{ c^T x : A_0^\ell + \sum_{j=1}^n x_j A_j^\ell \succeq 0,\ \ell = 1, ..., L \right\}.$$

There are, however, important special cases where the robust counterpart of an uncertain semidefinite program with a "nontrivial" uncertainty set is computationally tractable. Let us look at two most important examples.

4.3.1 Example: "Rank 2" ellipsoidal uncertainty and robust truss topology design

Consider a "nominal" (certain) semidefinite problem

$$\min_x \left\{ c^T x : A^{\mathrm{n}}[x] \equiv A_0^{\mathrm{n}} + \sum_{j=1}^n x_j A_j^{\mathrm{n}} \succeq 0 \right\}, \tag{37}$$

where A_j^{n} are symmetric $M \times M$ matrices. Let d be a fixed nonzero M-dimensional vector, and let us call *rank 2 perturbation of* $A^{\mathrm{n}}[\cdot]$ *associated with* d a perturbation of the form

$$A^{\mathrm{n}}[x] \mapsto A^{\mathrm{n}}[x] + b(x)d^T + db^T(x),$$

where $b(x)$ is an affine function of x taking values in $\mathbf{R}^M$. Consider the uncertain semidefinite problem obtained from (37) by all possible rank 2 perturbations associated with a fixed vector $d \neq 0$ and with $b(\cdot)$ varying in a (bounded) ellipsoid:

$$\left\{ \min_x \left\{ c^T x : A^{\mathrm{n}}[x] + [\sum_{\ell=1}^L u_\ell b^\ell(x)]d^T + d[\sum_{\ell=1}^L u_\ell b^\ell(x)]^T \succeq 0 \right\} | u^T u \leq 1 \right\}, \tag{38}$$

where $b^\ell(x)$ are given affine functions of x taking values in $\mathbf{R}^M$.

Proposition 4.1 [[4], Proposition 3.1] *The robust counterpart of the uncertain semidefinite problem* (38) *is equivalent to the following SDP problem:*

$$\min_{x,\lambda} \left\{ c^T x : \begin{pmatrix} \lambda I & [b^1(x); b^2(x); ...; b^k(x)]^T \\ [b^1(x); b^2(x); ...; b^k(x)] & A^{\mathrm{n}}[x] - \lambda dd^T \end{pmatrix} \succeq 0 \right\}. \tag{39}$$

□

Application: Robust Truss Topology Design. The situation described in Proposition 4.1 arises, e.g., in the truss topology design problem. In this problem, we are interested to design a *truss* – a construction comprised of thin elastic *bars* of a given total weight linked with each other at *nodes* from a given finite 2D or 3D nodal set – in such a way that the resulting construction is most rigid w.r.t. a given external load (a collection of forces distributed along the nodes). For a detailed description of the model, see, e.g., [1]. In the Truss Topology Design problem, the "nominal" program (37) can be posed as the following SDP:

$$\min_{\tau,t}\left\{\tau:\ \begin{array}{c}\left(\begin{array}{c|c|c}\tau & f^T & \\ \hline f & \sum\limits_{i=1}^{p} t_i A^i & \\ \hline & & \mathrm{Diag}(t)\end{array}\right)\succeq 0,\\ \sum\limits_{i=1}^{p} t_i = 1,\end{array}\right\}; \tag{40}$$

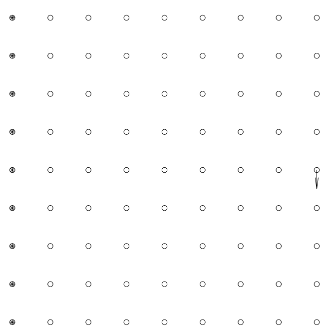
here p is the number of tentative bars, the design variables $t_1,...,t_p$ are volumes of these bars, $A^i \succeq 0$ are matrices readily given by the geometry of the nodal set, f represents the load of interest, and τ stands for the *compliance* of the truss w.r.t. f (the less is the compliance, the more rigid is the truss when loaded by f). In the robust setting of the TTD problem, we treat the load f as an uncertain data which runs through a given ellipsoid

$$E = \{Qu \mid u \in \mathbf{R}^k, u^T u \leq 1\}.$$

This is nothing but a rank 2 perturbation of the nominal problem (40) with properly chosen $b^\ell(x)$ (in fact independent of x). The robust counterpart of the resulting uncertain semidefinite problem is the semidefinite program

