Efficient Simulation for Rare Events

Jose Blanchet

Columbia Department of IEOR

Lunteren Conference 2010

Agenda

- **Introduction**
- A Simple Random Walk Example
- Systematic Approach
- Testing Efficiency
- Counter-examples, Heavy-tails and Beyond

Rare Event Analysis: Areas of Applicability

Rare events are consequential in many areas

- Insurance / Finance

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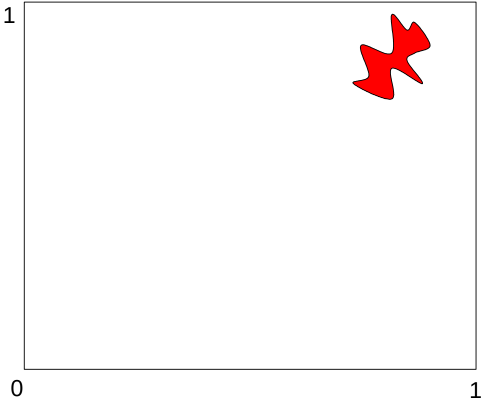
- Statistics

Why are Rare Events Difficult to Assess?

- Typically no closed forms (complex systems)
- But crudely implemented simulation might not be good



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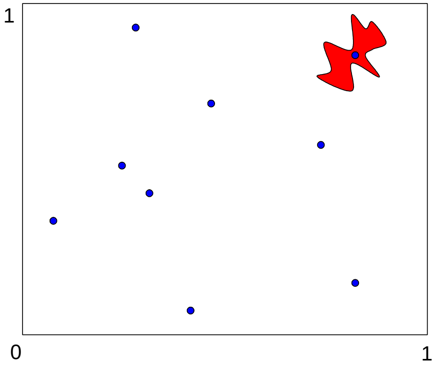

Why are Rare Events Difficult to Assess?

- **Relative mean squared error (RMSE)** PER TRIAL = stdev / mean

$$\frac{\sqrt{P(\text{RED})(1 - P(\text{RED}))}}{P(\text{RED})} \approx \frac{1}{\sqrt{P(\text{RED})}}$$

Performance Analysis

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- **STRONG EFFICIENCY:**

$$\textbf{RMSE} \;\; \text{for } P(A) = O(1)$$

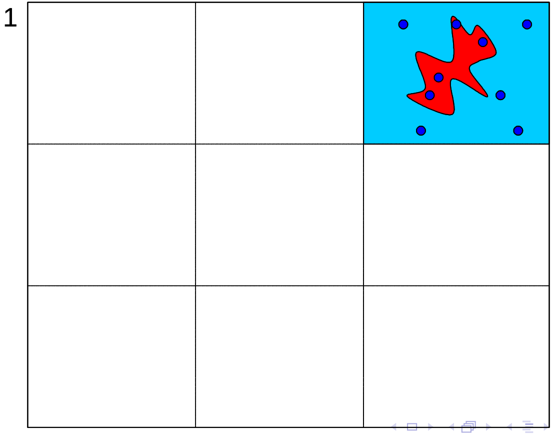
- **WEAK EFFICIENCY:** For each $\varepsilon > 0$

$$\textbf{RMSE} \;\; \text{for } P(A) = O\left(1/P(A)^{\varepsilon}\right)$$

Graphical Interpretation of Importance Sampling

- **Importance sampling (I.S.)**: sample from the important region and correct via likelihood ratio

$$\text{RED AREA} \approx \text{PROPORTION DARTS IN RED AREA} \times \frac{1}{9}$$



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$$I.S.Estimator = L(\omega)\, I(\omega \in A),$$

where $L(\omega)$ is the likelihood ratio (i.e. $L(\omega) = P(\omega)/\widetilde{P}(\omega)$).

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- **NOTE:** $\widetilde{P}$ is called a change-of-measure

- Suppose we choose $\widetilde{P}(\cdot) = P(\cdot|\, A)$

$$L(\omega) = \frac{P(\omega)\, I(\omega \in A)}{P(\omega)\, I(\omega \in A)\, / P(A)} = P(A)$$

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- **Lesson:** *Try choosing* $\widetilde{P}(\cdot)$ *close to* $P(\cdot | A)$*!*

