On the Power of Robust solutions in Dynamic Optimization

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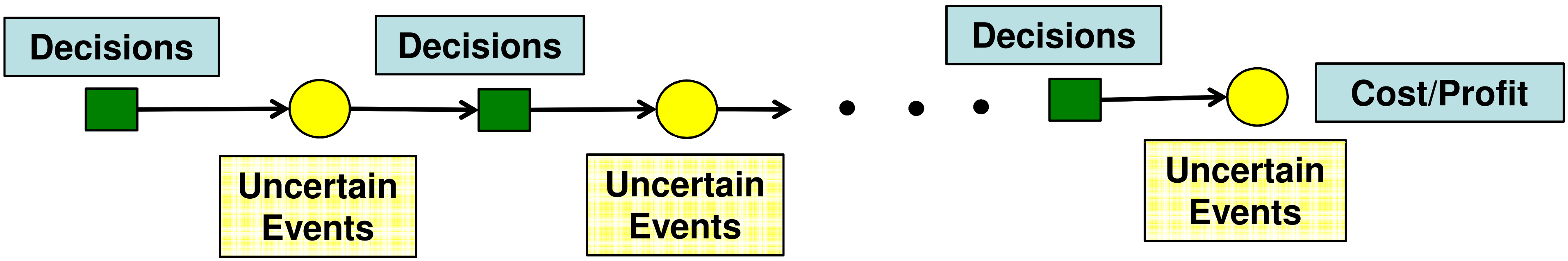
Why Optimization under Uncertainty?

GEORGE B. DANTZIG (2001)

"Planning under uncertainty. This, I feel, is the real field that we should be all working in."



- **Stochastic Optimization**
  - Several approaches : (Dantzig(1955), Birge and Louveaux (1997), Prekopa (2005), Shapiro (2005))
- **Deterministic Optimization**
  - EXPRESS bought by Fair Isaac
  - CPLEX bought by ILOG which was acquired recently by IBM
- But no commercial solver like **CPLEX or EXPRESS!**

Motivation/Philosophy

- Performance analysis given that primitives are probability distributions is often intractable; (Performance of queueing networks)

- Combining probability theory and optimization often leads to the ``Curse of dimensionality''

- **What is available in practice is data, not probability distributions**

Proposal

- Replace probability distributions as primitives with uncertainty sets

- Use worst case analysis: Robust optimization

- **To define the uncertainty sets use the conclusions of probability theory (CLT for example)**

Concretely:

- Let X_i be demand in period i.
- Traditional modeling: X_i iid random variables..
- Proposed modeling: (CLT based)

$$\left|\sum_{i=1\ldots m} (x_i - \mu)/ \sigma \sqrt{m} \right| < 2$$

Robust Optimization

- **Uncertainty: set based, Objective: optimize worst-case**

- **Previous Work**
  - Introduced by Soyster (1973)
  - Studied recently by Ben-Tal and Nemirovski (1998, 2000, 2002), Bertsimas and Sim (2003, 2004)

- **Tractable Approach**

- **No performance bounds known**
  - Widely perceived to produce **highly conservative** solutions

Stochastic Model

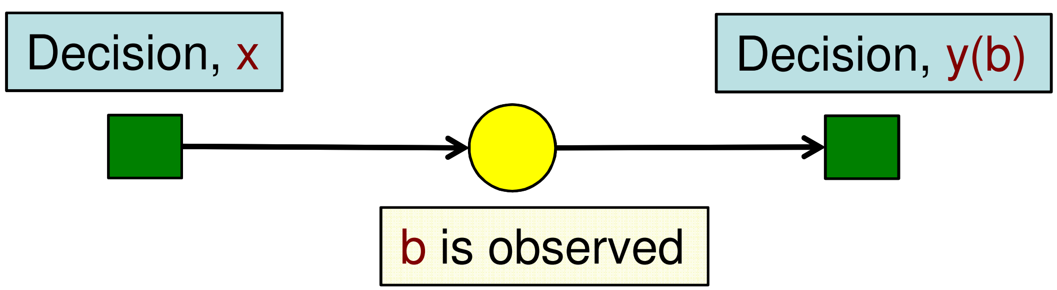
- **Two-stage Stochastic Optimization Model**

$$
\begin{aligned}
\textbf{zStoch} = \min c^T x + \mathbb{E}[d^T y(b)] & \\
Ax + By(b) &\geq b, \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

(Minimize **expected cost**)

(Uncertainty Set)

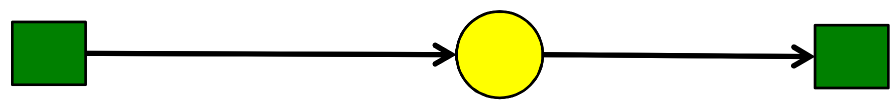
(Uncertain Right Hand Side)



Inventory Management

Order Quantity, x

y(b): Excess



Uncertain Demand, b

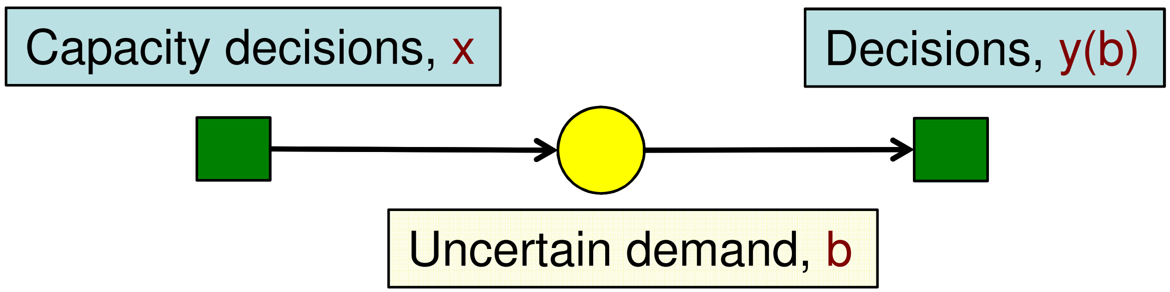
Minimize $\mathbb{E}_b\left[h(x-b)^+ \;+\; p(b-x)^+\right]$

Holding Cost — Backorder Penalty

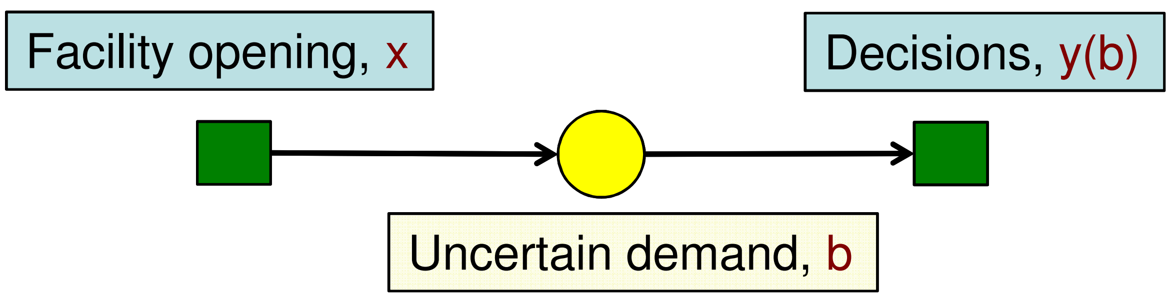
$$
\begin{aligned}
\min \;\; \mathbb{E}_b\Big[\underbrace{h\big(x+y(b)-b\big)}_{\text{Holding Cost}} \;+\; \underbrace{p\cdot y(b)}_{\text{Backorder Penalty}}\Big] & \\
x+y(b) \;&\geq\; b, \;\forall\, b \\
x, y(b) \;&\geq\; 0
\end{aligned}
$$

More Applications

- Capacity Planning



- Facility Location



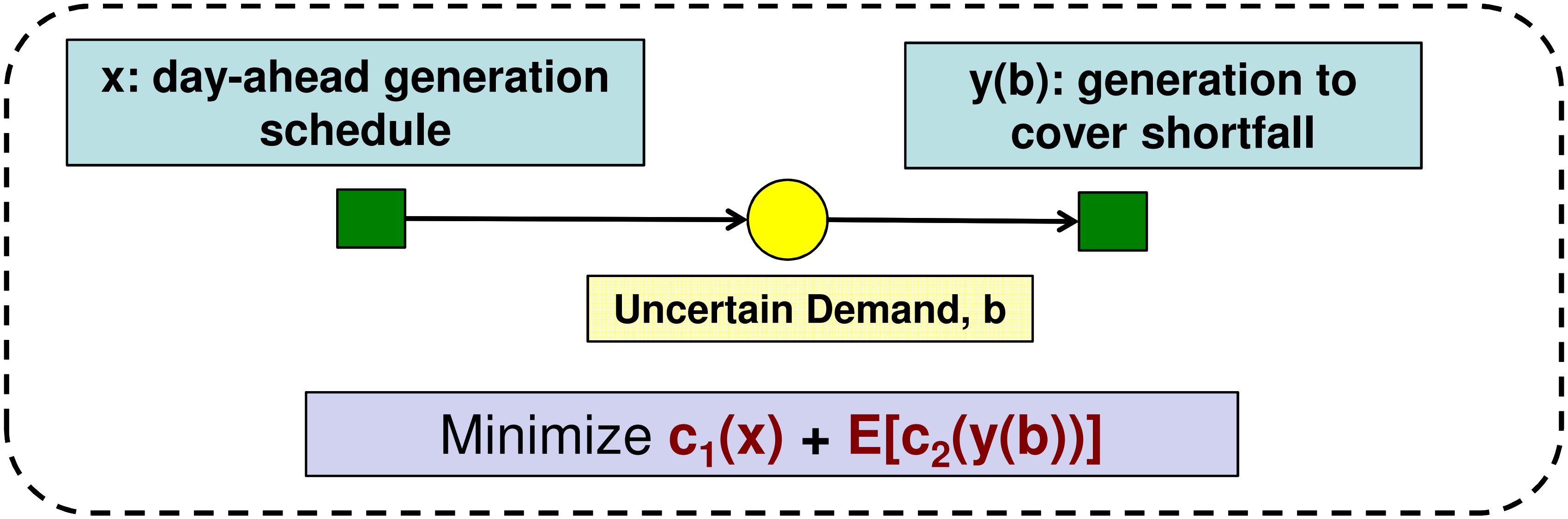
Electricity Markets: Planning for Uncertain Demand

- System operators (New England ISO) need to **plan today** for **tomorrow's uncertain demand**
  - Most generators have a high startup time (few hours)

- **Today (1st Stage)**
  - schedule (or commit) generators for each hour tomorrow
  - decide how much each with produce in each hour

- **Tomorrow (2nd Stage)**
  - Uncertain demand is realized
  - ISO may use high cost (quick-start) generators to cover shortfall