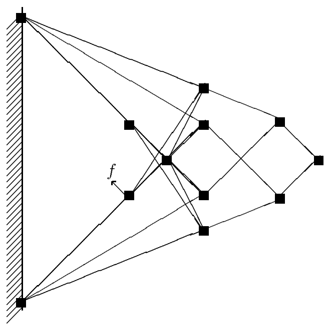
$$\min_{\tau,t}\left\{\tau:\ \begin{array}{c}\left(\begin{array}{c|c|c}\tau I & Q^T & \\ \hline Q & \sum\limits_{i=1}^{p} t_i A^i & \\ \hline & & \mathrm{Diag}(t)\end{array}\right)\succeq 0,\\ \sum\limits_{i=1}^{p} t_i = 1,\end{array}\right\}; \tag{41}$$

For discussing the role played by the Robust Optimization methodology in the context of Truss Topology Design and for illustrative examples, see [2]; here we present a single example as follows. We want to design a minimum-compliance 2D cantilever arm with the 9×9 nodal set, boundary conditions and a loading scenario as follows:



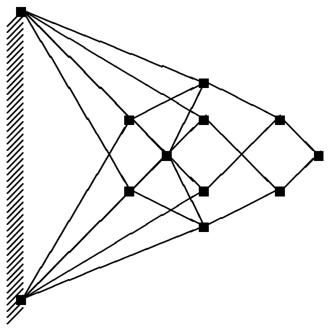
9×9 nodal grid and the load of interest f^{n}
[the most left nodes are fixed]

Solving the associated problem (40), we come to the *Nominal design* which is completely unstable w.r.t. small occasional loads:



Nominal Design: Dream and Reality
The compliance w.r.t. $f^{\rm n}$ is 1.000
The compliance w.r.t. f with $\| f \|_2 = 0.01 \| f^{\rm n} \|_2$ is 17 times larger!

In order to get a "reliable" truss, we reduce the original 9×9 nodal set to its 12-element subset formed by the nodes which are present in the nominal design, and then replace the load of interest $f^{\rm n}$ with the uncertainty set chosen as the smallest volume ellipsoid of loads $\mathcal{F}$ spanned by $f^{\rm n}$ and all "small occasional loads" f, $\| f \|_2 \leq 0.1 \| f^{\rm n} \|_2$, distributed along the reduced nodal set. Solving the associated robust counterpart (41), we come to the Robust design as follows



Robust Design: Dream and Reality
The compliance w.r.t. $f^{\rm n}$ is 1.0024

Design	Compliance w.r.t. $f^{\rm n}$	Maximum compliance w.r.t. $f \in \mathcal{F}$
Nominal	1.000	>3,360
Robust	1.002	1.003

We see that the truss corresponding to the Robust design is nearly as good w.r.t. the load of interest as the truss obtained from the Nominal design; at the same time, the robust design is incomparably more stable w.r.t. small occasional loads distributed along the nodes of the truss.

4.3.2 Example: norm-bounded perturbations and Lyapunov stability analysis

Consider a pair of specific uncertain LMI's

$$\begin{array}{ll} (a) & \left\{ R^T Z P \Delta Q + Q^T \Delta^T P^T Z^T R \preceq Y \mid \| \Delta \|_2 \leq \rho \right\}, \\ (b) & \left\{ R^T Z P \Delta Q + Q^T \Delta^T P^T Z^T R \preceq Y \mid \| \Delta \| \leq \rho \right\}; \end{array} \qquad (42)$$

here $Y \in \mathbf{S}^n$ and $Z \in \mathbf{R}^{p\times q}$ are variable matrices, $\Delta \in \mathbf{R}^{\mu\times\nu}$ is a perturbation, and $Q \in \mathbf{R}^{\nu\times n}$, $P \in \mathbf{R}^{q\times\mu}$ are constant matrices, $\| \Delta \|_2 = \sqrt{\mathrm{Tr}(\Delta\Delta^T)}$ is the Frobenius norm, and $\| \Delta \| = \max\{\| \Delta x \|_2 : \| x \|_2 \leq 1\}$ is the spectral norm of a matrix.

