Randomization has become a pervasive technique in combinatorial optimization. Randomization has been used to develop algorithms that are faster, simpler, and/or better-performing than previous deterministic algorithms. I will survey some applications of randomization to cut and flow problems in undirected graphs. Our work uses four important randomization techniques:

Random Selection, which lets us easily choose a "typical" element of a set, avoiding rare "bad" elements;

Random Sampling, which provides a quick way to build a small, representative subproblem of a larger problem for quick analysis;

Randomized Rounding, which lets us transform fractional problem solutions into integral ones; and

Monte Carlo Simulation, which lets us estimate the probabilities of interesting events.

We apply these techniques to various optimization problems on undirected graphs. Among the graph problems we address are finding (or approximating) a maximum flow, determining the connectivity (minimum cut) of a graph, network design, and estimating the reliability (disconnection probability) of a network with random edge failures. We will attempt to cover material sufficient that demonstrates all four of the techniques listed above.

We will begin by presenting a simple minimum cut algorithm [KS96] that demonstrates the power of random selection while laying the groundwork for several other techniques we will use later. While the best deterministic bound for minimum cuts is roughly $O(mn)$ on an m-edge, n-vertex graph, this randomized algorithm runs in roughly $O(n^2)$ time.

The minimum cut algorithm extends to some combinatorial and algorithmic results on near-minimum cuts; applying these ideas to some Monte Carlo simulation methods yields a *fully polynomial randomized approximation scheme* for the problem of estimating the failure (disconnection) probability of a network suffering random edge failures [Kar99]. Although the reliability problem is NP-complete, our algorithm produces an estimate that is accurate to within an arbitrary relative error of $(1 \pm \epsilon)$ in time polynomial in n and $1/\epsilon$.

Another application of these results is to show good performance for a class of maximum flow *approximation* algorithms based on *random sampling* [Kar94, BK96, Kar98a]. By accepting a flow which is merely *approximately* maximum, we are able to solve flow problems substantially faster than the best known exact bounds. In a separate application, we give *randomized rounding* algorithms for *network design* problems that aim to build a network of minimum cost satisfying certain connectivity properties.

By developing techniques for "repairing" our approximate solutions, we can also solve certain problems exactly. In one application [Kar96], we give an algorithm for finding a minimum cut in essentially linear ($O(m \log^3 m)$) time. We also develop exact maximum flow algorithms [Kar97, Kar98a, KL98] that improve on the best know bounds for finding flow in graphs with unit capacity edges—instead of the classical $O(n^3)$ bound, we achieve a bound of $O(n^{2.3})$.

Some of this work is joint with András A. Benczúr and Matt Levine.

While we will not be able to detail all these results, we will survey as many as possible while diving deeply into a few results to give a sense of the techniques. We will also attempt to touch upon related work [Kar98b, KKT95, KM97, CGK+97, KT97, AKK99, BK98]

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