Randomized Assortment Optimization

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The Assortment Optimization Problem

£3 profit

£10 profit

£1 profit
The Assortment Optimization Problem

£3 profit

£10 profit

£1 profit

Group 1 (50%)

100% 100%

Group 2 (50%)

75% 25%

25% 75%
The Assortment Optimization Problem

£5.25 expected profit

£3 profit

£10 profit

£1 profit

Group 1 (50%)

100% → 100% → 12:20

Group 2 (50%)

12:20

75% → 75% → 25% → 25%
The Assortment Optimization Problem

- **£2.00** expected profit
- **£3** profit
- **£10** profit
- **£1** profit

**Group 1 (50%)**
- 100% → 100% → 12:20

**Group 2 (50%)**
- 75% → 75%
- 25% → 25%
The Assortment Optimization Problem

- **£5.50 expected profit**
- **£3 profit**
- **£10 profit**
- **£1 profit**

Group 1 (50%)
- 100% of Group 1 customers purchase £3 phones
- 100% of Group 1 customers purchase £10 phones

Group 2 (50%)
- 25% of Group 2 customers purchase £3 phones
- 75% of Group 2 customers purchase £10 phones

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- **£5.50 expected profit**
- **£3 profit**
- **£10 profit**
- **£1 profit**

Group 1 (50%)
- 100% of Group 1 customers purchase £3 phones
- 100% of Group 1 customers purchase £10 phones

Group 2 (50%)
- 25% of Group 2 customers purchase £3 phones
- 75% of Group 2 customers purchase £10 phones
The Assortment Optimization Problem

★ Multinomial logit model:
  ★ Choice model: Luce ('59), Plackett ('75)
  ★ Optimization: Talluri & van Ryzin ('04), Rusmevichientong et al. ('10) and Davis et al. ('13) for cardinality constraints

★ Markov chain model:
  ★ Choice model: Zhang & Cooper ('05), Blanchet et al. ('16), Simsek & Topaloglu ('18) for estimation
  ★ Optimization: Blanchet et al. ('16), Feldman & Topaloglu ('17), Désir et al. ('20) for cardinality constraints

★ Preference ranking model:
  ★ Choice model: Farias et al. ('13), van Ryzin & Volcano ('15, '17)
  ★ Optimization: Honhon et al. ('12), Aouad et al. ('18, '21), Paul et al. ('18), Bertsimas & Mišić ('19)
The Assortment Optimization Problem

Group 1 (50%)

Group 2 (50%)

£3 profit

£10 profit

£1 profit

£5.50 exp. profit
The Assortment Optimization Problem

Group 1 (50%)

Group 2 (50%)

Group 1 (25%)

Group 2 (75%)

£3 profit

£10 profit

£1 profit

£5.50 exp. profit
The Assortment Optimization Problem

Group 1 (50%)
£3 profit

Group 2 (50%)
£10 profit

£3.25 exp. profit
Group 2 (75%)

£5.50 exp. profit
Group 1 (25%)

£1 profit
The bias-variance tradeoff in choice models:
The bias-variance tradeoff in choice models:
The bias-variance tradeoff in choice models:

- **MNL model**
  - large bias
  - small variance

- **MC/PR models**
  - small bias
  - large variance

Uncertainty

\[
\begin{array}{c|cccc}
\lambda_0 & \emptyset & 1 & 0 & 0 \\
\lambda_1 & p_{10} & 0 & p_{12} & p_{13} \\
\lambda_2 & p_{20} & p_{21} & 0 & p_{23} \\
\lambda_3 & p_{30} & p_{31} & p_{32} & 0 \\
\end{array}
\]
The bias-variance tradeoff in choice models:

MNL model

- large bias
- small variance

MC/PR models

- small bias
- large variance

Bishop (2006), Pattern Recognition and Machine Learning (Springer).
Hastie et al. (2009), The Elements of Statistical Learning: Data Mining, Inference, and Prediction (Springer).
Combining estimation with optimization amplifies errors:

MC/PR models

- Small bias
- Large variance

"Post-decision disappointment"  
"error-maximization effect of optimization"

The robust optimization paradigm to combat estimation errors:
The robust optimization paradigm to combat estimation errors:

