New (Practical) Complementary Pivot Algorithms for Market Equilibria

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Leon Walras, 1874

- Pioneered general equilibrium theory
General Equilibrium Theory
Occupied center stage in Mathematical Economics for over a century
Central tenet

- Markets should operate at equilibrium
Central tenet

- Markets should operate at equilibrium

  i.e., prices s.t.

  Parity between supply and demand
Do markets even admit equilibrium prices?
Do markets even admit equilibrium prices?

Easy if only one good!
Supply-demand curves
Do markets even admit equilibrium prices?

What if there are multiple goods and multiple buyers with diverse desires and different buying power?
Arrow-Debreu Model

- $n$ agents and $g$ divisible goods.

- Agent $i$: has initial endowment of goods $R^g_+$

  and a concave utility function $u_i: R^g_+ \rightarrow R_+$

  (yields convexity!)
Arrow-Debreu Model

- $n$ agents and $g$ divisible goods.

- Agent $i$: has initial endowment of goods $R_+^g$ and a concave utility function $u_i : R_+^g \rightarrow R_+$ piecewise-linear, concave (PLC)
Agent $i$ comes with an initial endowment
At given prices, agent $i$ sells initial endowment $p_1$, $p_2$, and $p_3$. 
… and buys optimal bundle of goods, i.e, $\max u_i(\text{bundle})$
Several agents with own endowments and utility functions.

Currently, no goods in the market.
Agents sell endowments at current prices.
Each agent wants an optimal bundle.
Equilibrium

- Prices $p$ s.t. market clears,

i.e., there is no deficiency or surplus of any good.
Arrow-Debreu Theorem, 1954

- Celebrated theorem in Mathematical Economics

- Established existence of market equilibrium under very general conditions using a deep theorem from topology - Kakutani fixed point theorem.
Kenneth Arrow

- Nobel Prize, 1972
Gerard Debreu

Nobel Prize, 1983
Arrow-Debreu Theorem, 1954

- Celebrated theorem in Mathematical Economics
- Established existence of market equilibrium under very general conditions using a theorem from topology - Kakutani fixed point theorem.
- Highly non-constructive!
Inherently algorithmic notion!

Leon Walras (1774):
Tatonnement process:
Price adjustment process to arrive at equilibrium

- Deficient goods: raise prices
- Excess goods: lower prices
Leon Walras

- Tatonnement process: Price adjustment process to arrive at equilibrium
  - Deficient goods: raise prices
  - Excess goods: lower prices

- Does it converge to equilibrium?
GETTING TO ECONOMIC EQUILIBRIUM: A PROBLEM AND ITS HISTORY

For the third International Workshop on Internet and Network Economics

Kenneth J. Arrow
OUTLINE

I. BEFORE THE FORMULATION OF GENERAL EQUILIBRIUM THEORY
II. PARTIAL EQUILIBRIUM
III. WALRAS, PARETO, AND HICKS
IV. SOCIALISM AND DECENTRALIZATION
V. SAMUELSON AND SUCCESSORS
VI. THE END OF THE PROGRAM
Part VI: THE END OF THE PROGRAM

A. Scarf’s example

B. Saari-Simon Theorem: For any dynamic system depending on first-order information (z) only, there is a set of excess demand functions for which stability fails. (In fact, theorem is stronger).

C. Uzawa: Existence of general equilibrium is equivalent to fixed-point theorem

D. Assumptions on individual demand functions do not constrain aggregate demand function (Sonnenschein, Debreu, Mantel)
Centralized algorithms for equilibria

- Scarf, Smale, …, 1970s: Nice approaches!
Centralized algorithms for equilibria

- Scarf, Smale, …, 1970s: Nice approaches!
  
  (slow and suffer from numerical instability)
Theoretical Computer Science

- Primal-dual paradigm
- Convex programs
- Complementary pivot algorithms
(Complementary) Pivot Algorithms

- Dantzig, 1947: Simplex algorithm for LP

- Lemke-Howson, 1964: 2-Nash Equilibrium

- Eaves, 1975: Equilibrium for Arrow-Debreu markets under linear utilities

\[ f(x) = \sum_{j} c_j x_j \]
(Complementary) Pivot Algorithms

- Very fast in practice (even though exponential time in worst case).

- Work on rational numbers with bounded denominators, hence no instability issues.

- Reveal deep structural properties.
(Complementary) Pivot Algorithms

- Eaves, 1975: Equilibrium for linear Arrow-Debreu markets (based on Lemke’s algorithm)

- Until very recently, no extension to more general utility functions!
(Complementary) Pivot Algorithms

- Eaves, 1975: Equilibrium for linear Arrow-Debreu markets

- Until very recently, no extension to more general utility functions!