System Operator: Planning Problem



b : hourly demand vector for tomorrow

x_t^i : day-ahead generation from plant i in period t

$y_t^j(b)$: generation from plant j in period t for demand is b

Stochastic Model

- **Two-stage Stochastic Optimization Model**

$$
\begin{aligned}
\textbf{zStoch} = \min c^T x + \mathbb{E}[d^T y(b)] \\
Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

- **Intractability: #P-hard**

Adaptive Optimization Model

- **Two-stage Adaptive Optimization Model**

$$
\begin{aligned}
\textbf{zAdapt} = \min c^T x + \max_{b \in \mathcal{U}} d^T y(b) & \\
Ax + By(b) &\geq b, \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

(Minimize worst-case cost)

(Uncertainty Set)

(Uncertain Right Hand Side)

- **Studied in Literature**
  - Bertsekas (1970s)
  - Bemporad and Morari (1999), Bemporad et al. (2003)
  - Ben-Tal et al. (2003), Iyengar (2005), Bertsimas and Caramanis (2005)

- **Still computationally intractable in general**

- **Even approximating LP within an factor of O(log m) is NP-hard [Feige et al.'07]**

Robust Optimization Model

$$
\begin{aligned}
\text{zRob} = \min\; & c^T x + d^T y \\
Ax + By &\geq b, \; \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

(Minimize Cost of a **static solution**)

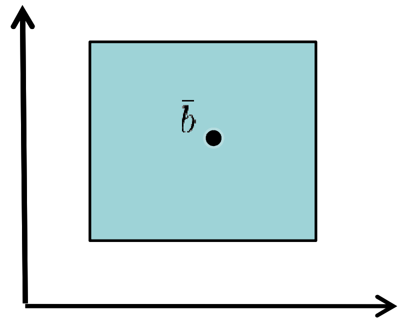
(Uncertainty Set)

(Uncertain Right Hand Side)

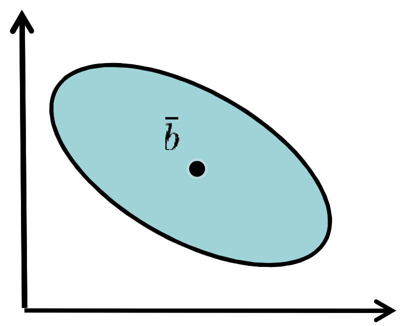
Solution **y** does **not depend on b**

- **Computationally tractable**
- **But does it give a highly conservative solution?**

Uncertainty Sets



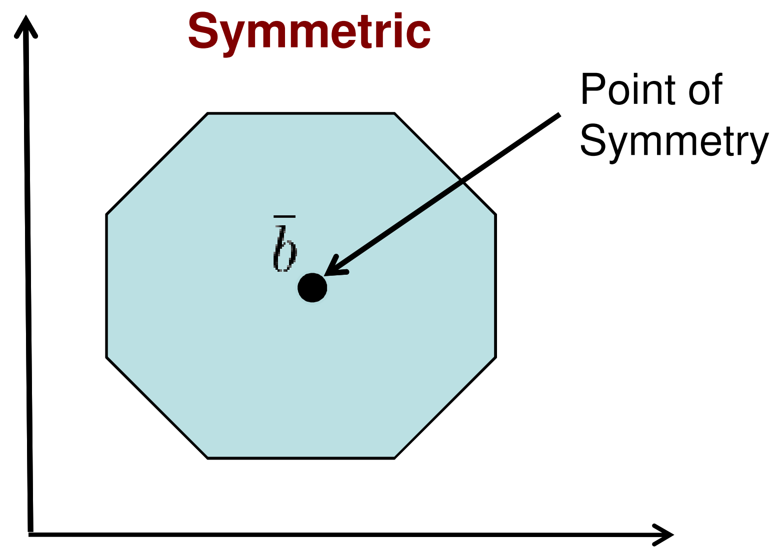
Hypercube : $||b - \bar{b}||_\infty \leq \beta$



Ellipsoid : $||\Sigma(b - \bar{b})||_2 \leq \beta$

Norm-Ball : $||\Sigma(b - \bar{b})||_p \leq \beta$

Symmetric Sets



Set $\mathcal{U}$ is **symmetric** if $\exists\, \bar{b} \in \mathcal{U}$, s.t.

$$(\bar{b} - \delta) \in \mathcal{U} \iff (\bar{b} + \delta) \in \mathcal{U}, \ \forall \delta \in \mathbb{R}^m$$

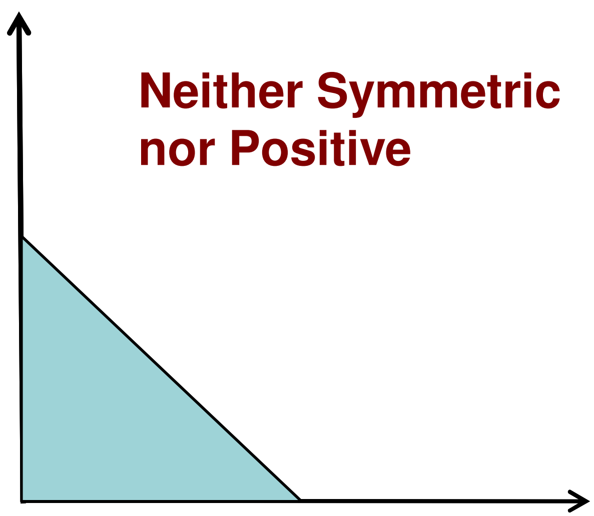
Examples: hypercubes, ellipsoids, norm-balls

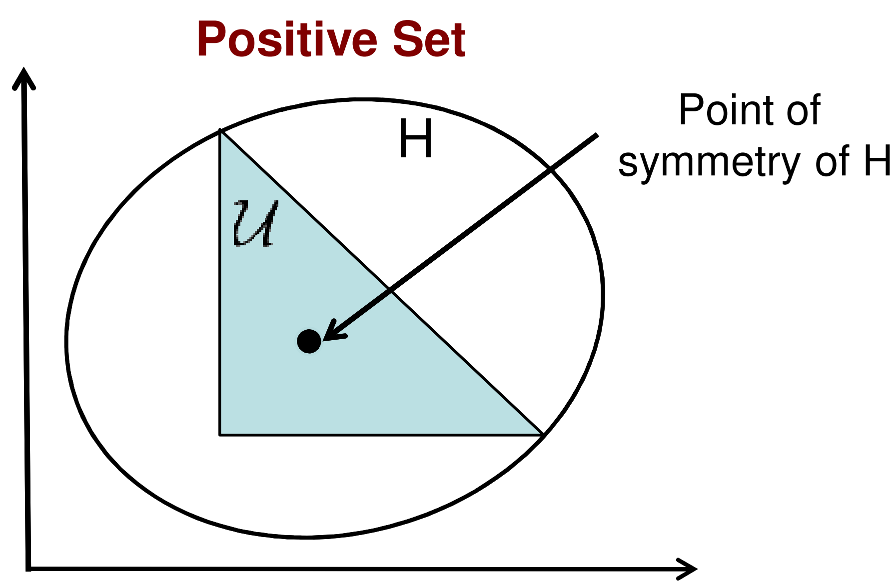
Positive Sets

A set $\mathcal{U} \subseteq \mathbb{R}^m_+$ is **positive**,

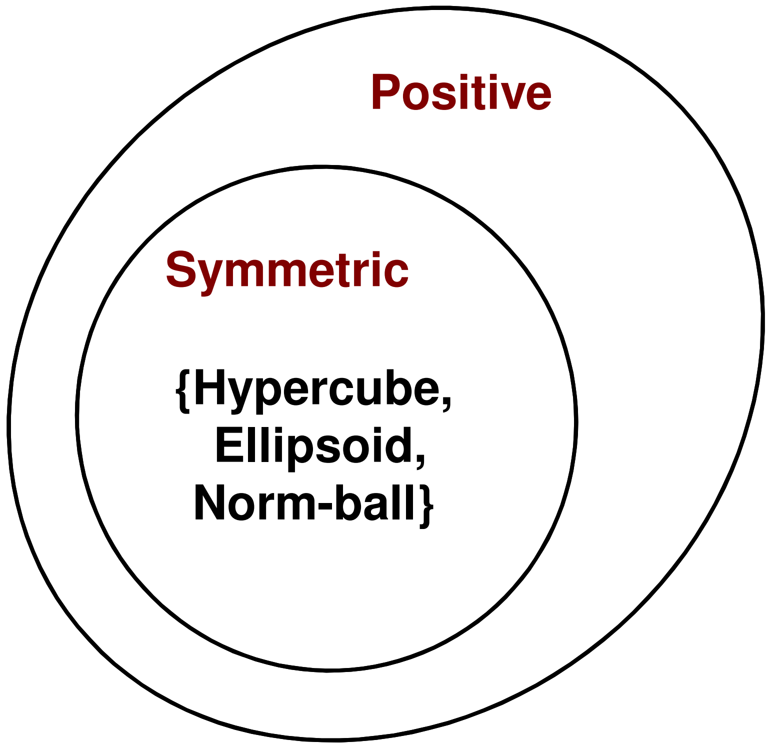
i) a symmetric set $H \subseteq \mathbb{R}^m$ contains $\mathcal{U}$, and,

ii) the point of symmetry of H belongs to $\mathcal{U}$.





