INTERIOR-POINT METHODS FOR SECOND-ORDER CONE PROGRAMMING AND SEMIDEFINITE PROGRAMMING

ROBERT J. VANDERBEI

We begin the talk by defining for the nonspecialist semidefinite (SDP) and second-order cone (SOCP) programming. Examples, with lots of pictures, will be given of real-world problems that fall into these categories of optimization problems.

Recently, a variety of customized interior-point methods for SDP and SOCP have appeared. In this talk we consider the question of how to apply a general purpose interior-point method, i.e. one designed for nonlinear programming (NLP), to the solution of these problems.

SOCPs are characterized by having constraints of the form

$$\|u\| \le t,$$

where u is a vector of variables in $\mathbb{R}^k$, t is a scalar variable, and the norm is the Euclidean norm. Using constraints expressed in this way, a general purpose interior-point method works fine except in cases where some of the constraints have $u = 0$ at optimality. When this happens, and that is often, the fact that the Euclidean norm fails to be differentiable at $u = 0$ becomes a serious issue. We shall show that reexpressing SOCP constraints as

$$\begin{aligned} t - \frac{\|u\|^2}{t} &\ge 0 \\ t &\ge 0 \end{aligned}$$

provides an effect solution to this nonsmoothness problem.

In SDP, the variables form a symmetric matrix X and all constraints are linear equalities except for the constraint that X be positive semidefinite. To formulate an SDP as a general nonlinear programming problem, we must express the positive semidefiniteness of X using a system of inequalities. The system we propose to use is

$$d_j(X) \ge 0, \qquad j = 1, 2, \ldots, n,$$

where d_j denotes the j-th diagonal element of the matrix D in an LDL^T decomposition of X. The advantage provided by these functions is that they are smooth and concave in the interior of the space of positive semidefinite matrices. For a 2×2 matrix,

$$X = \begin{bmatrix} z & y \\ y & x \end{bmatrix},$$

these conditions can be written explicitly as

$$\begin{aligned} x - \frac{y^2}{z} &\ge 0 \\ z &\ge 0, \end{aligned}$$

revealing the close connection between the proposed methods for handling SDP and SOCP.

In the talk we shall describe the changes made to LOQO, a code for general NLP, to accomodate these problems and we shall present preliminary computational results.

This work is joint with Hande Yurttan Benson.

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