Why?
Separable, piecewise-linear concave utility functions
Separable utility function

For a single buyer:

Utility from good $j$, $f_j : \square + \rightarrow \square +$

Total utility from bundle, $f(x) = \sum_j f_j(x_j)$
$f_j$ : piecewise-linear, concave
Arrow-Debreu market under separable, piecewise-linear linear concave (SPLC) utilities

- Can Eaves’ algorithm be extended to this case?
- Eaves, 1975 Technical Report:
Also under study are extensions of the overall method to include piecewise-linear utilities, production, etc., if successful, this avenue could prove important in real economic modeling.

- Eaves, 1976 Journal Paper:
... Now suppose each trader has a piecewise-linear, concave utility function. Does there exist a rational equilibrium? Andreu Mas-Colell generated a negative example, using Leontief utilities. Consequently, one can conclude that Lemke’s algorithm cannot be used to solve this class of exchange problems.
Leontief utility

\[ u(x) = \min \left( \frac{x_1}{a_1}, \frac{x_2}{a_2}, \ldots, \frac{x_n}{a_n} \right) \]
Leontief utility: is non-separable!

- Utility = min\{#bread, 2 #butter\}

- Only bread or only butter gives 0 utility!
Rationality for SPLC Utilities

- Devanur & Kannan, 2007,
  V. & Yannakakis, 2007:

  If all parameters are rational numbers, there is a rational equilibrium.
Theorem (Garg, Mehta, Sohoni & V., 2012): Complementary pivot algorithm for Arrow-Debreu markets under SPLC utility functions. (based on Lemke’s algorithm)
Experimental Results

- Inputs are drawn uniformly at random.

| |A|x|G|x#Seg | #Instances | Min Iters | Avg Iters | Max Iters |
|---|---|---|---|---|---|---|---|
|10 x 5 x 2 | 1000 | 55 | 69.5 | 91 |
|10 x 5 x 5 | 1000 | 130 | 154.3 | 197 |
|10 x 10 x 5 | 100 | 254 | 321.9 | 401 |
|10 x 10 x 10 | 50 | 473 | 515.8 | 569 |
|15 x 15 x 10 | 40 | 775 | 890.5 | 986 |
|15 x 15 x 15 | 5 | 1203 | 1261.3 | 1382 |
|20 x 20 x 5 | 10 | 719 | 764 | 853 |
|20 x 20 x 10 | 5 | 1093 | 1143.8 | 1233 |
Linear Complementarity & Lemke’s Algorithm
Linear complementarity problem

- Generalizes LP
\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax = b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\max & \quad b^T z \\
\text{s.t.} & \quad A^T z = c \\
& \quad z \geq 0
\end{align*}
\]
\[
\min \quad c^T x \\
\text{s.t.} \quad Ax \leq b \\
\]
\[
\max \quad b^T z \\
\text{s.t.} \quad A^T z \leq c \\
\]
\[
x \geq 0 \\
z \geq 0 \\
\]
Complementary Slackness:

Let \( x \) , \( z \) both be feasible.

Then both are optimal iff

\[
i: \quad z_i = 0 \quad \text{or} \quad (Ax)_i = b_i
\]

\[
 j: \quad x_j = 0 \quad \text{or} \quad (A^T z)_j = c_j
\]
$x, \ z$ are both optimal iff

\[
\begin{align*}
Ax & \geq b & A^Tz & \leq c \\
x & \geq 0 & z & \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
z^T(Ax & \ b) = 0 & x^T(c & \ A^Tz) = 0 \\
\end{align*}
\]
Let

\[
M = \begin{pmatrix}
A & 0 \\
0 & A^T
\end{pmatrix}
\]

\[
y = \begin{pmatrix}
x \\
z
\end{pmatrix}
\]

\[
q = \begin{pmatrix}
b \\
c
\end{pmatrix}
\]

Then \( y \) gives optimal solutions iff

\[
M y \leq q
\]

\[
y \geq 0
\]

\[
y. (q \quad M y) = 0
\]
Let

\[
M = \begin{pmatrix} A & 0 \\ 0 & A^T \end{pmatrix} \quad y = \begin{pmatrix} x \\ z \end{pmatrix} \quad q = \begin{pmatrix} b \\ c \end{pmatrix}
\]

Then \( y \) gives optimal solutions iff

\[
M y \leq q \\
y \geq 0 \\
y.(q \quad M y) = 0
\]
Linear Complementarity Problem

Given $n \times n$ matrix $M$ and vector $q$ find $y$ s.t.

