Computational MIP: Selected Topics

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Outline, Assumptions and Notation

- We consider a general Mixed Integer Program in the form:

$$\min\{c^T x : Ax \geq b, x \geq 0,\ x_j \text{ integer},\ j \in \mathcal{I}\}$$

where matrix A does not have a special structure.

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  and we also suggested that there is a strong relationship, very little understood, among them.

- Today, the talk discusses some new and sophisticated ideas for both components.

1: Traditional Branching

- The traditional way of partitioning the problem into sub-problems is the so-called variable branching. Pick a variable $x_j,\ j \in \mathcal{I}$ whose value x_j^* is fractional in the current LP relaxation and generate two sub-MIPs:

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- Moreover, in the context of knapsack equality constraints with large coefficients and bounds branching on variables is not effective.
- This is the same for other types of MIPs, like symmetric ones.

- Finally, even for 0-1 combinatorial optimization problems it is often the case that fixing a variable to 1 is typically very strong <u>while</u> fixing it to 0 can have little or no effect for difficult instances.

1: Less Traditional Branching

- One recent line of research concerns branching on more complicated general disjunctions (in contrast with *elementary* ones, i.e., variable branching):

$$\alpha^T x \leq \alpha_0 \quad \bigvee \quad \alpha^T x \geq \alpha_0 + 1, \tag{2}$$

where α is an integer vector with 0-entries for variables $x_j,\ j \notin \mathcal{I}$ and α_0 is an integer scalar.

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- The overall goal of any branching scheme is to reduce running time.
  As a proxy, most branching schemes try to maximize the (estimated) bound increase resulting from imposing the disjunction.

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- The disjunction selection problem can sometimes be formulated as a bilevel program:
  - The upper-level variables can be used to model the choice of the disjunction (we'll see an example shortly).
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- For general disjunctions, different authors have suggested different quality measures either circumventing the bilevel nature (Karamanov & Cornuéjols, and Cornuéjols, Liberti & Nannicini) or formulating it as a single level program (Mahajan & Ralphs).

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- We are interested in a set $S = \{i_1, \ldots, i_{|S|}\} \subseteq I^n = \{1, \ldots, n\}$, to impose the (clearly valid) multi-variable disjunction

$$x_{i_1} = 1 \vee (x_{i_2} = 1 \wedge x_{i_1} = 0) \vee \ldots \vee (x_{i_{|S|}} = 1 \wedge x_{i_1} = 0 \wedge \ldots \wedge x_{i_{|S|-1}} = 0) \vee \sum_{i \in S} x_i = 0.$$

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- In particular we want sets S playing a fundamental role for improving the current incumbent solution value, say $\bar{z}$.

1: Interdiction Branching (cont.d)

- Such a goal can be achieved by solving the Interdiction Branching Problem (IBP) (at node a):

$$\min \sum_{i \in I^n} y_i \tag{3}$$

$$\text{s.t.} \sum_{i \in I^n} c_i x_i \geq \bar{z} \tag{4}$$

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- In other words,

(i) the last term of the branching disjunction does not have to be explored because it contains NO improving solution, while,

(ii) by the minimality of (3), it follows that any other child node contains at least one improving solution.

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- Interdiction branching takes into account both the current incumbent value and the bound provided by the LP relaxation and can thus be seen as targeting improvements in both the upper and lower bounds.
- As in the traditional branching on variables, child subproblems are generated by imposing variable bounds, without introducing additional constraints.
- In all but one of the $|S|$ children, at least two variables are fixed, often yielding a remarkable improvement in the bound provided by the LP relaxation.

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- Extensive computational experiments on difficult 0-1 knapsack instances have shown a very favorable behavior wrt traditional branching on variables although this has to be confirmed within state-of-the-art solvers.

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- In the most natural setting, the aggregation of step 1 above is done by using a single row of the simplex tableau as mixed-integer set.

- Very recently, since 2007, a new line of research (strongly connected to a very old one) has been followed by considering multiple rows of the simplex tableau at the same time.