Uncertainty Sets



Our Results: Robust Solutions

Stochastic (zStoch)

$$\min c^T x + \mathbb{E}[d^T y(b)]$$
$$\begin{aligned} Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

Adaptive (zAdapt)

$$\min c^T x + \max_b d^T y(b)$$
$$\begin{aligned} Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube		
Symmetric		
Positive		

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Our Results: Robust Solutions

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Adaptive (zAdapt)

$$\begin{aligned} \min c^T x + \max_b d^T y(b) & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube	2	
Symmetric		
Positive		

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Our Results: Robust Solutions

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}[d^T y(b)] & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

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Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube	2	1
Symmetric		
Positive		

Adaptability Gap = 1 for **hypercube uncertainty sets**

(Intuition) Each coordinate can achieve its worst-possible simultaneously

Our Results: Robust Solutions

Stochastic (zStoch)

$$\begin{aligned} \min c^T x + \mathbb{E}[d^T y(b)] & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\geq 0 \end{aligned}$$

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Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube	2	1
Symmetric	2	2
Positive	2	2

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Integer Variables

Stochastic (zStoch)

$$
\begin{aligned}
\min c^T x + \mathbb{E}[d^T y(b)] & \\
Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) &\in \mathbb{R}^n_+
\end{aligned}
$$

Adaptive (zAdapt)

$$
\begin{aligned}
\min c^T x + \max_b d^T y(b) & \\
Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\
x &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \\
y(b) &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+
\end{aligned}
$$

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube	2	1
Symmetric	2	2
Positive	2	2

- **Assumption**: **E[b] =** $\bar{b}$ where $\bar{b}$ is the **point of symmetry**

Symmetric Sets

For any symmetric set $\mathcal{U}$ s.t. $b^0 \in \mathcal{U}$ is the point of symmetry,

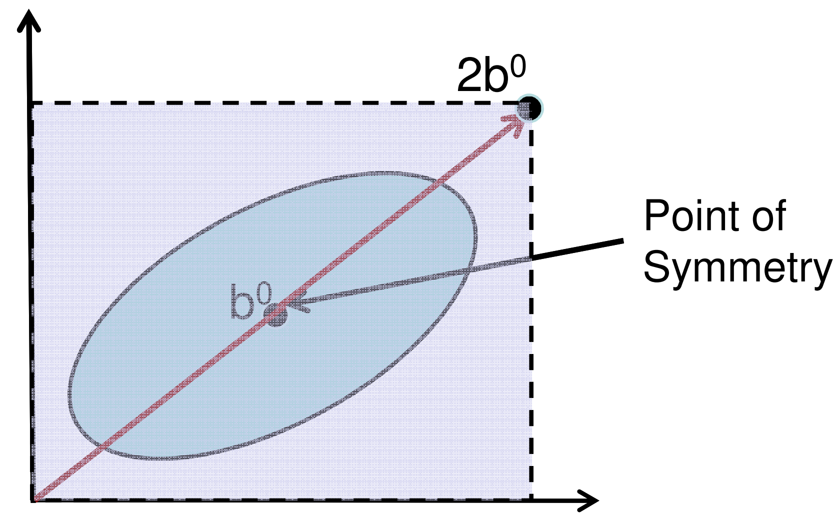
$$b \leq 2b^0, \ \forall \ b \in \mathcal{U}$$

$$b = (b^0 + \delta) \in \mathcal{U}$$

$$(b^0 - \delta) \in \mathcal{U}$$

$$(2b^0 - b) \in \mathcal{U}$$

$$(2b^0 - b) \in \mathbb{R}^m_+ \Rightarrow b \leq 2b^0$$



Stochasticity Gap
(for symmetric RHS uncertainty sets)

zStoch =

$$\begin{aligned} \min\ c^T x + \mathbb{E}_b[d^T y(b)] & & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\in \mathbb{R}^n_+ \end{aligned}$$

Let **x*, y*(b)** be an **optimal solution** for the stochastic problem

Static Solution: **(2x*, 2y*(b^0))** where **b^0** is the **point of symmetry**

Feasibility of Static Robust Solution

zStoch =

$$\begin{aligned} \min c^T x &+ \mathbb{E}_b[d^T y(b)] \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\in \mathbb{R}^n_+ \end{aligned}$$

Static Solution: **$(2x^*, 2y^*(b^0))$** where **b^0** is the **point of symmetry**

$$\begin{aligned} A(2x^*) + B\,(2y^*(b^0)) &= 2\,(Ax^* + B\,y^*(b^0)) \\ &\geq 2b^0 \\ &\geq b \end{aligned}$$

$(2x^*, 2y^*(b^0))$ is a feasible solution for all **$b \in U$**

Cost of Static Robust Solution

Cost: $zRob \leq 2(c^T x^* + d^T y^*(b^0))$ $zStoch = (c^T x^* + d^T E_b[y^*(b)])$

$Ax^* + B\, y^*(b) \geq b$

$E_b[Ax^* + B\, y^*(b)] \geq E_b[b]$

$Ax^* + B(E_b[y^*(b)]) \geq b^0$

$E_b[y^*(b)]$ is a feasible solution for scenario **b^0**

$d^T y^*(b^0) \leq d^T E_b[y^*(b)]$

$$
\begin{aligned}
zRob &\leq 2(c^T x^* + d^T y^*(b^0)) \\
&\leq 2\,(c^T x^* + d^T E_b[y^*(b)]) \\
&= 2\, zStoch
\end{aligned}
$$

Stochasticity Gap ≤ 2

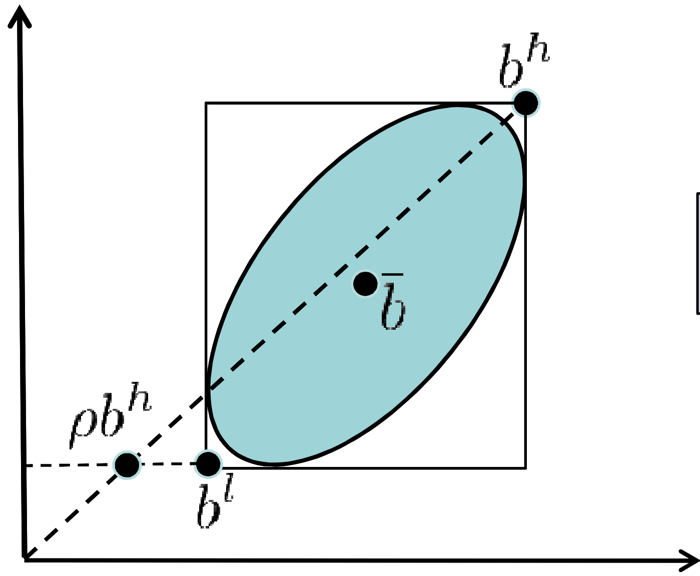
Adaptability Gap (for symmetric RHS uncertainty)

zAdapt ≥ zStoch

Worst-case cost is at least the Expected cost

Adaptability Gap ≤ 2

Improved Parametric Bounds



$$b^l \geq \rho \cdot b^h,\ 0 \leq \rho \leq 1$$

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube	$2/(1+\rho)$	**1**
Symmetric	$2/(1+\rho)$	$2/(1+\rho)$
Positive	$2/(1+\rho)$	$2/(1+\rho)$

Rest of the Talk

- Uncertainty in both Cost and RHS
- Multi-stage problems
- Electricity Markets: Revisited

Our Results: Cost, RHS uncertainty

Stochastic (zStoch)

$$\min c^T x - \mathbb{E}_{(b,d)}[d^T y(b,d)]$$
$$\begin{aligned} Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

Adaptive (zAdapt)

$$\min c^T x + \max_{(b,d)} d^T y(b,d)$$
$$\begin{aligned} Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

Uncertainty Set (U) (Cost and RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube		
Symmetric		
Positive		

Assume: $E_{b,d}$[(b,d)] = $(\bar{b},\bar{d})$ where $(\bar{b},\bar{d})$ is the **point of symmetry**

Our Results: Cost, RHS uncertainty

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$$\begin{aligned} \min c^T x - \mathbb{E}_{(b,d)}[d^T y(b,d)] \\ Ax + By(b,d) &\geq b, \ \forall (b,d) \in \mathcal{U} \\ x, y(b,d) &\in \mathbb{R}^n_+ \end{aligned}$$

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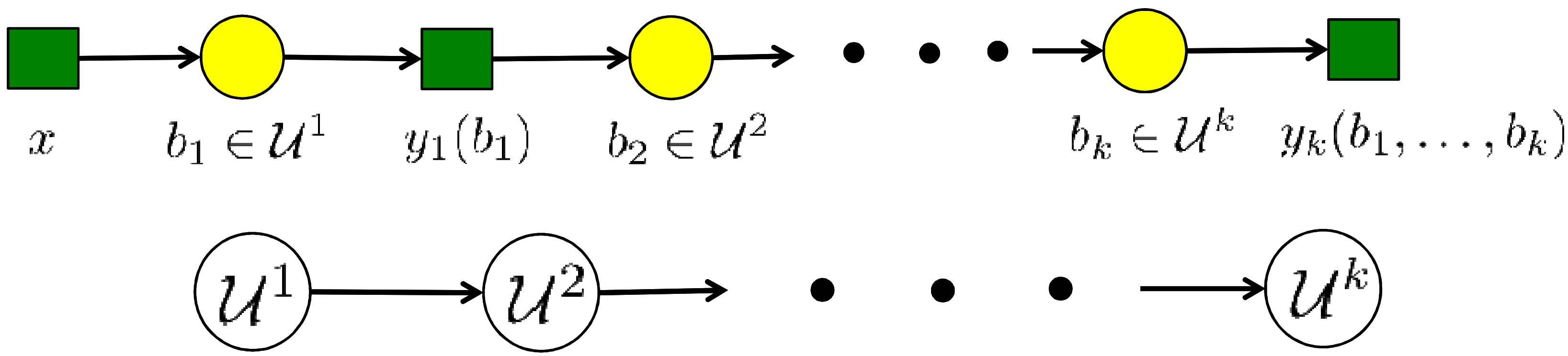
Uncertainty Set (U) (Cost and RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube	Ω(m)	1
Symmetric	Ω(m)	4
Positive	Ω(m)	4

Assume: $E_{b,d}$[(b,d)] = $(\bar{b}, \bar{d})$ where $(\bar{b}, \bar{d})$ is the **point of symmetry**

Multi-Stage Stochastic Model

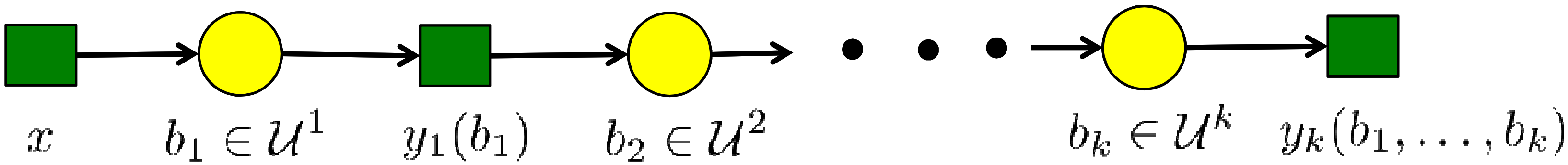


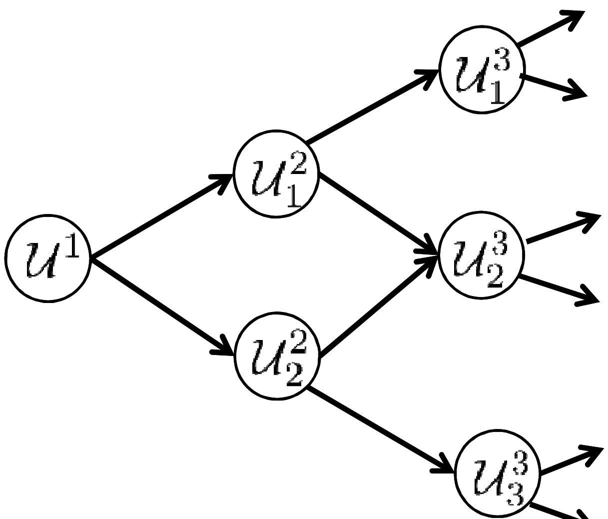
Bounds for Multi-stage Problems

Uncertainty Set (U) (RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube	2	1
Symmetric	2	2
Positive	2	2