$M y \leq q$

$y \geq 0$

$y \cdot (q - M y) = 0$
Given $n \times n$ matrix $M$ and vector $q$, find $y$ s.t.

\[ M y \leq q \]
\[ y \geq 0 \]
\[ y. (q - M y) = 0 \]
Linear Complementarity Problem

Given \( n \times n \) matrix \( M \) and vector \( q \) find \( y \) s.t.
\[
M y \leq q \\
0 \\
y. (q - M y) = 0
\]
Clearly, \( q - M y \geq 0 \)

i.e., for each \( i \):
\( y_i = 0 \) or inequality \( i \) is satisfied with equality.
Examples of linear complementarity

- LP: complementary slackness
- 2-Nash: For row player,
  either $\text{Pr}[\text{row } i] = 0$ or row $i$ is a best response.
Nonlinear complementarity

- Plays a key role in KKT conditions for convex programs
Linear Complementarity Problem

Given \( n \times n \) matrix \( M \) and vector \( q \) find \( y \) s.t.

\[
M \begin{pmatrix} y \\ q \end{pmatrix} \leq 0 \\
y \begin{pmatrix} y \end{pmatrix} \geq 0 \\
y \cdot (q - M y) = 0
\]

i.e., for each \( i \):

\( y_i = 0 \) or inequality \( i \) is satisfied with equality.
Linear Complementarity Problem

Given \( n \times n \) matrix \( M \) and vector \( q \) find \( y \) s.t.

\[
M y \leq q
\]
\[
y \geq 0
\]
\[
y.(q - M y) = 0
\]

Introduce slack variables \( v \)

\[
M y + v = q
\]
\[
y \geq 0, \quad v \geq 0 \quad (\text{Since } q \geq M y \geq 0)
\]
\[
y.v = 0
\]

i.e., for each \( i \): \( y_i = 0 \) or \( v_i = 0 \).
\[ M\ y + \ v = \ q \]
\[ y \geq 0 \]
\[ v \geq 0 \]
\[ y \cdot v = 0 \]

Assume polyhedron in \( \mathbb{R}^{2n} \) defined by red constraints is non-degenerate.
Solution to LCP satisfies \( 2n \) equalities is a vertex of the polyhedron.
Possible scheme

- Find one vertex of polyhedron and walk along 1-skeleton, via pivoting, to a solution.
Possible scheme

- Find one vertex of polyhedron and walk along 1-skeleton, via pivoting, to a solution.

- But in which direction is the solution?
Lemke’s idea

Add a new dimension:

\[ M \mathbf{y} + \mathbf{v} \quad z_1 = q \]

\[
\begin{align*}
  y & \quad 0 \\
  v & \quad 0 \\
  z & \quad 0 \\
  y \cdot v & = 0
\end{align*}
\]
Lemke’s idea

Add a new dimension:

\[ M_y + v - z_1 = q \]

\[
y, v \\ 0
\]

\[
y, v = 0
\]

\[
z, 0
\]

\[
y, v = 0
\]

Note: Easy to get a solution to augmented LCP:

Pick \( y = 0 \), \( z \) large and \( v = q + z_1 \). Then \( v \geq 0 \).
Lemke’s idea

Add a new dimension:

\[ M \bar{y} + \bar{v} \quad z_1 = q \]

\[ \begin{align*}
  \bar{y} &\geq 0 \\
  \bar{v} &\geq 0 \\
  z &\geq 0 \\
  y \cdot v &= 0
\end{align*} \]

Want: solution of augmented LCP with \( z = 0 \).

Will be solution of original LCP!
- $S$: set of solutions to augmented LCP, each satisfies $2n$ equalities.

- Polyhedron is in $2n+1$ space.

- Hence, $S$ is a subset of 1-skeleton, i.e., consist of edges and vertices.

- Every solution is **fully labeled**, i.e.,

\[ i : y_i = 0 \text{ or } v_i = 0 \]
Vertices of polyhedron lying in $S$

- Two possibilities:

1). Has a double label, i.e.,

$$i : y_i = 0 \quad \text{and} \quad v_i = 0$$

- Only 2 ways of relaxing double label, hence this vertex has exactly 2 edges of $S$ incident.
Vertices of polyhedron lying in $S$

- Two possibilities:

2). Has $z = 0$

- Only 1 way of relaxing $z = 0$, hence this vertex has exactly 1 edge of $S$ incident.
Vertices of polyhedron lying in $S$

- Two possibilities:

2). Has $z = 0$

- Only 1 way of relaxing $z = 0$, hence this vertex has exactly 1 edge of $S$ incident.