2: Cuts from Multiple Rows of the Simplex Tableau

- We consider a mixed-integer set of the form

$$S = \{x \in \mathbb{R}^n \ : \ Ax = b, x \geq 0, x_j \in \mathbb{Z} \ \forall j \in \mathcal{I}\}$$

with $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$.

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- Given a basis $B \subset \{1, \ldots, n\}$ corresponding to a vertex x^* of the continuous relaxation of S, the set S can be rewritten as

$$\begin{array}{rcl} x_B & = & x^*_B + \sum_{j \in N} r^j x_j, \\ x & \geq & 0, \\ x_j & \in & \mathbb{Z}, \ j \in \mathcal{I}, \end{array} \tag{9}$$

where N denotes the set of nonbasic variables.

2: Cuts from Multiple Rows of the Simplex Tableau (cont.d)

- A first relaxation of S can be obtained by dropping the nonnegativity restrictions on all the basic variables and considering a subset Q (with $|Q| = q$) of rows of (9) associated with basic integer-constrained variables, thus getting

$$\begin{array}{llll} (S_Q) & x_i & = & f_i + \sum_{j \in N} r_i^j x_j, \;\; i \in Q \\ & x_j & \geq & 0, \;\; j \in N \\ & x_j & \in & \mathbb{Z}, \;\; j \in \mathcal{I} \end{array} \qquad (10)$$

with $f_i = x_i^* - \lfloor x_i^* \rfloor$ for any $i \in Q$ and $f_i > 0$ for some $i \in Q = \{1, \ldots, q\}$.

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- A key step is further relaxing set S_Q by dropping all the integrality requirements on the nonbasic variables, thus getting a system of the form

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\begin{array}{lll}
x & = & f + \sum_{j=1}^{k} r^j s_j, \\
x & \in & \mathbb{Z}^q, \\
s & \in & \mathbb{R}^k_+,
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  where all the continuous variables have been renamed as s and $|N| = k$.

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$$\begin{array}{rcl} x & = & f + \sum_{j=1}^{k} r^j s_j, \\ x & \in & \mathbb{Z}^q, \\ s & \in & \mathbb{R}_+^k, \end{array} \tag{11}$$

  where all the continuous variables have been renamed as s and $|N| = k$.

- Denote as $R_f(r^1, \ldots, r^k)$ the convex hull of all vectors $s \in \mathbb{R}^k$ for which there exists $x \in \mathbb{R}^q$ such that (x, s) satisfies (11).

2: Cuts from Multiple Rows of the Simplex Tableau (cont.d)

- Borozan & Cornuéjols considered relaxing the k-dimensional space of variables $s = (s_1, \ldots, s_k)$ to an infinite-dimensional space, where the variables s_r are defined for any $r \in \mathbb{Q}^q$.

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- In addition, Borozan & Cornuéjols proved that any valid inequality for R_f can be written as

$$\sum_{r \in \mathbb{Q}^q} \psi(r) s_r \geq 1 \tag{12}$$

where $\psi : \mathbb{Q}^q \longrightarrow \mathbb{Q} \cup \{+\infty\}$ is said to be a valid function if the corresponding inequality (12) is valid for R_f.