1. $v_1, v_2, v_3$

2. assortment optimization

parameter estimation
The robust optimization paradigm to combat estimation errors:

\[ (v_1, v_2, v_3) \in \mathcal{V} \]

\[ v^* \in \mathcal{V} \]

1. Parameter estimation
2. Assortment optimization
3. Uncertainty set estimation

Ben-Tal et al. (2009), Robust Optimization (Princeton University Press).
Bertsimas & den Hertog (2022), Robust and Adaptive Optimization (Dynamic Ideas).
The Robust Assortment Optimization Problem

Group 1 (25-75%)

Group 2 (25-75%)

£3 profit

£10 profit

£1 profit
The Robust Assortment Optimization Problem

£3.25 worst-case expected profit

£3 profit

£10 profit

£1 profit

worst-case: 25% : 75%

Group 1 (25-75%)

100% → 100%

Group 2 (25-75%)

75% → 75% → 25% → 25%
The Robust Assortment Optimization Problem

£1.50 worst-case expected profit

£3 profit

£10 profit

£1 profit

worst-case: 25% : 75%

Group 1 (25-75%)

100% → 100% → 12:20

Group 2 (25-75%)

75% ← 25% ← 75% ← 25%
The Robust Assortment Optimization Problem

£4.13 worst-case expected profit

- £3 profit
- £10 profit
- £1 profit

Group 1 (25-75%)

Group 2 (25-75%)

worst-case: 75% : 25%
The Robust Assortment Optimization Problem

**Multinomial logit model:**
- Rusmevichientong & Topaloglu ('12) solve robust assortment optimization problem under uncertain product valuations; revenue-ordered assortments remain optimal

**Markov chain model:**
- Désir et al. ('21) use robust MDP-type algorithms to solve robust assortment optimization problem under uncertain arrival rates and transition probabilities

**Preference ranking model:**
- Farias et al. ('13) estimate worst-case revenues for fixed assortment under uncertain preference distributions
- Bertsimas & Mišić ('17) solve robust assortment optimization problem under uncertain preference distributions
The Randomized Robust Assortment Optimization Problem

\[
\text{w. p. } \frac{1}{2}
\]
The Randomized Robust Assortment Optimization Problem
The Randomized Robust Assortment Optimization Problem

$\Pr[\text{expected profit}] \geq 1/2 \times \£4.81 + 1/2 \times \£5.38$

$\Pr[\text{nominal}] \geq \£4.81 \text{ worst-case expected profit}$

$\Pr[\text{nominal}] \geq (\£5.38 \text{ nominal})$
The Randomized Robust Assortment Optimization Problem

![Diagram showing two scenarios with probabilities and expected profits](image)

- w. p. $\frac{1}{3}$: £5.33 worst-case expected profit
- w. p. $\frac{2}{3}$: (£5.42 nominal)
The Randomized Robust Assortment Optimization Problem

\[
\frac{1}{3} \times + \frac{2}{3} \times \]

\[\£5.33 \text{ worst-case expected profit}\]

profits of nominal policy decrease by up to 41% in the worst case
The Randomized Robust Assortment Optimization Problem

$$\frac{1}{3} * \frac{1}{5.33} \text{ worst-case expected profit} + \frac{2}{3} * \frac{1}{5.33} \text{ worst-case expected profit}$$

16% uplift in worst-case profits if we implement robust instead of nominal policy
The Randomized Robust Assortment Optimization Problem

\[
\frac{1}{3} \times \text{profit} + \frac{2}{3} \times \text{profit} = \boxed{\text{\£5.33 worst-case expected profit}}
\]

- 16% uplift in worst-case profits
- 22% uplift in worst-case profits if we implement random instead of deterministic policy
The **Randomized Robust** Assortment Optimization Problem

\[
\frac{1}{3} * + \frac{2}{3} * \quad \text{£5.33 worst-case expected profit}
\]

- **16% uplift in worst-case profits**
- **22% uplift in worst-case profits**
- **3% less than nominal performance**
The **Randomized Robust** Assortment Optimization Problem

Why does randomization help?
Why does randomization help?

1. Mathematical Interpretation:
The **Randomized Robust** Assortment Optimization Problem

Why does randomization help?

1. **Mathematical Interpretation:**

   - £4.13
   - £3.25
   - £1.50
The Randomized Robust Assortment Optimization Problem

Why does randomization help?