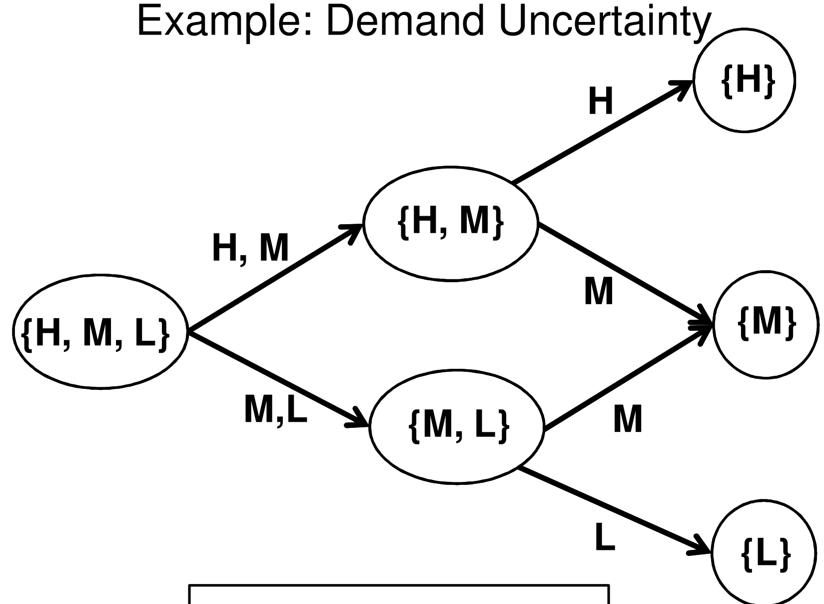
Uncertainty Set (U) (Cost and RHS)	Stochasticity Gap zRob/zStoch	Adaptability Gap zRob/zAdapt
Hypercube	$\Omega(m)$	1
Symmetric	$\Omega(m)$	4
Positive	$\Omega(m)$	4

Multi-Stage Stochastic Model



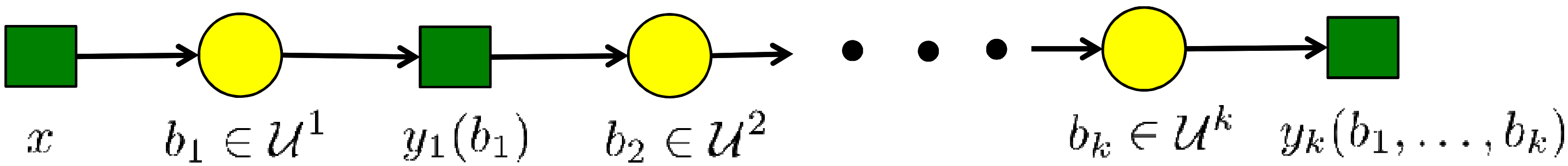


Example: Demand Uncertainty



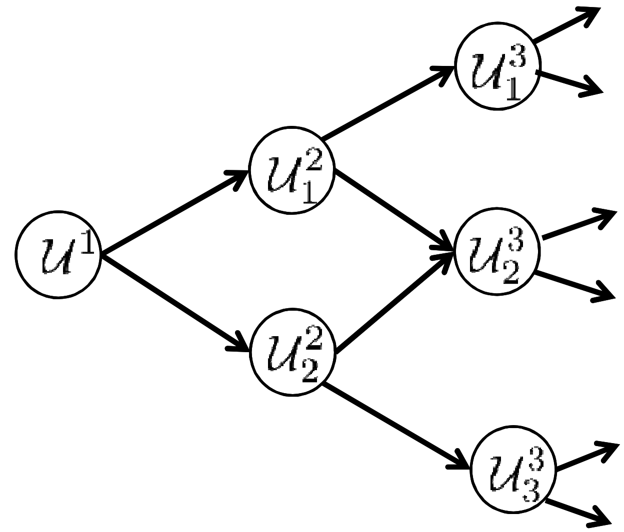
H: High demand
M: Medium demand
L: Low demand

Finitely Adaptable Solution



- Choose a small collection of solutions for each stage

- Select the best feasible solution from the collection after uncertainty has realized



- **Finitely adaptable solution** is a **good approximation** for symmetric and positive uncertainty sets

- number of solutions is equal to the number of uncertainty sets

Outline

- Uncertainty in both Cost and RHS
- Multi-stage problems
- **General Convex Uncertainty (Affine policies)**
- Electricity Markets: Revisited

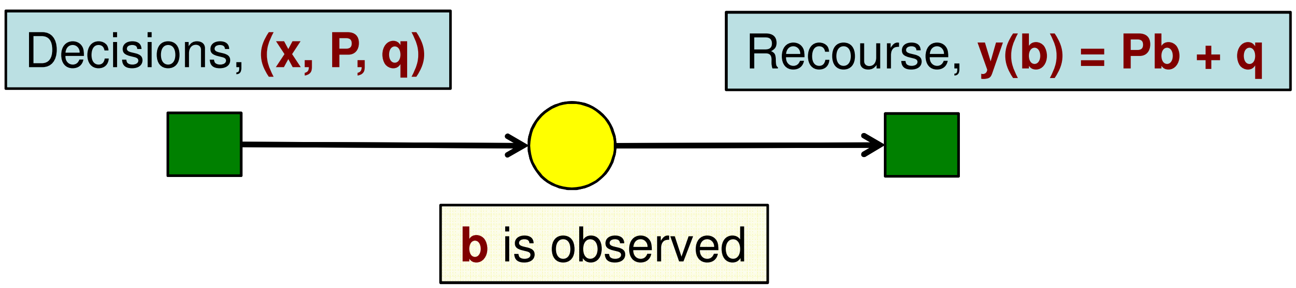
General Convex Uncertainty

$$\begin{aligned}
\textbf{zAdapt} = \min c^T x + \max_{b\in\mathcal{U}} \; & d^T y(b) \\
Ax + By(b) \; & \geq \; b, \; \forall b \in \mathcal{U} \\
x, y(b) \; & \geq \; 0
\end{aligned}$$

Affine Policies

$$\begin{aligned} \textbf{zAdapt} = \min c^T x + \max_{b \in \mathcal{U}} \ & d^T y(b) \\ Ax + By(b) & \geq b, \ \forall b \in \mathcal{U} \\ x, y(b) & \geq 0 \end{aligned}$$

$$y(b) = Pb + q$$ → (Affine function of RHS, b)



Affine Policies: Previous Work

- Extensively studied in literature
  - Gartska and Wets (1974), Rockafellar and Wets (1978)
  - Bemporad and Morari (1999)
  - Bertsimas et al. (2009), Skaf and Boyd (2009)

- **Computationally tractable**

- Perform extremely well in practice
  - Kalman filtering (Kalman (1960))
  - Linear decision rules for approximate DP (Bertsekas (2001), de Farias and Van Roy (2003))
  - Retailer-supplier flexible commitment contracts (Ben-Tal et al. (2005))

Affine Policies: Simplex Uncertainty Sets

Simplex $\mathcal{U} = \mathbf{conv}(b^1, \ldots, b^{m+1})$

Affine policies are optimal if the uncertainty set is a simplex

$$y(b) = \underbrace{\begin{bmatrix} & P & \end{bmatrix}}_{\text{m columns}} b + \begin{bmatrix} q \end{bmatrix}$$

- Simplex has (m+1) extreme points
- Enough **degrees of freedom** to obtain an **optimal solution**

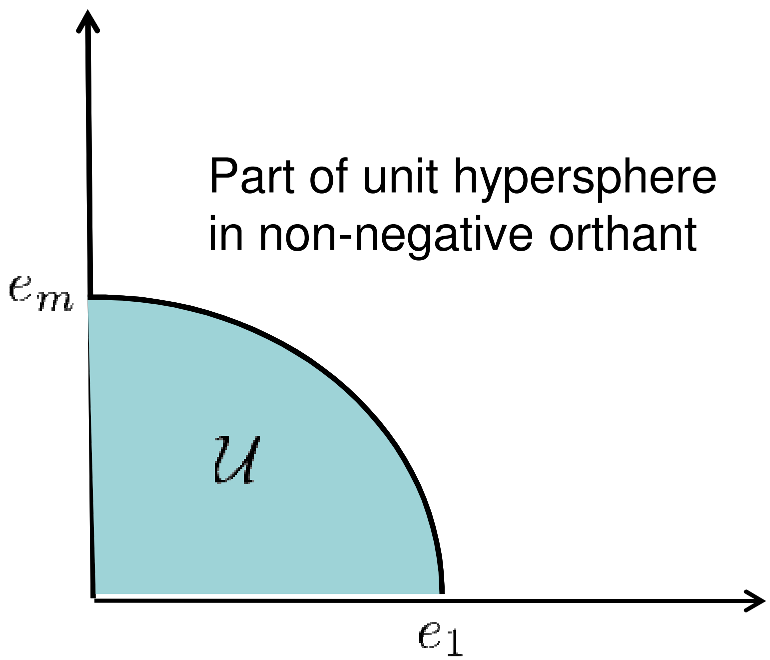
Affine Policies: General Convex Sets

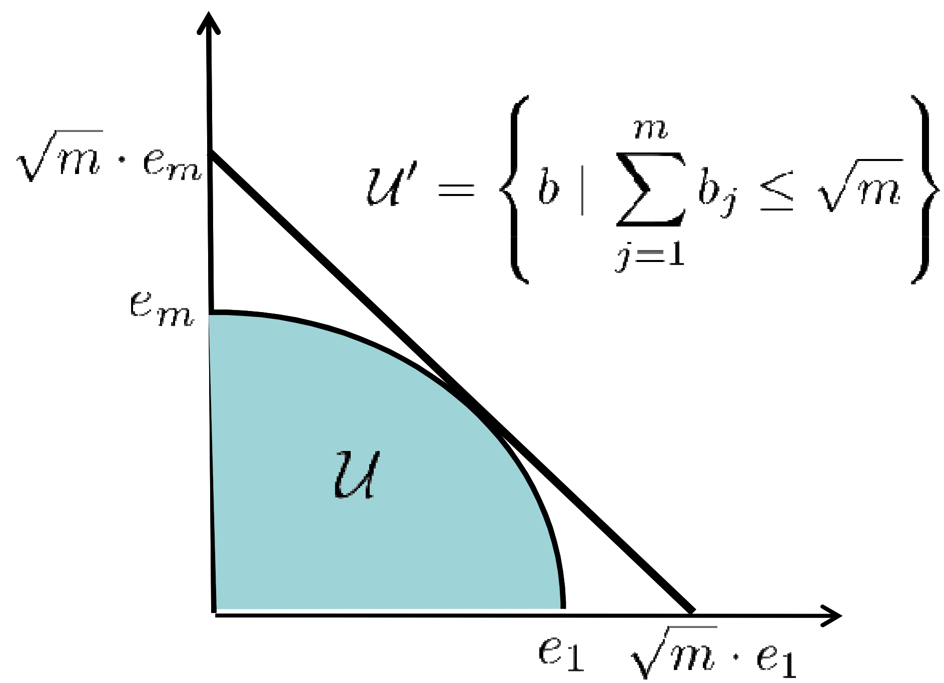
- **Cost of optimal affine policy is at most $\sqrt{m}$ times the optimal adaptive problem (zAdapt)**