- This is a solution to original LCP!
Hence $S$ consists of paths and cycles!
- **ray**: unbounded edge of $S$.

- **principal ray**: each point has $y = 0$.

- **secondary ray**: rest of the rays.
Lemke’s idea

Add a new dimension:

\[ M \underline{y} + \underline{v} + z_1 = q \]

\[
\begin{align*}
\underline{y} & = 0 \\
\underline{v} & = 0 \\
z & = 0 \\
\end{align*}
\]

\[ y \cdot v = 0 \]

Note: Easy to get a solution to augmented LCP:

Pick \( y = 0 \), \( z \) large and \( v = q + z_1 \). Then \( v \neq 0 \).
principal ray

\[ y = 0 \]

\[ y_i = 0 \]

\[ v_i = 0 \]
principal ray

\[ y = 0 \]

\[ y_i = 0 \]
\[ v_i = 0 \]

\[ y_k = 0 \]
\[ v_k = 0 \]
principal ray

\[ y = 0 \]

\[ y_i = 0 \]
\[ v_i = 0 \]

\[ y_k = 0 \]
\[ v_k = 0 \]
principal ray

Path lies in $S$!

$y = 0$

$y_i = 0$

$v_i = 0$
principal ray

\[ y = 0 \]

\[ y_i = 0 \]

\[ v_i = 0 \]

\[ z = 0 \]
\[ y = 0 \]

\[ y_i = 0 \]

\[ v_i = 0 \]
Problem with Lemke’s algorithm

- No recourse if path starting with primary ray ends in a secondary ray!
Problem with Lemke’s algorithm

- No recourse if path starting with primary ray ends in a secondary ray!

- We show that for each of our LCPs, associated polyhedron has no secondary rays!
Dramatic change!

Polyhedron of

- original LPC: no clue where solution is.
- augmented LCP: know a path leading to solution!
Theorem (Garg, Mehta, Sohoni & V., 2012):
1). Derive LCP whose solutions correspond to equilibria.
2). Polyhedron of LCP has no secondary rays.

Corollary: The number of equilibria is odd, up to scaling.
Theorem (Garg, Mehta, Sohoni & V., 2012):
1). Derive LCP whose solutions correspond to equilibria.
2). Polyhedron of LCP has no secondary rays.
3). If no. of goods or agents is a constant, polyn. vertices of polyhedron are solutions
=> strongly polynomial algorithm
Derive LCP (assume linear utilities)

- **Market clearing**
  - Every good fully sold
  - Every agent spends all his money

- **Optimal bundles**
  - Every agent gets a utility maximizing bundle
Model

Utility of agent $i$: $u_{ij} x_{ij}$

Initial endowment of agent $i$: $w_{ij}, \quad j \in G$

W.l.o.g. assume 1 unit of each good in the market.
Variables

\[ p_j : \text{ price of good } j \]
\[ q_{ij} : \text{ amount of money spent by } i \text{ on } j \]
Guaranteeing optimal bundles

- Agent $i$ spends only on $S_i = \arg \max_j \left\{ \frac{u_{ij}}{p_j} \right\}$

- bang-per-buck of $i$ $= \max_j \left\{ \frac{u_{ij}}{p_j} \right\}$

  $(= \frac{1}{i} \text{ at equilibrium})$
Optimal bundles, guaranteed by:

\[ j : \quad \frac{u_{ij}}{p_j} > 0 \quad \iff \quad \frac{u_{ij}}{p_j} = \frac{1}{l_i} \]

\[ q_{ij} > 0 \quad \frac{u_{ij}}{p_j} = \frac{1}{l_i} \]
Optimal bundles, guaranteed by:

\[ j : \]

\[ \frac{u_{ij}}{p_j} = \frac{1}{q_{ij} > 0} \]

\[ \frac{u_{ij}}{p_j} = \frac{1}{i} \]

\[ q_{ij} > 0 \]

or

\[ \frac{u_{ij}}{p_j} = \frac{1}{q_{ij} > 0} \]

\[ \frac{u_{ij}}{p_j} = \frac{1}{i} \]

\[ q_{ij} > 0 \]
Optimal bundles, via complementarity

\[ i: j: \quad u_{ij} \quad i \quad p_j \]

\[ q_{ij} (u_{ij} \quad i \quad p_j) = 0 \]
LCP for linear utilities

\[ j : q_{ij} \quad p_j \quad \text{comp} \quad p_j \]

\[ i : w_{ij} p_j \quad q_{ij} \quad \text{comp} \quad i \]

\[ i, j : u_{ij} \quad p_j \quad \text{comp} \quad q_{ij} \]

& non-negativity for \( p_j, \quad q_{ij}, \quad i \)