1 Mathematical Interpretation:

- Probability of 1+2: £1.50
- Probability of 2+3: £4.13
- Probability of 1+2: £3.25

[Diagram showing probabilities and profits]
The **Randomized Robust** Assortment Optimization Problem

Why does randomization help?

1. **Mathematical Interpretation:**

   - **Probability of 1+2:** £3.25 (25%)
   - **Probability of 2+3:** £4.13 (75%)
   - **Probability of 1+2:** £1.50 (25%)

   **Graphical Representation:**
   - 3-dimensional space with axes labeled "probability of 1+2," "probability of 2+3," and "profits.
   - Blue pyramid showing the distribution of profits.
   - Red circle indicating the optimal point with the given probabilities and outcomes.
The Randomized Robust Assortment Optimization Problem

Why does randomization help?

1. Mathematical Interpretation:

- Probability of 1+2: 25%  
- Probability of 2+3: 75%

- £3.25
- £4.13
- £1.50

Probability of 1+2: 25%
Probability of 2+3: 75%
Why does randomization help?

Mathematical Interpretation:

- Probability of 1+2: £3.25
- Probability of 2+3: £4.13
- Probability of 1+2: £1.50

Income distribution:

- 25%: £3.25
- 75%: £4.13
- 25%: £1.50
- 75%: £0

1/3 + 2/3
Why does randomization help?

2. “Game-Theoretic” Interpretation:
Why does randomization help?

2. "Game-Theoretic" Interpretation:

deterministic robust
The \textbf{Randomized Robust} Assortment Optimization Problem

Why does \textit{randomization} help?

\begin{itemize}
  \item \textbf{2} \textit{“Game-Theoretic”} Interpretation:
\end{itemize}

\begin{itemize}
  \item \textbf{deterministic robust}
  \item \textbf{randomized robust}
\end{itemize}

\begin{itemize}
  \item $p_1$
  \item $p_2$
  \item $p_3$
\end{itemize}
Why does randomization help?

2. “Game-Theoretic” Interpretation:

deterministic robust  \rightarrow \text{randomized robust}
The Randomized Robust Assortment Optimization Problem

Why does randomization help?

3 “Managerial” Interpretation:
The Randomized Robust Assortment Optimization Problem

Why does randomization help?

3 “Managerial” Interpretation:

- $p_1$
- $p_2$
- $p_3$

randomization = diversification
The Randomized Robust Assortment Optimization Problem

Why does randomization help?

“Managerial” Interpretation:

randomization = diversification

But divide your investments among many places, for you do not know what risks might lie ahead. (Book of Ecclesiastes)

My ventures are not in one bottom trusted, Nor to one place; nor is my whole estate Upon the fortune of this present year: Therefore, my merchandise makes me not sad. (Merchant of Venice)
1. When does randomization improve the worst-case profit?
1. When does \textit{randomization} improve the \textit{worst-case profit}?

2. Is \textit{in-sample improvement} = \textit{out-of-sample improvement}?
1. When does randomization improve the worst-case profit?

2. Is in-sample improvement = out-of-sample improvement?

3. How can we compute optimal randomized assortments?
When does randomization improve the worst-case profit?

Is in-sample improvement = out-of-sample improvement?

How can we compute optimal randomized assortments?

How can we implement optimal randomized assortments?
1. Implementing Randomized Assortments
2. When Does Randomization Help?
3. Computing Randomized Assortments
4. Numerical Experiments
Implementation: The E-Commerce Setting

Randomization across different users:

Each user's experience can be kept consistent via cookies.
Implementation: The Brick-and-Mortar Setting

Randomization across retail stores:

Possible for larger chains, not suitable for individual stores.

https://www.bain.com/insights/successful-a-b-tests-in-retail-hinge-on-these-design-considerations/
Agenda

1. Implementing Randomized Assortments
2. When Does Randomization Help?
3. Computing Randomized Assortments
4. Numerical Experiments
(Nominal) Assortment optimization problem: \((N, S, C, r)\) where

\[N = \{1, \ldots, N\}\]: set of products

\[S \subseteq \{S : S \subseteq N\}\]: set of admissible assortments

\[C : S \rightarrow \Delta(N_0)\]: choice model; \(C(i | S) = 0\) if \(i \notin S\)

\[r = (r_1, \ldots, r_N)\]: product prices
(Nominal) Assortment optimization problem: \((\mathcal{N}, \mathcal{S}, \mathcal{C}, r)\) where