- **Cost of optimal affine policy is at least $\Omega(\sqrt{m})$ times the optimal adaptive problem (zAdapt)**

Performance of affine policies $\Theta(\sqrt{m})$ times the optimal

Geometric Intuition

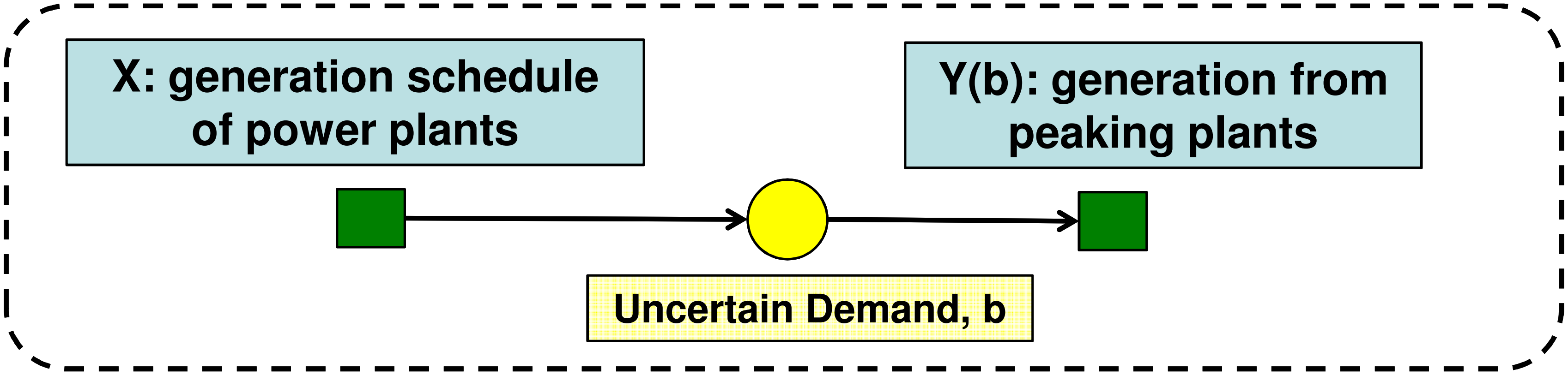




Outline

- Uncertainty in both Cost and RHS
- Multi-stage problems
- **Electricity Markets: Revisited**

Electricity Markets Revisited



Deterministic Model Minimize $c_1(X) + c_2(Y(b^0))$

- **Forecast demand (b^0) treated as deterministic**
- **$(X, Y(b^0))$ satisfies demand b^0**
- **Cover shortfall in real-time using costly peaking plants**

Our Model and Results

Adaptive Model Minimize $c_1(X)$ + $\max_{b \in U} c_2(Y(b))$

- **Demand uncertainty modeled as hypercube**

- $\mathcal{U} = [b^0 - \delta e, b^0 + \delta e]$
  - b^0: day-ahead forecast vector
  - δ: std. dev. of forecast error

- **zAdapt = zRob**
  - can solve the adaptive problem **optimally**

- **Richer Model** that **handles uncertainty** in day-ahead problem

- **Adaptive model improves cost** on average **~2%** as compared to the deterministic model

Conclusions and Future Directions

- Robust Optimization
  - **Tractable approach**
  - **Good approximation** for fairly general dynamic opt. problems

- Potential for **commercial success** similar to deterministic optimization

- **(Future Directions)** Multi-stage problems in Operations Research both methodologically and practically
  - Energy, supply chain management, pricing and revenue management

Related Papers

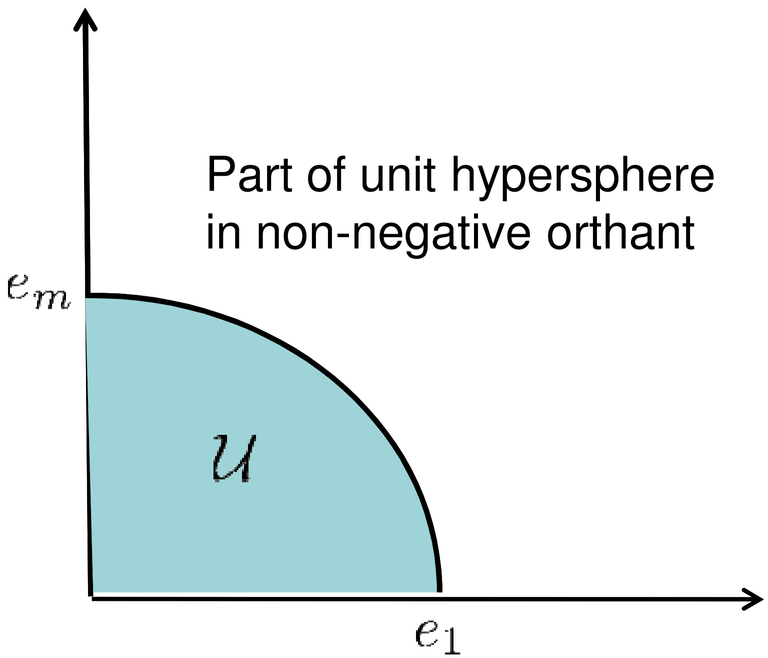
[1] D. Bertsimas and V. Goyal. On the Power of Robust Solutions in Two-stage Stochastic and Adaptive Problems. To Appear in *Math of Operations Research*

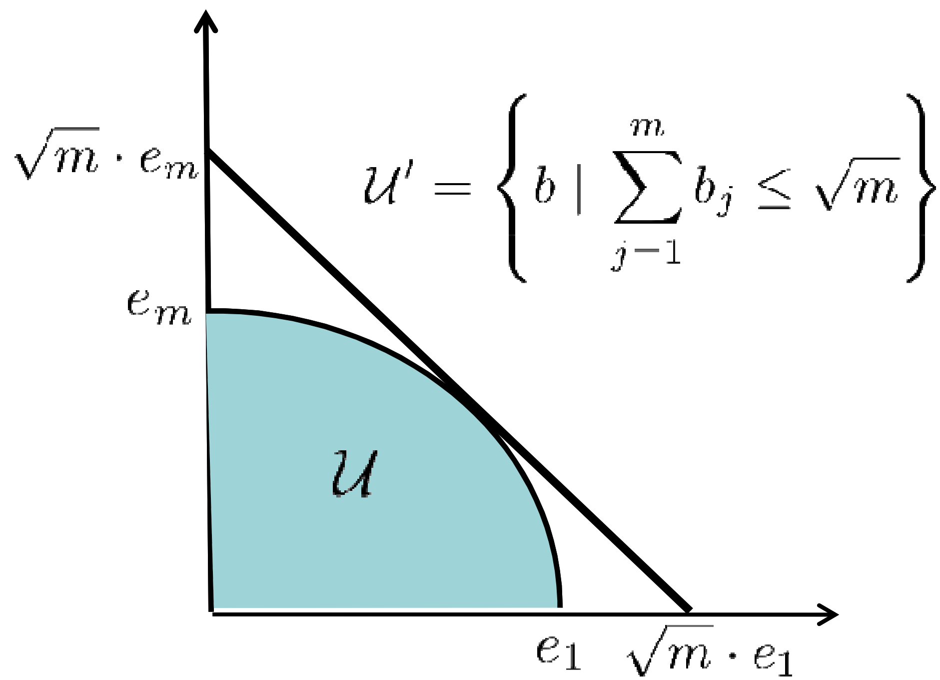
[2] D. Bertsimas and V. Goyal. On the Power and Limitations of Affine Policies in Two-stage Adaptive Optimization. Submitted to *Math Programming*.

[3] D. Bertsimas and V. Goyal. On the Power of Finite Adaptability in Multi-stage Stochastic and Adaptive Optimization Problems. *In preparation*

[4] D. Bertsimas and V. Goyal. An Adaptive Optimization Approach to Unit-Commitment under Demand and Capacity Uncertainty. *In preparation*

Affine Policies: Convex Sets





Cost of optimal affine policy is at most $\sqrt{m}$ times the optimal

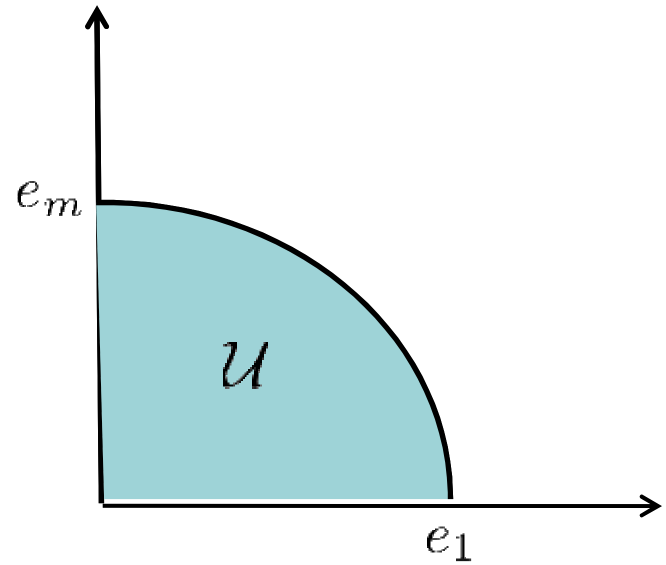
Outline

- Models (Stochastic, Adaptive, and Robust)
- An Example from Electricity Markets
- Performance Bounds for Two-stage problems
- Performance bounds for Multi-stage problems
- Affine Policies and their Performance
- Electricity Problem: Revisited
- Conclusions