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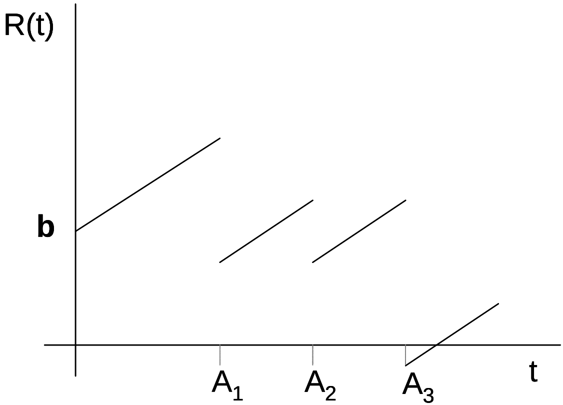
- $N(t) = \#$ arrivals up to time t

Insurance

Plot of risk reserve



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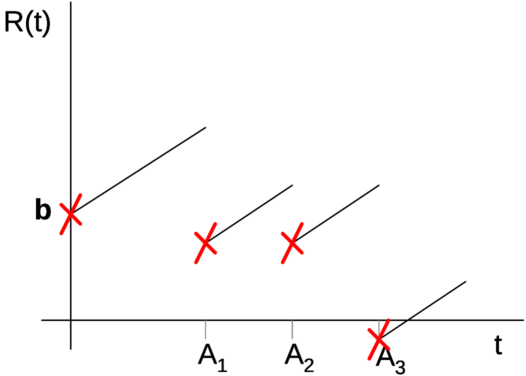
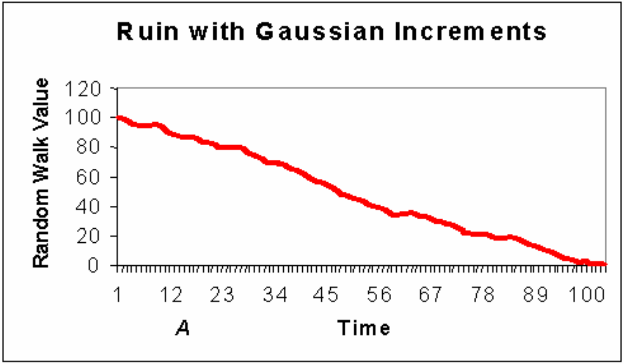
- $R(A_n) = S(n)$ reserve at arrival times with $Y_n = p\tau_n - V_n$.



Insurance Process Conditioned on Ruin



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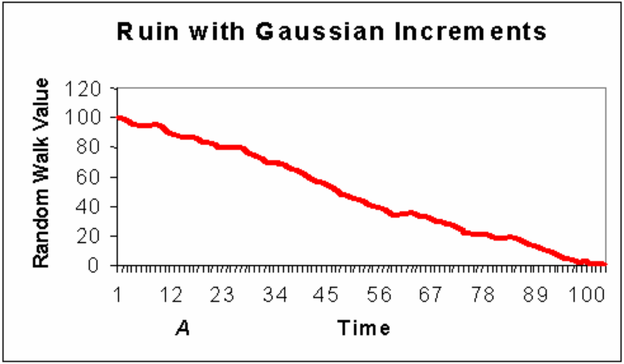

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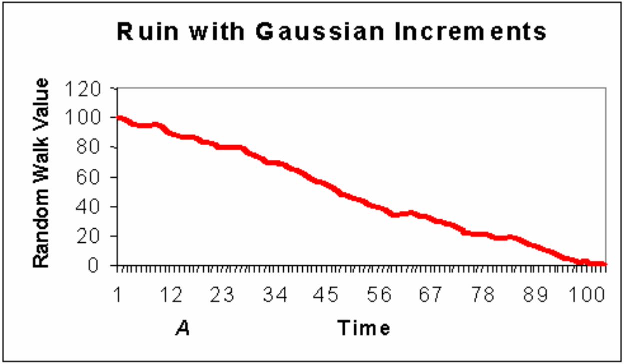

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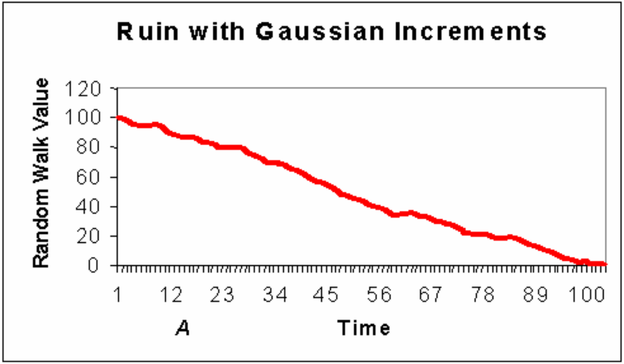

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- Picture generated with Siegmund's 76 algorithm

Light Tails Setting: Asymptotic Conditional Distributions

- In light-tailed cases there is large deviations theory (ref. Dembo and Zeitouni '99).