\* \(\mathcal{N} = \{1, \ldots, N\}\): set of products
\* \(\mathcal{S} \subseteq \{S : S \subseteq \mathcal{N}\}\): set of admissible assortments
\* \(\mathcal{C} : \mathcal{S} \rightarrow \Delta(\mathcal{N}_0)\): choice model; \(\mathcal{C}(i | S) = 0\) if \(i \notin S\)
\* \(r = (r_1, \ldots, r_N)\): product prices

\[
R_{\text{nom}}^* = \max_{S \in \mathcal{S}} R(S)
\]

where \(R(S) = \sum_{i \in S} r_i \cdot \mathcal{C}(i | S)\)
Robust Assortment optimization problem: \((\mathcal{N}, \mathcal{S}, \mathcal{C}, \mathcal{U}, r)\) where

\(\mathcal{U}\): uncertainty set

\(\mathcal{C}: \mathcal{S} \times \mathcal{U} \to \Delta(\mathcal{N}_0)\): choice model; \(\mathcal{C}(i|S, u) = 0\) if \(i \notin S\)
Robust Assortment optimization problem: $(\mathcal{N}, \mathcal{S}, \mathcal{C}, \mathcal{U}, r)$ where

- $\mathcal{U}$: uncertainty set
- $\mathcal{C} : \mathcal{S} \times \mathcal{U} \to \Delta(\mathcal{N}_0)$: choice model; $\mathcal{C}(i \mid S, u) = 0$ if $i \notin S$

\[
R^{*}_{\text{det}}(\mathcal{U}) = \max_{S \in \mathcal{S}} \min_{u \in \mathcal{U}} R(S, u)
\]

where $R(S, u) = \sum_{i \in S} r_i \cdot \mathcal{C}(i \mid S, u)$
Robust Assortment optimization problem: \((\mathcal{N}, \mathcal{S}, \mathcal{C}, \mathcal{U}, r)\) where

- \(\mathcal{U}\): uncertainty set
- \(\mathcal{C} : \mathcal{S} \times \mathcal{U} \rightarrow \Delta(\mathcal{N}_0)\): choice model; \(\mathcal{C}(i \mid S, u) = 0\) if \(i \notin S\)

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R^*_{\text{det}}(\mathcal{U}) = \max_{S \in \mathcal{S}} \min_{u \in \mathcal{U}} R(S, u)
\]

where \(R(S, u) = \sum_{i \in S} r_i \cdot \mathcal{C}(i \mid S, u)\)

Randomized Assortment optimization problem:

\[
R^*_{\text{rand}}(\mathcal{U}) = \max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u)
\]
Robust Assortment optimization problem: \((\mathcal{N}, \mathcal{S}, \mathcal{C}, \mathcal{U}, r)\) where

- \(\mathcal{U}\): uncertainty set
- \(\mathcal{C}: \mathcal{S} \times \mathcal{U} \to \Delta(\mathcal{N}_0)\): choice model; \(\mathcal{C}(i | S, u) = 0\) if \(i \not\in S\)

\[ R^*_{\text{det}}(\mathcal{U}) = \max_{S \in \mathcal{S}} \min_{u \in \mathcal{U}} R(S, u) \]

where \(R(S, u) = \sum_{i \in S} r_i \cdot \mathcal{C}(i | S, u)\)

Randomized Assortment optimization problem:

\[ R^*_{\text{rand}}(\mathcal{U}) = \max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) \]

**Definitions**
The Multinomial Logit Model (Luce ‘59, McFadden ‘80)

Luce (1959), Individual Choice Behavior: A Theoretical Analysis (Wiley, New York)
The Multinomial Logit Model (Luce ‘59, McFadden ‘80)

Luce (1959), Individual Choice Behavior: A Theoretical Analysis (Wiley, New York)
The Multinomial Logit Model (Luce ‘59, McFadden ‘80)

Purchase probability:

$$\frac{v_i}{v_0 + \sum_{j \in S} v_j} \quad \text{if} \quad i \in S$$

Selected assortment

Outside option

Luce (1959), Individual Choice Behavior: A Theoretical Analysis (Wiley, New York)
Popular Choice Models