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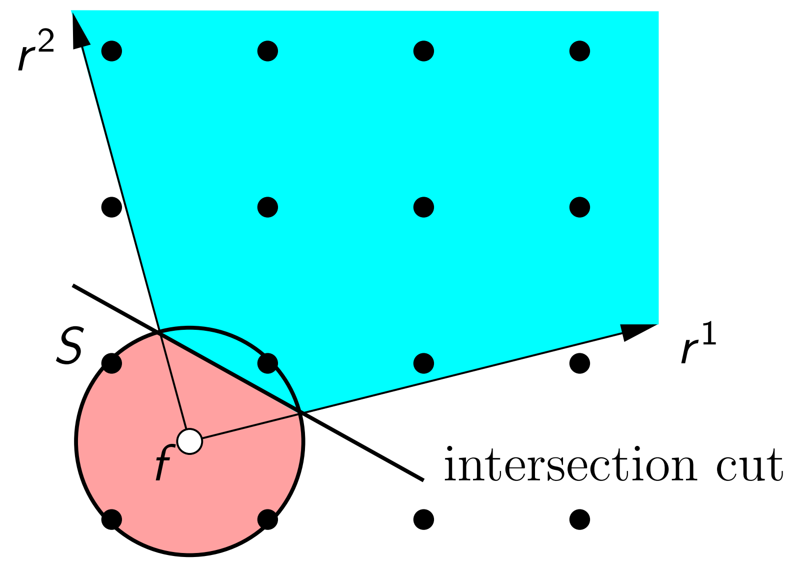
- Finally, Borozan & Cornuéjols provided a strong correspondence between minimal valid inequalities and maximal lattice-free convex sets.

- Zambelli proved that one does not have to worry about lattice-free convex sets with f on their boundary.

2: Cuts from TWO Rows of the Simplex Tableau

Andersen, Louveaux, Weismantel & Wolsey have shown that all the facets of $R_f(r^1, \ldots, r^k)$ are intersection cuts arising from two-dimensional lattice-free convex sets. [Balas, 1971]



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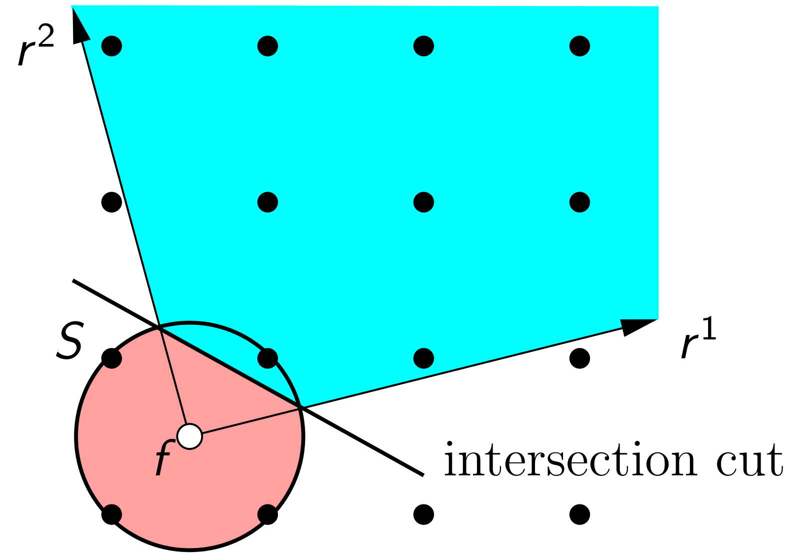

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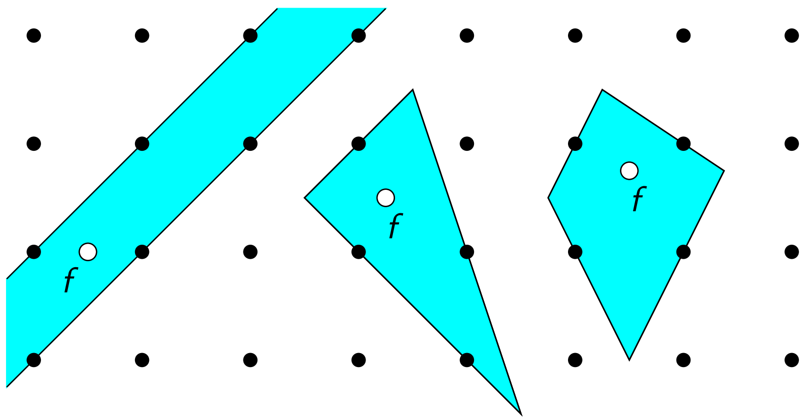

Cornuéjols & Margot proved that only three types of maximal lattice-free convex sets are sufficient, namely, splits, triangles and quadrilaterals.