Affine Policies: Lower Bound

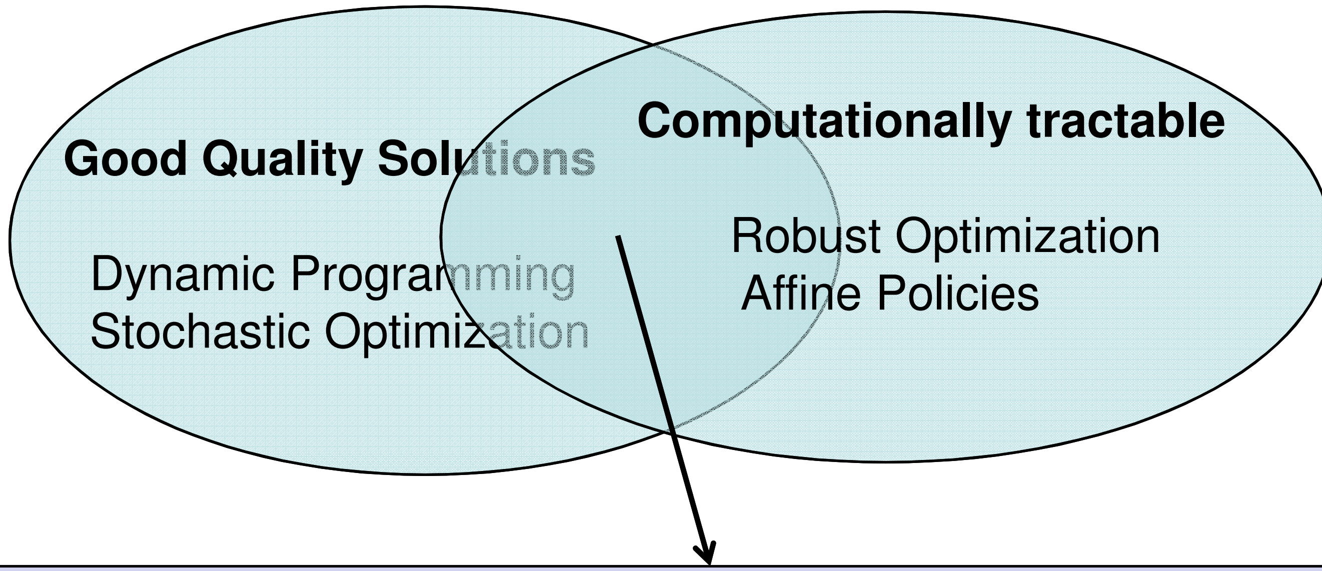
$$\min_{y(b)\geq 0} \max_b \, (1,\ldots,1)^T y(b)$$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & \frac{1}{m^{1/2-\epsilon}} & \\ & & \cdot & & \\ & \frac{1}{m^{1/2-\epsilon}} & & \cdot & \\ & & & & \cdot \\ & & & & & 1 \end{bmatrix} y(b) \geq b, \ \forall\, b \in \mathcal{U}$$



Cost of optimal affine policy is at least $\Omega(\sqrt{m})$ times the optimal

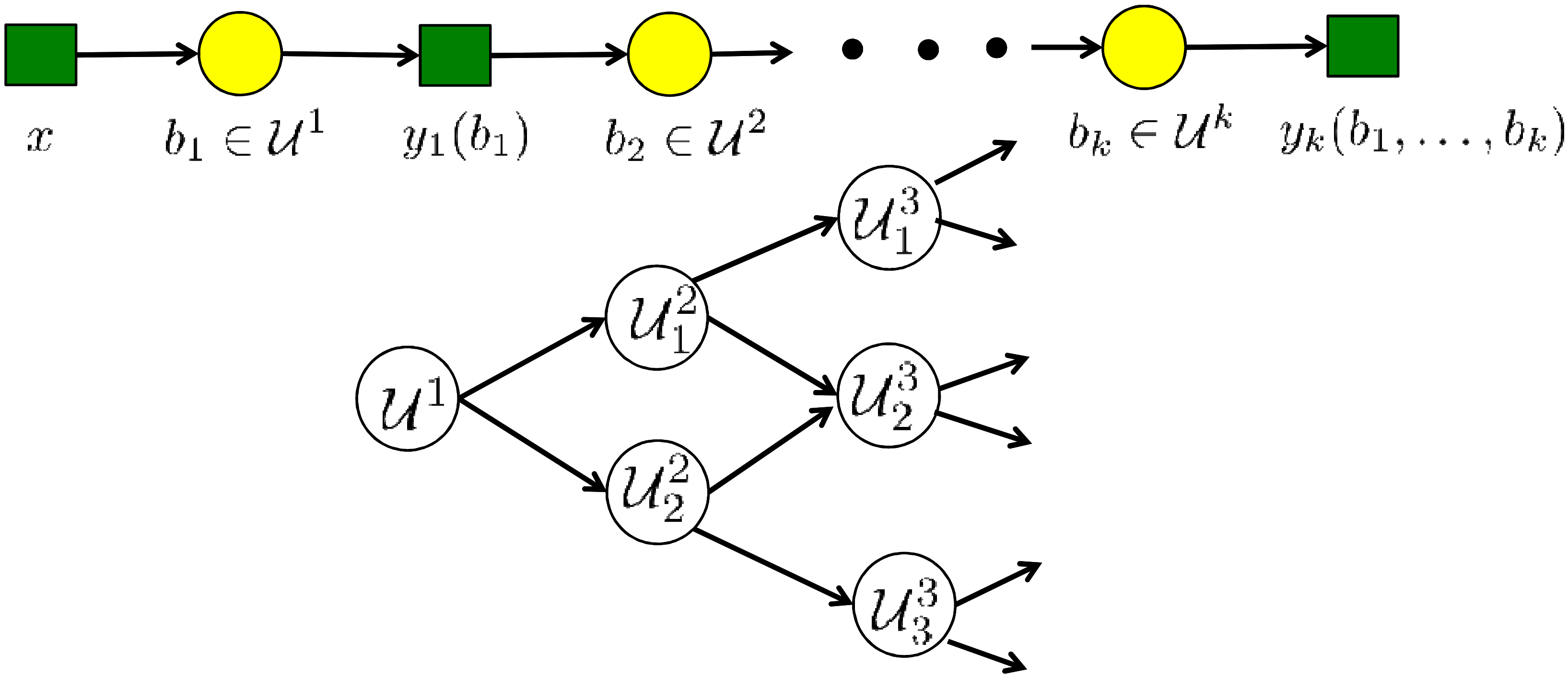
Conclusions and Future Directions



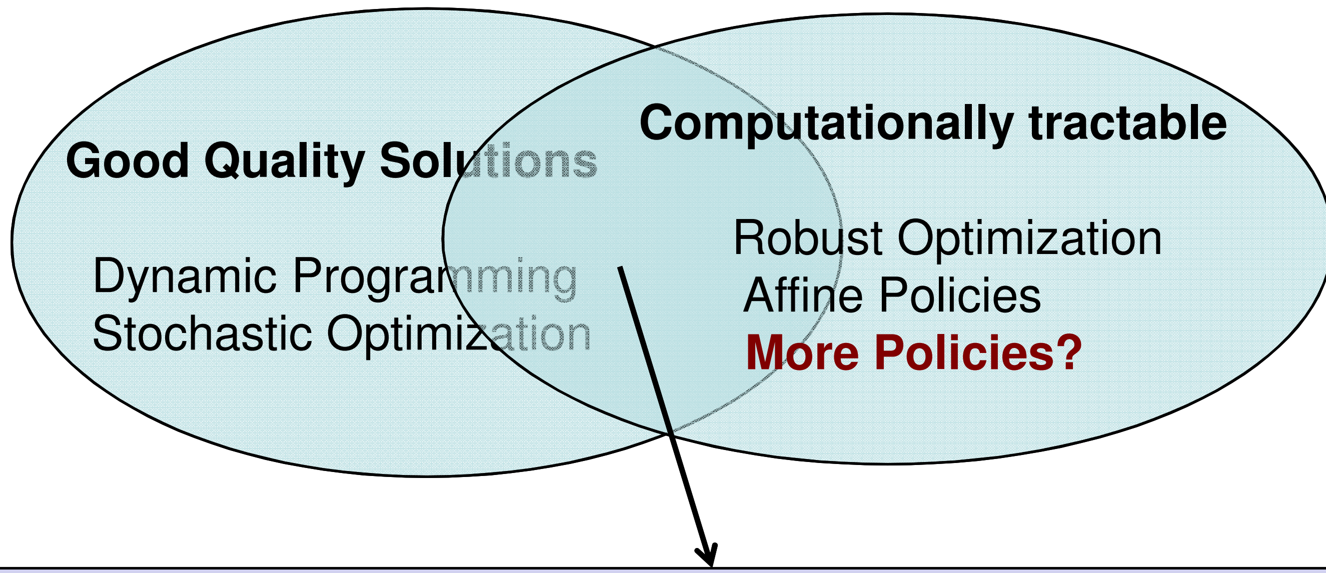
- **Robust optimization** is a **practical approach** to dynamic optimization
  - if uncertainty is **distributed symmetrically** in a **symmetric set**

- **Affine policies** perform well for **symmetric** and **simplex** uncertainty sets

Multi-Stage Stochastic Model



Conclusions and Future Directions



- **Robust optimization** is a **practical approach** to dynamic optimization
  - if uncertainty is **distributed symmetrically** in a **symmetric set**

- **Affine policies** perform well for **symmetric** and **simplex** uncertainty sets

- **More policies and more classes of problems?**

Previous Work in Stochastic Optimization

- Studied extensively in literature
  - Dantzig (1955), Rockafellar and Wets (1978), Birge and Louveaux (1997), Prekopa (1995), Shapiro (2008)
  - Combinatorial Problems: Shmoys and Swamy (2006), Ravi and Sinha (2004)
- Computationally intractable in general
  - Dyer and Stougie (2005), Shapiro and Nemirovski (2005)

Symmetric Sets

For any symmetric set $\mathcal{U}$ s.t. $b^0 \in \mathcal{U}$ is the point of symmetry,

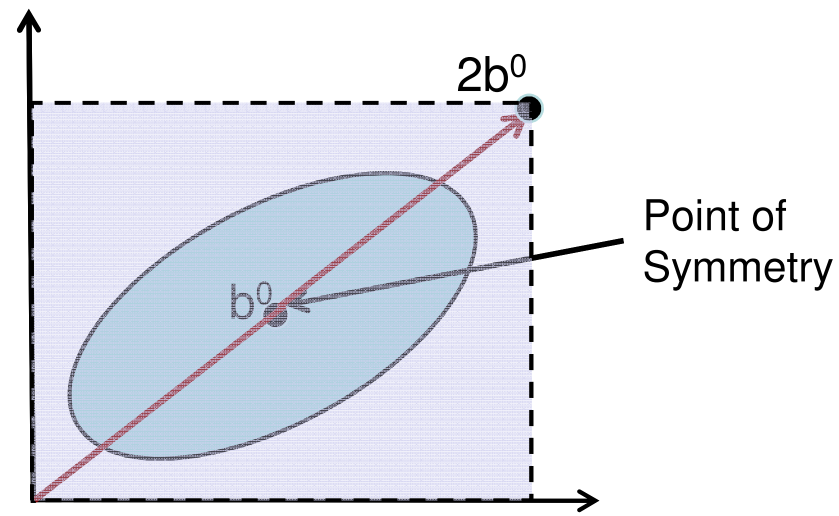
$$b \leq 2b^0, \ \forall \ b \in \mathcal{U}$$

$$b = (b^0 + \delta) \in \mathcal{U}$$

$$(b^0 - \delta) \in \mathcal{U}$$

$$(2b^0 - b) \in \mathcal{U}$$

$$(2b^0 - b) \in \mathbb{R}^m_+ \Rightarrow b \leq 2b^0$$



Stochasticity Gap (for symmetric RHS uncertainty sets)

zStoch =

$$\begin{aligned} \min\ c^T x + \mathbb{E}_b[d^T y(b)] & & \\ Ax + By(b) &\geq b, \ \forall b \in \mathcal{U} \\ x, y(b) &\in \mathbb{R}^n_+ \end{aligned}$$