Proposition 4.2 [8] *Let $Q \neq 0$. The sets of robust feasible solutions of the uncertain LMI's (42.a), (42.b) coincide with each other and with the set*

$$\left\{ (Y,Z) | \exists \lambda \in \mathbf{R} : \quad \underbrace{\begin{pmatrix} Y - \lambda Q^T Q & \rho R^T Z P \\ \rho P^T Z^T R & \lambda I \end{pmatrix} \succeq 0}_{(*)} \right\}. \tag{43}$$

Thus, the robust counterparts of both (42.a), (42.b) are equivalent to the explicit LMI $(*)$ in variables Y, Z, λ. □

As an application example, consider the Lyapunov Stability Analysis problem for an uncertain controlled linear dynamic system

$$\begin{array}{c} \boxed{\begin{array}{rcll} \frac{d}{dt}x(t) & = & Ax(t) + B(t)u(t) & \text{[state equations]} \\ y(t) & = & Cx(t) & \text{[observer]} \\ u(t) & = & K(t)y(t) & \text{[feedback]} \end{array}} \\ \Downarrow \\ \boxed{\frac{d}{dt}x(t) = [A + BK(t)C]x(t) \quad \text{[closed loop system]}} \end{array} \tag{44}$$

with $K(t)$ varying in the uncertainty set

$$\mathcal{U} = \{K \mid \| K - K^{\mathrm{n}} \| \leq \rho\}. \tag{45}$$

Our goal is to find, if possible, a quadratic Lyapunov function $L(x) = x^T X x$ for the resulting uncertain closed loop system, i.e., to find a matrix $X \succ 0$ such that

$$\frac{d}{dt}\left[x^T(t)Xx(t)\right] < 0$$

for all t and all nonzero trajectories of the closed loop system, whatever is $K(t)$ taking values in $\mathcal{U}$.

It is easily seen that a necessary and sufficient condition for a matrix $X \succ 0$ to yield a quadratic Lyapunov function for the system (44) – (45) is to satisfy the semi-infinite system of strict matrix inequalities

$$[A + BKC]^T X + X[A + BKC] \prec 0 \quad \forall K \in \mathcal{U}; \tag{46}$$

by homogeneity reasons, we loose nothing when restricting ourselves with X's normalized by the requirement $X \succeq I$ and satisfying the following normalized version of (46):

$$[A + BKC]^T X + X[A + BKC] \preceq -I \quad \forall K \in \mathcal{U}. \tag{47}$$

Thus, what we are looking for is a robust feasible solution of the uncertain system of LMI's

$$\left\{ \left\{ \begin{array}{c} X \succeq I, \\ [A + B(K^{\mathrm{n}} + \Delta)C]^T X + X[A + B(K^{\mathrm{n}} + \Delta)C] \preceq -I \end{array} \right\} \middle| \; \| \Delta \| \leq \rho \right\} \tag{48}$$

in matrix variable X. According to Proposition 4.2, the robust counterpart of the uncertain system (48) is equivalent to the explicit system of LMI's

$$\begin{array}{c}\begin{pmatrix} -I-[A+BK^{\rm n}C]^TX-X[A+BK^{\rm n}C]-\lambda C^TC & \rho XB \\ \rho B^TX & \lambda I \end{pmatrix} \succeq 0, \\ X \succeq I \end{array} \tag{49}$$

in variables X, λ.

Note that what is important in the above construction is that as a result of the uncertainty affecting (44), the possible values of the matrix $\widetilde{A} = A + BKC$ of the closed loop system form the set of the type $\{P + Q\Delta R \mid \| \Delta \| \leq \rho\}$. Note that we would get drifts of $\widetilde{A}$ of exactly this type when assuming, instead of a norm-bounded perturbation in the feedback, that among the four matrices A, B, C, K participating in (44) three are fixed, and the remaining matrix, let it be called S, is affected by a norm-bounded uncertainty, specifically, S runs through the set

$$\{S^{\rm n} + U\Delta V \mid \| \Delta \| \leq \rho\}$$

or through the set

$$\{S^{\rm n} + U\Delta V \mid \| \Delta \|_2 \leq \rho\}.$$

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