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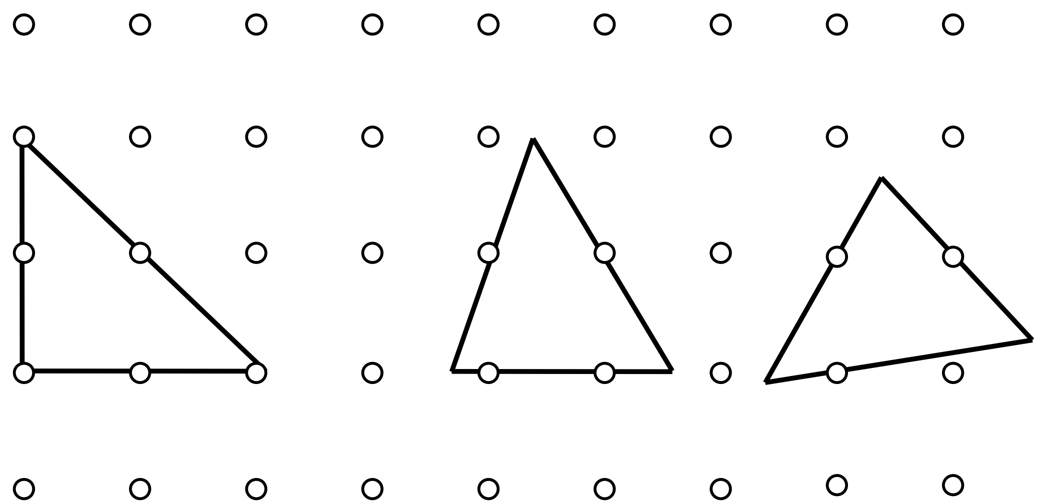
2. One side of Π contains more than one integral point in its relative interior.
3. The vertices are non-integral and each side contains one integral point in its relative interior.



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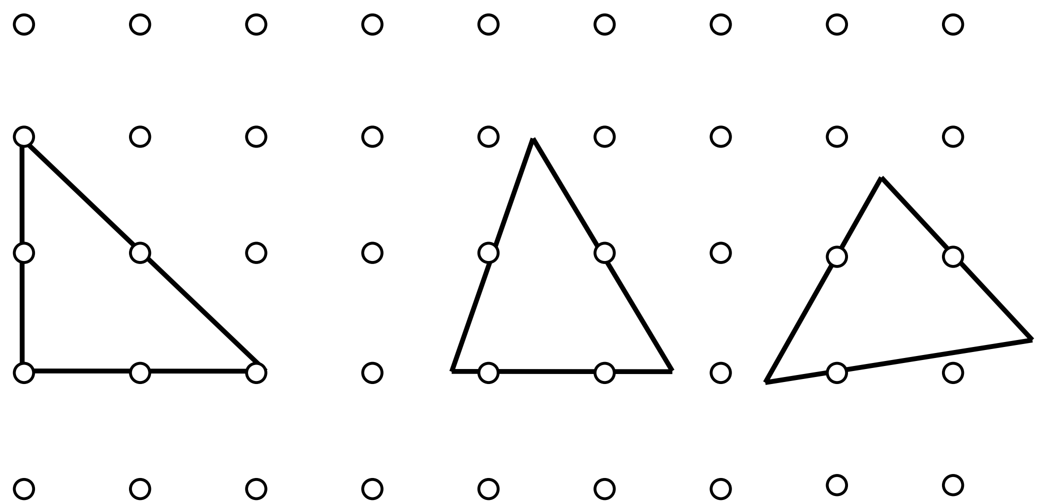
  2. One side of Π contains more than one integral point in its relative interior.

  3. The vertices are non-integral and each side contains one integral point in its relative interior.



- For the triangles of type 1 and 2 above they also defined the best possible way of lifting the coefficients of the nonbasic integer variables.

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  Before really asserting their effectiveness one has to find a clever way of doing cut selection which also includes understanding their relationship with Mixed-Integer Gomory cuts.