Let **x*, y*(b)** be an **optimal solution** for the stochastic problem

Static Solution: **(2x*, 2y*(b^0))** where **b^0** is the **point of symmetry**

Feasibility of Static Robust Solution

zStoch =

$$\begin{aligned} \min\ & c^T x + \mathbb{E}_b[d^T y(b)] \\ Ax + By(b) &\geq b,\ \forall b \in \mathcal{U} \\ x, y(b) &\in \mathbb{R}^n_+ \end{aligned}$$

Static Solution: **$(2x^*, 2y^*(b^0))$** where **b^0** is the **point of symmetry**

$$\begin{aligned} A(2x^*) + B(2y^*(b^0)) &= 2(Ax^* + By^*(b^0)) \\ &\geq 2b^0 \\ &\geq b \end{aligned}$$

$(2x^*, 2y^*(b^0))$ is a feasible solution for all **$b \in U$**

Cost of Static Robust Solution

Cost: $zRob \leq 2(c^T x^* + d^T y^*(b^0))$ | $zStoch = (c^T x^* + d^T E_b[y^*(b)])$

$Ax^* + B\,y^*(b) \geq b$

$E_b[Ax^* + B\,y^*(b)] \geq E_b[b]$

$Ax^* + B(E_b[y^*(b)]) \geq b^0$

$E_b[y^*(b)]$ is a feasible solution for scenario **b^0**

$d^T y^*(b^0) \leq d^T E_b[y^*(b)]$

$$
\begin{aligned}
zRob &\leq 2(c^T x^* + d^T y^*(b^0)) \\
&\leq 2\,(c^T x^* + d^T E_b[y^*(b)]) \\
&= 2\,zStoch
\end{aligned}
$$

Stochasticity Gap ≤ 2

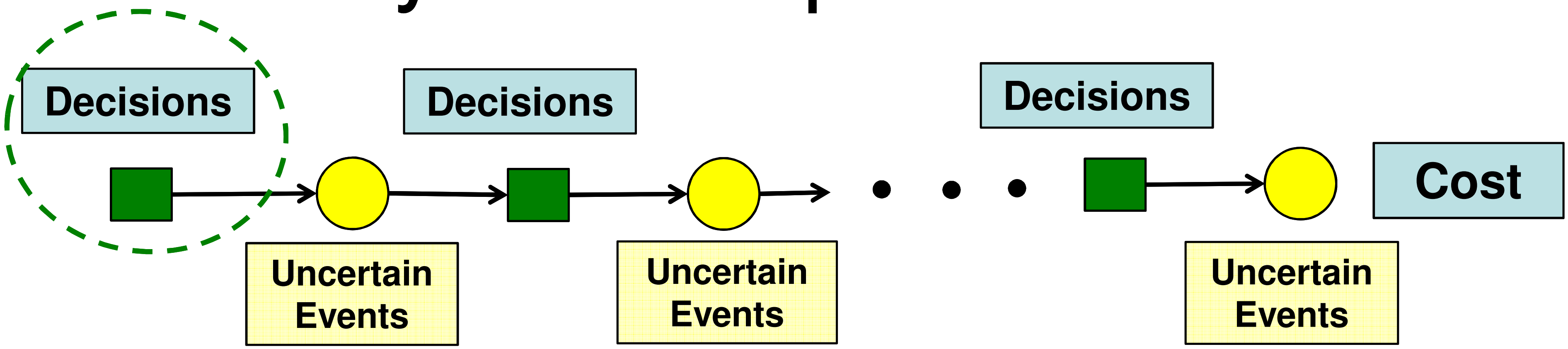
Adaptability Gap (for symmetric RHS uncertainty)

zAdapt ≥ zStoch

Worst-case cost is at least the Expected cost

Adaptability Gap ≤ 2

Dynamic Optimization



Good Quality Solutions

Dynamic Programming
Stochastic Optimization

Computationally tractable

Robust Optimization

When can we obtain the **best of both** worlds?

Conclusions

Uncertainty Set (U) (RHS)	Stochasticity Gap (zRob/zStoch)	Adaptability Gap (zRob/zAdapt)
Hypercube	2*	1*
Symmetric	2*	2*
Positive	2*	2*
General Convex	Ω(m)	Ω(m)

Uncertainty Set (U) (Cost and RHS)	Stochasticity Gap (zRob/zStoch)	Adaptability Gap (zRob/zAdapt)
Hypercube	Ω(m)	1*
Symmetric	Ω(m)	4
Positive	Ω(m)	4
General Convex	Ω(m)	Ω(m)

Performance of affine policies is $\Theta(m)$ worse than optimal adaptive solution for general convex uncertainty sets

Future Directions

- **Dynamic Optimization is important due to its wide applicability but computationally intractable**
- **Broad Goal**
  - **Understand models where a good approximation is possible (eg. symmetric uncertainty sets)**
  - **Also, the *dual* problem of identifying models where dynamic optimization will be intractable**

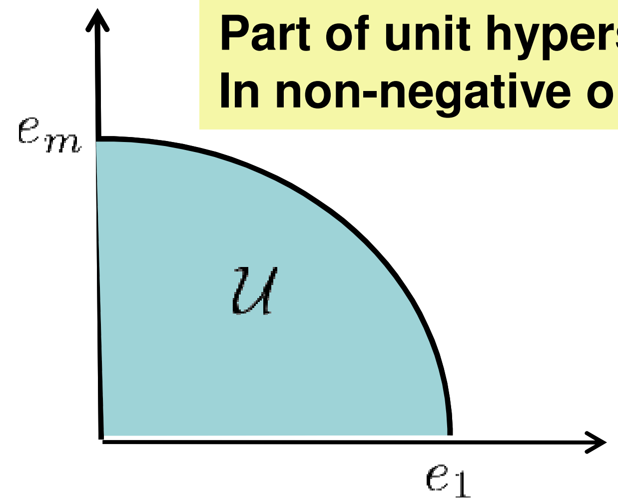
Upper Bound: Simple Case

$$\textbf{zAdapt} = \min c^T x \;|\; \max_{b \in \mathcal{U}} d^T y(b)$$

$$\begin{aligned} Ax + By(b) &\geq b, \; \forall b \in \mathcal{U} \\ x, y(b) &\in \mathbb{R}^n_+ \end{aligned}$$

$$y(b) = Pb + q$$

Part of unit hypersphere
In non-negative orthant



Let **x*, y*(b)** be an **optimal solution** for the adaptive problem

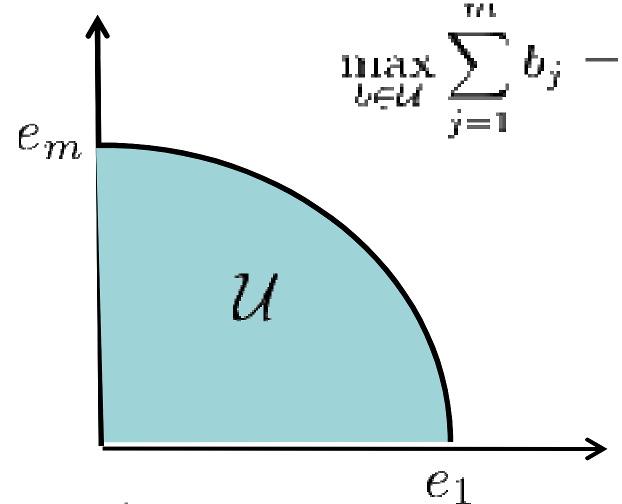
Affine Solution

$$\begin{aligned} \bar{x} &= \sqrt{m} \cdot x^* \\ \bar{y}(b) &= \sum_{j=1}^{m} y^*(e_j) \cdot b_j \end{aligned}$$

Upper Bound: Simple Case

Affine Solution

$$\bar{x} = \sqrt{m} \cdot x^*$$
$$\bar{y}(b) = \sum_{j=1}^{m} y^*(e_j) \cdot b_j$$



$$\max_{b \in \mathcal{U}} \sum_{j=1}^{m} b_j - \sqrt{m}$$

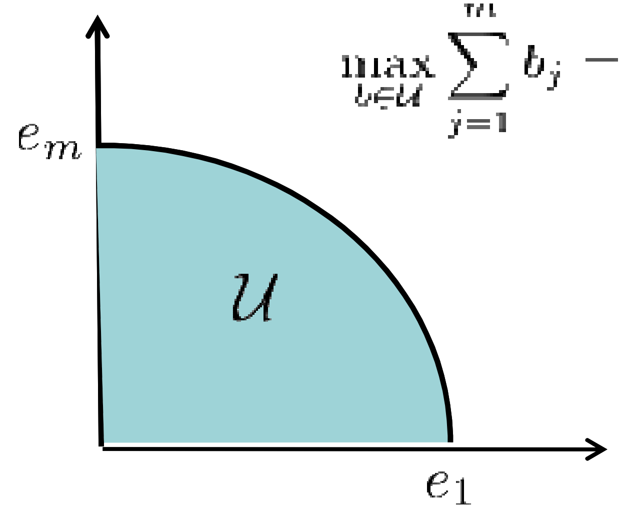
Feasibility

$$
\begin{aligned}
A(\sqrt{m} \cdot x^*) + B\left(\sum_{j=1}^{m} y^*(e_j) \cdot b_j\right) \\
= A\left(\sqrt{m} - \sum_{j=1}^{m} b_j\right) \cdot x^* + \sum_{j=1}^{m} (Ax^* + By^*(e_j)) \cdot b_j \\
\geq \sum_{j=1}^{m} e_j \cdot b_j \\
= b
\end{aligned}
$$