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$$P\left(Y_1 \leq x, ..., Y_k \leq x \right| \text{ruin starting at } b) \approx \widetilde{P}\left(Y_1 \leq x\right)...\widetilde{P}\left(Y_k \leq x\right),$$

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- Suggested change-of-measure: Sample Y_k's i.i.d. using $\widetilde{P}\left(\cdot\right)$

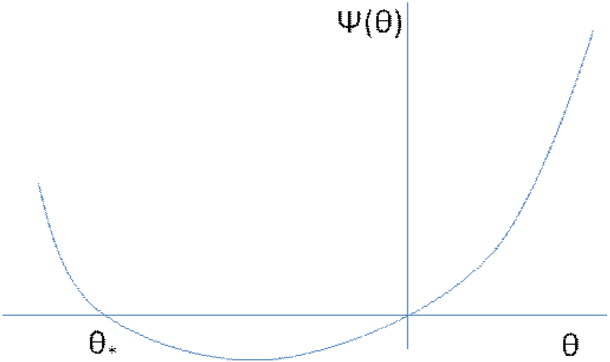
$$L = \frac{p\left(Y_1\right)}{\widetilde{p}\left(Y_1\right)} \cdot \frac{p\left(Y_2\right)}{\widetilde{p}\left(Y_2\right)} \cdot ... \cdot \frac{p\left(Y_{\text{ruin time}}\right)}{\widetilde{p}\left(Y_{\text{ruin time}}\right)}.$$

Light Tails Setting: Exponential Tilting

- More precisely if $p(\cdot)$ is the density of Y_i

$$\widetilde{p}(y) = p(y)\exp(\theta_* y)$$

where $\theta_* < 0$ solves $\psi(\theta_*) = \log E \exp(\theta_* Y_i) = 0$.



Siegmund's Algorithm for Random Walks

Theorem (Siegmund '76)

Assume $\psi(\theta_* - \delta) < \infty$ *for some* $\delta > 0$. *Then, the estimator*

$$L = \frac{p(Y_1)}{\widetilde{p}(Y_1)} \cdot \frac{p(Y_2)}{\widetilde{p}(Y_2)} \cdot \ldots \cdot \frac{p(Y_{ruin\ time})}{\widetilde{p}(Y_{ruin\ time})}$$
$$= \exp\left(-\theta_*[Y_1 + \ldots + Y_{ruin\ time}]\right)$$

is STRONGLY EFFICIENT. Moreover, $\widetilde{p}(\cdot)$ *is the ONLY STATE-INDEPENDENT change-of-measure that achieves efficiency.*

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3. Items 1) and 2) provide systematic tools for rare-event simulation

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Description of Systematic Approach

- State-dependent random walk: $t \in \{0, \Delta, 2\Delta, \ldots\}$

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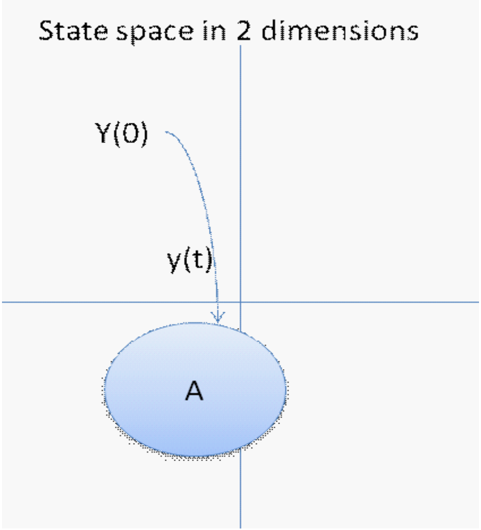
- As $\Delta \longrightarrow 0$ under mild assumptions $Y_{\cdot} \longrightarrow y(\cdot)$ so that

$$\dot{y}(t) = E[X(y(t))].$$

Fluid Limit of State-dependent Random Walk



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- Associated action integral

$$J(z) = \int_0^{\infty} I(\dot{z}(t), z(t))\, dt.$$

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- A generic rare-event estimation problem:

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\begin{aligned}
& \Delta \log P\,(\text{Hit } B \text{ prior to } A) \\
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- **Tracking optimal path: Optimal Exponential Tilting $\theta_*\left(t\right)$ solves**