Multinomial Logit
Popular Choice Models

Multinomial Logit

Helpful?
Popular Choice Models

Multinomial Logit

Helpful?
Popular Choice Models

Multinomial Logit

Helpful?  ☠️

Complexity  ☠️
Popular Choice Models

Multinomial Logit

Helpful? n/a
Complexity n/a
Popular Choice Models

- Multinomial Logit
- Markov Chain
- Preference Ranking

### Helpful?
- Multinomial Logit: No
- Markov Chain: No
- Preference Ranking: No

### Complexity
- Multinomial Logit: n/a
- Markov Chain: n/a
- Preference Ranking: ???

### Other
- ???
Agenda

1. Implementing Randomized Assortments
2. When Does Randomization Help?
3. Computing Randomized Assortments
4. Numerical Experiments
Randomized Assortment optimization problem:

\[ R^\star_{\text{rand}}(\mathcal{U}) = \max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) \]
Randomized Assortment optimization problem:

\[ R^\star_{\text{rand}}(\mathcal{U}) = \max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) \]

Robust optimization problem
Randomized Assortment optimization problem:

\[
R^\star_{\text{rand}}(U) = \max_{p \in \Delta(S)} \min_{u \in U} \sum_{S \in S} p_S \cdot R(S, u)
\]

Robust optimization problem with two challenges:
Computing Randomized Assortments

**Randomized Assortment optimization problem:**

\[ R^\star_{\text{rand}}(\mathcal{U}) = \max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) \]

**Robust optimization problem with two challenges:**

1. exponentially many decision variables
Randomized Assortment optimization problem:

\[ R^*_{\text{rand}}(\mathcal{U}) = \max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) \]

Robust optimization problem with two challenges:

1. exponentially many decision variables
2. (typically) hard-to-evaluate objective function
The Randomized RO problem satisfies the following strong duality:

\[
\max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) = \min_{\kappa \in \Delta(\mathcal{U})} \max_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} R(S, u) \kappa(du)
\]
Two-Layer Primal-Dual Solution Approach

The **Randomized RO** problem satisfies the following **strong duality**:

\[
\max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) = \min_{\kappa \in \Delta(\mathcal{U})} \max_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} R(S, u) \kappa(du)
\]

Indeed:

\[
\max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) = \max_{p \in \Delta(\mathcal{S})} \min_{\kappa \in \Delta(\mathcal{U})} \sum_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} p_S \cdot R(S, u) \kappa(du)
\]
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\[
\max_{p \in \Delta(S)} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) = \min_{\kappa \in \Delta(\mathcal{U})} \max_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} R(S, u) \kappa(\text{d}u)
\]

Indeed:

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\max_{p \in \Delta(S)} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) = \max_{p \in \Delta(S)} \min_{\kappa \in \Delta(\mathcal{U})} \sum_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} p_S \cdot R(S, u) \kappa(\text{d}u)
\]

\[
= \min_{\kappa \in \Delta(\mathcal{U})} \max_{p \in \Delta(S)} \sum_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} p_S \cdot R(S, u) \kappa(\text{d}u)
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\max_{p \in \Delta(S)} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) = \min_{\kappa \in \Delta(\mathcal{U})} \max_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} R(S, u) \kappa(du)
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Indeed:

\[
\max_{p \in \Delta(S)} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) = \max_{p \in \Delta(S)} \min_{\kappa \in \Delta(\mathcal{U})} \sum_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} p_S \cdot R(S, u) \kappa(du)
\]

\[
= \min_{\kappa \in \Delta(\mathcal{U})} \max_{p \in \Delta(S)} \sum_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} p_S \cdot R(S, u) \kappa(du)
\]

\[
= \min_{\kappa \in \Delta(\mathcal{U})} \max_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} R(S, u) \kappa(du)
\]

---

We use this **strong duality** in the **outer layer** of our solution approach:
We use this **strong duality** in the **outer layer** of our solution approach:

\[
\begin{align*}
\text{primal} & \quad \text{solve restricted primal} \\
\max_{p \in \Delta(\hat{\mathcal{S}})} \min_{u \in \mathcal{U}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)
\end{align*}
\]

where \( \hat{\mathcal{S}} \subseteq \mathcal{S} \) is “small”
Two-Layer Primal-Dual Solution Approach

We use this strong duality in the outer layer of our solution approach:

**Primal**

\[
\max_{p \in \Delta(\hat{\mathcal{S}})} \min_{u \in \hat{\mathcal{U}}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)
\]

where \(\hat{\mathcal{S}} \subseteq \mathcal{S}\) is “small”

**Dual**

\[
\min_{\kappa \in \Delta(\hat{\mathcal{U}})} \max_{S \in \mathcal{S}} \sum_{u \in \hat{\mathcal{U}}} \kappa_u \cdot R(S, u)
\]

where \(\hat{\mathcal{U}} \subseteq \mathcal{U}\) is “small”
We use this strong duality in the outer layer of our solution approach:

**primal**

\[
\max_{p \in \Delta(\hat{\mathcal{S}})} \min_{u \in \hat{\mathcal{U}}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)
\]

where \( \hat{\mathcal{S}} \subseteq \mathcal{S} \) is “small”

**dual**

\[
\min_{\kappa \in \Delta(\hat{\mathcal{U}})} \max_{S \in \mathcal{S}} \sum_{u \in \hat{\mathcal{U}}} \kappa_u \cdot R(S, u)
\]

where \( \hat{\mathcal{U}} \subseteq \mathcal{U} \) is “small”

*add worst-case \( u^* \)'s to \( \hat{\mathcal{U}} \)*
Two-Layer Primal-Dual Solution Approach

We use this strong duality in the outer layer of our solution approach:

**primal**

\[
\max_{p \in \Delta(\hat{\mathcal{S}})} \min_{u \in \mathcal{U}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)
\]

where \( \hat{\mathcal{S}} \subseteq \mathcal{S} \) is “small”

**dual**

\[
\min_{\kappa \in \Delta(\hat{\mathcal{U}})} \max_{S \in \mathcal{S}} \sum_{u \in \hat{\mathcal{U}}} \kappa_u \cdot R(S, u)
\]

where \( \hat{\mathcal{U}} \subseteq \mathcal{U} \) is “small”

**add worst-case \( S^* \)'s to \( \hat{\mathcal{S}} \)**
We use this **strong duality** in the **outer layer** of our solution approach:

**primal**

\[
\max_{\mathbf{p} \in \Delta(\hat{\mathcal{S}})} \min_{\mathbf{u} \in \hat{\mathcal{U}}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)
\]

where \( \hat{\mathcal{S}} \subseteq \mathcal{S} \) is “small”

**dual**

\[
\min_{\mathbf{\kappa} \in \Delta(\hat{\mathcal{U}})} \max_{S \in \mathcal{S}} \sum_{u \in \hat{\mathcal{U}}} \kappa_u \cdot R(S, u)
\]

where \( \hat{\mathcal{U}} \subseteq \mathcal{U} \) is “small”

**Theorem:** Finite \( \varepsilon \)-convergence to optimal \((\mathbf{p}^*, \kappa^*)\).
Two-Layer Primal-Dual Solution Approach

**Inner layer:** solve *restricted primal*

\[
\max_{p \in \Delta(\hat{\mathcal{S}})} \min_{u \in \mathcal{U}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)
\]

with \( \hat{\mathcal{S}} \subseteq \mathcal{S} \) “small”
**Two-Layer Primal-Dual Solution Approach**

**Inner layer:** solve restricted primal

\[
\begin{align*}
\max_{p \in \Delta(\hat{S})} & \min_{u \in \hat{U}} \sum_{S \in \hat{S}} p_S \cdot R(S, u) \\
\text{with } & \hat{S} \subseteq S \text{ “small”}
\end{align*}
\]

1. set LB = $-\infty$ and UB = $+\infty$; choose any \( p \in \Delta(\hat{S}) \)
2. while LB < UB:
   a. solve the evaluation problem
   \[
   \min_{u \in \hat{U}} \sum_{S \in \hat{S}} p_S \cdot R(S, u)
   \]
   \[
   \rightarrow \text{LB} \leftarrow \max\{\text{LB}, \text{obj}\}
   \]
   \[
   \rightarrow \hat{U} \leftarrow \hat{U} \cup \{u^*\}
   \]
   b. solve the optimization problem
   \[
   \max_{p \in \Delta(\hat{S})} \min_{u \in \hat{U}} \sum_{S \in \hat{S}} p_S \cdot R(S, u)
   \]
   \[
   \rightarrow \text{UB} \leftarrow \min\{\text{UB}, \text{obj}\}
   \]
   \[
   \rightarrow p \leftarrow p^*
   \]
Two-Layer Primal-Dual Solution Approach