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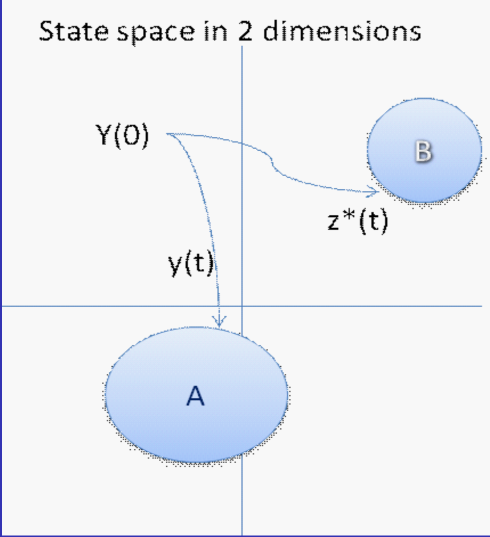

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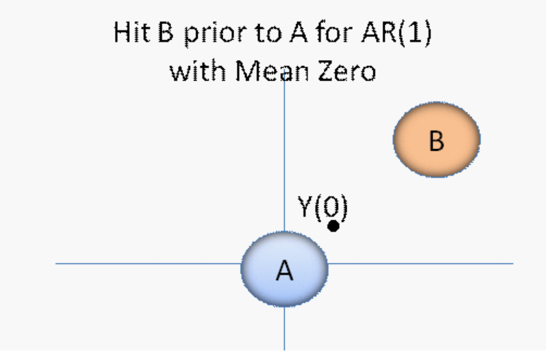
2. Follow path to approximate $z^*(t)$...

Example: Two dimensional AR(1) Process

- **Model:** $Y_{t+\Delta} - Y_t = -\Delta Y_t + \Delta X_{t+\Delta}$, X_t's i.i.d. $N(0, I)$ and $y_0 = (1, 1)$.
- **Estimate:** $P_{y_0}(T_B < T_A)$, with $B = \{x : \left\| x - (e + 1/\sqrt{2}, e + 1/\sqrt{2}) \right\|_2 \leq 1\}$ & $A = \{x : ||x||_2 \leq 1\}$.



Example: Computing the Optimal Path

- **Calculus of Variations Problem:**

$$\min_{z \in C} \frac{1}{2} \int_0^T ||\dot{z}(t) + z(t)||_2^2 \, dt$$

where

$$\begin{aligned} C &= \{ z : z(0) = (1,1), \left\| z(T) - (e + 2^{-1/2})(1,1) \right\|_2 \leq 1, \\ &\quad T < \infty, \|z(t)\|_2 > 1, \ t < T \} \end{aligned}$$

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- **Solution:**

$$\begin{aligned} z(t) &= (\exp(t), \exp(t)) \\ \dot{z}(t) &= z(t). \end{aligned}$$

Interpreting the Change-of-Measure

- **Importance Sampling Q:** W_t is i.i.d. N(0,1) under Q

$$\begin{aligned} Y_{t+\Delta} - Y_t &= \dot{z}(t)\,\Delta + \Delta W_{t+\Delta} \implies \text{Fluid } dy_t \approx \dot{z}(t)\,dt \\ Y_0 &= (1,1)\,, \\ X_{t+\Delta} &= W_{t+\Delta} + (\dot{z}(t) + Y_t). \end{aligned}$$

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- **Likelihood ratio** dP/dQ**:**

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L_0 = \exp\left(-\sum_{j=0}^{T_B/\Delta-1} (\dot{z}(j\Delta) + Y_{j\Delta}) \cdot X_{(j+1)\Delta} + \sum_{j=0}^{T_B/\Delta-1} \frac{\|\dot{z}(j\Delta) + Y_{j\Delta}\|_2^2}{2}
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Asymptotic Optimality

- **First note:** $Y_{t+\Delta} - Y_t = -\Delta Y_t + \Delta X_{t+\Delta} \Longrightarrow$

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- **Get likelihood ratio representation:**

$$L = \exp\left(- \sum_{j=0}^{T_B/\Delta - 1} 2 Y_{(j\Delta)} \cdot X_{(j+1)\Delta} + \sum_{j=0}^{T_B/\Delta - 1} 2 \left\| Y_{(j\Delta)} \right\|_2^2 \right)$$

$$= \exp\left(- \| Y_{T_B} \|_2^2 / \Delta + \| y_0 \|_2^2 / \Delta + \Delta \sum_{j=0}^{T_B/\Delta - 1} \left\| W_{(j+1)\Delta} + Y_{j\Delta} \right\|_2^2 \right)$$

Asymptotic Optimality

- **Bound Likelihood ratio:**

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$$E^Q[L^2 \times I(T_B < T_A)] = \exp\left(-2\left(\inf_{y\in B}\|y\|_2^2 - \|y_0\|_2^2\right)/\Delta + o(1/\Delta)\right)$$

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- **Asymptotic Optimality follows since:**

$$P_{y_0}(T_B < T_A) = \exp\left(-\left(\inf_{y\in B} \|y\|_2^2 - \|y_0\|_2^2\right) / \Delta + o(1/\Delta)\right).$$

Agenda

- Introduction
- A Simple Random Walk Example
- Systematic Approach
- Efficiency Argument
- **Counter-examples, Heavy-tails and Beyond**

Counter-examples

- **Model:** $Y_t = (-1,-1)t + X_t$; X_t is Brownian motion & $B_b = \{(x,y) : x \geq a_0 b \text{ or } y \geq a_1 b\}$

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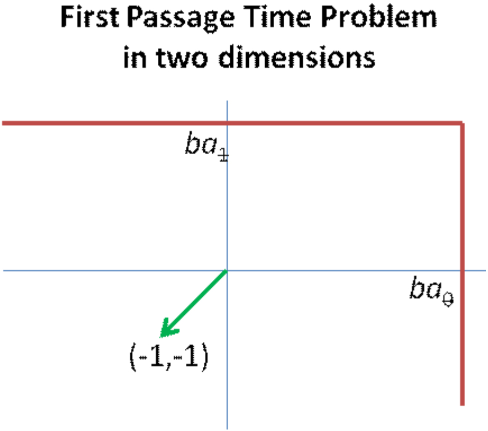
- **Estimate:** $u(b) = P(T_{B_b} < \infty)$ as $b \nearrow \infty$



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- **Let us assume** $a_0 < a_1$**...**

Counter-examples

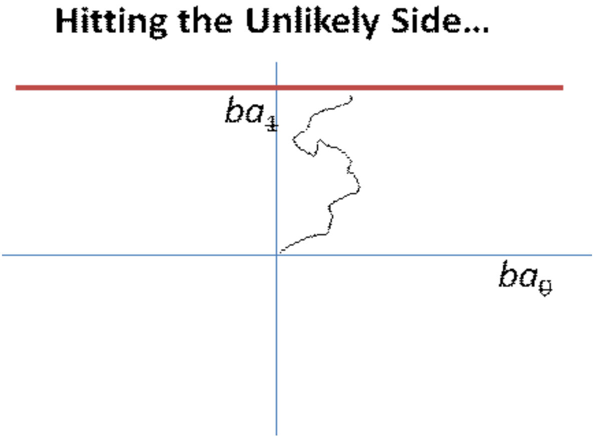
- **Change-of-measure Q_0:** Brownian Motion with drift $(1, -1)$.

Second Moment of Estimator:

$$\begin{aligned} \text{2nd Moment} &= E^{Q}[\exp\left(-(4,0)\cdot X_{T_{B_b}}\right) I\left(T_{B_b} < \infty\right)] \\ &= E[\exp\left(-(2,0)\, X_{T_{B_b}}\right) I\left(T_{B_b} < \infty\right)]. \end{aligned}$$

Counter-examples

- **What happens if a path exits $A_b = \{(x, y) : y \geq a_1 b\}$?**



Counter-examples

- **Lower bound:**

$$\begin{aligned} \text{2nd Mnt} &\geq E\left[\exp\left(-2X^{(1)}_{T_{A_b}}\right) I\left(T_{B_b} < \infty, T_{A_b} = T_{B_b}\right)\right] \\ &= E^{Q_1}\left[\exp\left(-2X^{(1)}_{T_{A_b}} - 2a_1 b\right) I\left(T_{A_b} = T_{B_b}\right)\right], \end{aligned}$$

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- Note that $X^{(1)}_{T_{A_b}} \approx -a_0 a_1 b$ and $T_{A_b} = T_{B_b}$ under Q_1
- **Consequently:** 2nd Moment at least roughly

$$\exp\left(-2a_1 b\left(1 - a_0\right)\right)$$

Can easily pick $a_0 < a_1$ to **break asymptotic optimality and even get HUGE variance**!!

Heavy-tails...

- **Model: Random walk...**

$$S_n = X_1 + ... + X_n,$$

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for $\alpha > 0$ and $L\left(t\beta\right)/L\left(t\right) \longrightarrow 1$ as $t \to \infty$ for $\beta > 0$ (e.g. $L\left(t\right) = \log(1+t)$).

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- Note $E \exp(\theta X_1) = \infty$ for $\theta > 0$.

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- **No clear way to apply the systematic approach!**

Solution? State-dependent Importance Sampling

As we shall see these examples can be addressed using

$$\textbf{State} - \textbf{dependent importance sampling}$$

which we will study in the second part of this lecture...

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- Optimal path in large deviations dictates tilting (several ways of interepreting)
- Approach fails (badly!) in non-convex problems and in heavy-tailed situations