**Inner layer:** solve restricted primal

\[
\begin{align*}
\max_{p \in \Delta(\mathcal{S})} & \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) \\
\text{with } \hat{\mathcal{S}} \subseteq \mathcal{S} \text{ “small”}
\end{align*}
\]

1. set LB = $-\infty$ and UB = $+\infty$; choose any \( p \in \Delta(\hat{\mathcal{S}}) \)
2. while LB < UB:
   a. solve the evaluation problem
      \[
      \min_{u \in \hat{\mathcal{U}}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)
      \]
      \[\rightarrow\] LB ← max{LB, obj}
      \[\rightarrow\] \( \hat{\mathcal{U}} \leftarrow \hat{\mathcal{U}} \cup \{u^*\} \)
   b. solve the optimization problem
      \[
      \max_{p \in \Delta(\hat{\mathcal{S}})} \min_{u \in \hat{\mathcal{U}}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)
      \]
      \[\rightarrow\] UB ← min{UB, obj}
      \[\rightarrow\] p ← p*
Two-Layer Primal-Dual Solution Approach

**Inner layer:** solve restricted primal

\[
\begin{align*}
\max_{p \in \Delta(\hat{S})} & \quad \min_{u \in \mathcal{U}} \sum_{S \in \hat{S}} p_S \cdot R(S, u) \\
\text{with } \hat{S} \subseteq S \text{ “small”}
\end{align*}
\]

1. set LB = −∞ and UB = +∞; choose any \( p \in \Delta(\hat{S}) \)
2. while LB < UB:
   a. solve the evaluation problem
      \[
      \min_{u \in \mathcal{U}} \sum_{S \in \hat{S}} p_S \cdot R(S, u)
      \]
      \( \Rightarrow \) LB ← \( \max \{ \text{LB, obj} \} \)
      \( \Rightarrow \) \( \hat{\mathcal{U}} \) ← \( \hat{\mathcal{U}} \cup \{ u^* \} \)
   b. solve the optimization problem
      \[
      \max_{p \in \Delta(\hat{S})} \min_{u \in \hat{\mathcal{U}}} \sum_{S \in \hat{S}} p_S \cdot R(S, u)
      \]
      \( \Rightarrow \) UB ← \( \min \{ \text{UB, obj} \} \)
      \( \Rightarrow \) \( p \) ← \( p^* \)
Two-Layer Primal-Dual Solution Approach

**Inner layer:** solve restricted primal

\[
\max_{p \in \Delta(\hat{S})} \min_{u \in \mathcal{U}} \sum_{S \in \hat{S}} p_S \cdot R(S, u) \quad \text{with } \hat{S} \subseteq S \text{ “small”}
\]

1. set \( \text{LB} = -\infty \) and \( \text{UB} = +\infty \); choose any \( p \in \Delta(\hat{S}) \)
2. while \( \text{LB} < \text{UB} \):
   a. solve the evaluation problem
      \[
      \min_{u \in \hat{\mathcal{U}}} \sum_{S \in \hat{S}} p_S \cdot R(S, u)
      \]
      \( \Rightarrow \) \( \text{LB} \leftarrow \max\{\text{LB}, \text{obj}\} \)
      \( \Rightarrow \) \( \hat{\mathcal{U}} \leftarrow \hat{\mathcal{U}} \cup \{u^*\} \)
   b. solve the optimization problem
      \[
      \max_{p \in \Delta(\hat{S})} \min_{u \in \hat{\mathcal{U}}} \sum_{S \in \hat{S}} p_S \cdot R(S, u)
      \]
      \( \Rightarrow \) \( \text{UB} \leftarrow \min\{\text{UB}, \text{obj}\} \)
      \( \Rightarrow \) \( p \leftarrow p^* \)
Agenda

1. Implementing Randomized Assortments
2. When Does Randomization Help?
3. Computing Randomized Assortments
4. Numerical Experiments
Data-driven experiment for MNL model:
- random MNL instances with 10 products
- purchase samples for random assortments under true model
- MLE estimation (with budget uncertainty set for robust approaches)

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