Upper Bound: Simple Case

Affine Solution

$$\bar{x} = \sqrt{m} \cdot x^*$$
$$\bar{y}(b) = \sum_{j=1}^{m} y^*(e_j) \cdot b_j$$



$$\max_{b \in \mathcal{U}} \sum_{j=1}^{m} b_j = \sqrt{m}$$

Cost

$$\begin{aligned} c^T(\sqrt{m} \cdot x^*) - \max_{b \in \mathcal{U}} \sum_{j=1}^{m} d^T y^*(e_j) \cdot b_j &\le \sqrt{m} \cdot c^T x^* + \max_{b \in \mathcal{U}} \sum_{j=1}^{m} \left(\max_{k \in [m]} d^T y^*(e_k) \right) \cdot b_j \\ &= \sqrt{m} \cdot c^T x^* + \left(\max_{b \in \mathcal{U}} \sum_{j=1}^{m} b_j \right) \cdot \left(\max_{k \in [m]} d^T y^*(e_k) \right) \\ &\le \sqrt{m} \cdot c^T x * - \sqrt{m} \cdot \left(\max_{k \in [m]} d^T y^*(e_k) \right) \\ &\le \sqrt{m} \cdot \text{zAdapt} \end{aligned}$$

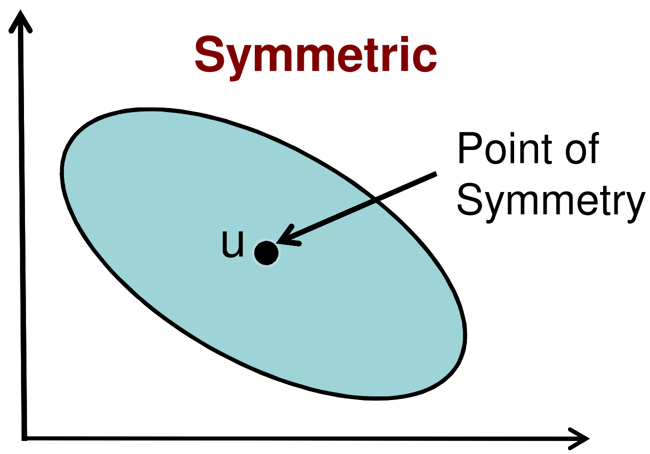
Uncertainty Sets

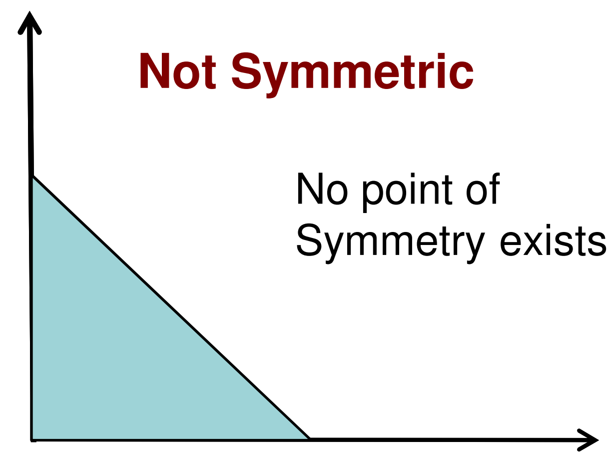
A set S is a **hypercube** if **S = { x | l ≤ x ≤ u}** for some vectors **l, u**

A set S is a **symmetric** if there exists a point **u ε S** such that,

(u-z) ε S if and only if (u+z) ε S for all z

Note that **u** is the **point of symmetry**





Upper Bound: General Case

- Assume $\mathcal{U}$ is in the unit-hypercube by scaling the constraint matrices

- Can partition the set [m] in to [J_1; J_2] such that

$$\sum_{j \in J_1} b_j \le \sqrt{m}, \ \forall b \in \mathcal{U}$$

$$\exists\, \bar{b} \in \mathcal{U} \text{ s.t. } b_j \le \sqrt{m} \cdot \bar{b}_j, \ \forall j \in J_2, \ \forall b \in \mathcal{U}$$

Feasible Solution

$$\hat{x} = \sqrt{m} \cdot x^*, \ \hat{y}(b) = \sum_{j \in J_1} y^*(e_j) \cdot b_j + \sqrt{m} \cdot y^*(\bar{b})$$

Large Stochasticity Gap Example (Cost and RHS uncertainty)

$$
\begin{aligned}
\text{zStoch} = \min\ & \mathbb{E}_d[d^T y(d)] \\
y_1 + y_2 + \ldots + y_m &\geq 1 \\
y(d) &\in \mathbb{R}^m_+,\ \forall d \in U
\end{aligned}
$$

- **U** (cost uncertainty only): **0-1 hypercube**, i.e., $U = [0,1]^m$
- d_j: **uniformly distributed** between 0 and 1 **independent** of others

Optimal Static Robust	Optimal Stochastic
$\text{zRob} = \mathbb{E}_d[d^T y] = \mathbb{E}[d_1 y_1 + \ldots + d_m y_m] = \mathbb{E}[d_1]\cdot y_1 + \ldots + \mathbb{E}[d_m]\cdot y_m = (y_1 + \ldots + y_m)/2 \geq 1/2$	$y_j^*(d) = \begin{cases} 1 & \text{if } d_j = \min_k d_k \\ 0 & \text{otherwise} \end{cases}$ $\text{zStoch} = \mathbb{E}_d[d^T y^*(d)] = \mathbb{E}_d[\min(d_1, \ldots, d_m)] = \frac{1}{m+1}$

Adaptability Gap
(Cost and RHS uncertainty)

zAdapt =

$$
\begin{aligned}
\min c^T x + \max_{(b,d)} \ d^T y(b,d) & \\
Ax + By(b,d) & \geq b, \ \forall (b,d) \in \mathcal{U} \\
x, y(b,d) & \in \mathbb{R}^n_+
\end{aligned}
$$

Let **x*, y*(b,d)** be an **optimal solution** for the adaptive problem

Static Solution: (2x*, 2y*(b^0,d^0)) where **(b^0,d^0)** is the **point of symmetry**

Feasibility: A (2x*) + B (2y*(b^0,d^0)) ≥ **2b^0** ≥ **b**, for all **b ε U**

Cost:

$$
\begin{aligned}
\text{zRob} & \leq c^T(2x^*) + \max_{(b,d)\in U} d^T(2y^*(b^0, d^0)) \\
& \leq 2c^T x^* + (2d^0)^T(2y^*(b^0, d^0)) \\
& = 2c^T x^* + 4(d^0)^T y^*(b^0, d^0) \\
& \leq 4(c^T x^* + 4(d^0)^T y^*(b^0, d^0)) \leq 4 \cdot \text{zAdapt}
\end{aligned}
$$

Stochasticity Gap: Tight Example

zStoch =

$$\begin{aligned} \min\ & \mathbb{E}_b\left[(1,\ldots,1)^T y(b)\right] \\ & I_m y(b) \geq b,\ \forall b \in U \\ & y(b) \in \mathbb{R}^m_+ \end{aligned}$$

- **I_m: m x m identity matrix**

Stochasticity Gap = 2 (is tight)

Even for Hypercube uncertainty sets

Proof Sketch

- Show that zAdapt ≤ 1
- Show existence of a "symmetric" optimal affine solution

$$y(b) = Pb + q$$

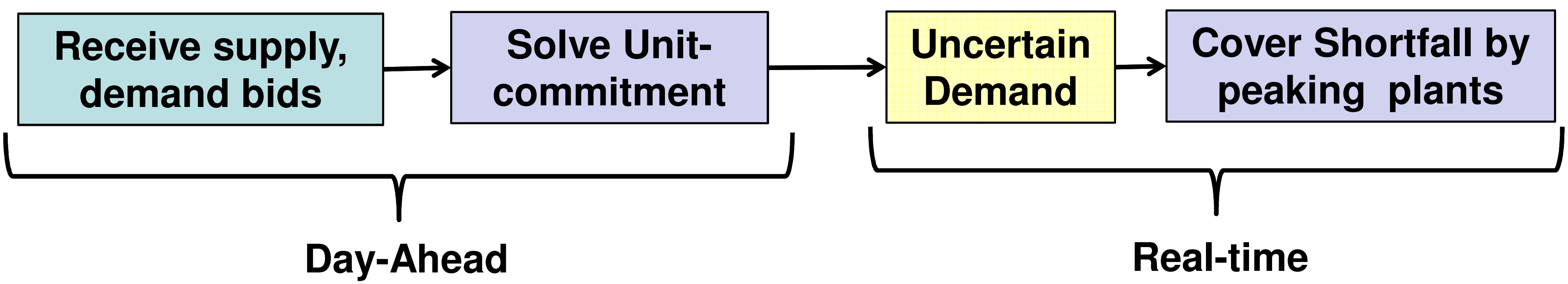
$$P_{ii} = \lambda,\ P_{ij} = \theta,\ \forall i, j = 1, \ldots, m$$

$$q_j = \beta,\ \forall j = 1, \ldots, m$$

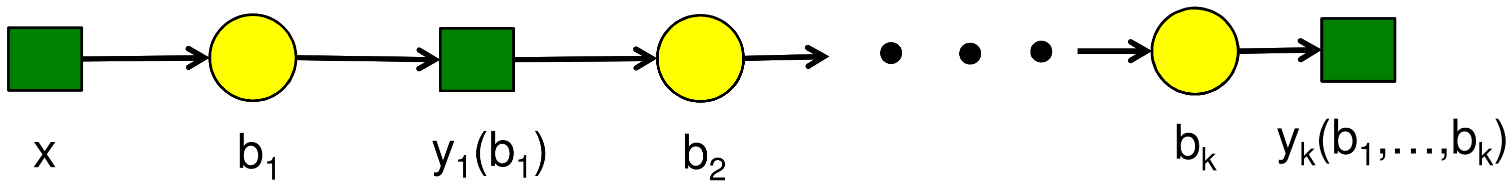
$$\text{zAffine} \geq \frac{m^{1/2-\delta}}{4}$$

System Operator: Unit Commitment Problem

- **Schedule or Commit generators** for each hour of the next day to satisfy an uncertain demand
  - **Minimize total expected cost**
  - **Operational and security constraints** are not violated
  - Real-time **energy balance** achieved by costly **peaking units**
  - **Multi-stage optimization problem** – hard to solve



Multi-Stage Stochastic Model



$$\text{zStoch} = \min c^T x + \mathbb{E}_{b_1}\left[d_1^T y_1(b) + E_{b_2}\left[d_2^T y_2(b_1, b_2) + \ldots + E_{b_k}\left[d_k^T y_k(b_1, \ldots, b_k)\right] \ldots\right]\right]$$

$$\begin{aligned} Ax + \sum_{t=1}^{k} B_t y_t(b_1, \ldots, b_t) &\geq \sum_{t=1}^{k} b_t, \ \forall b_t \in \mathcal{U}_t(b_1, \ldots, b_{t-1}) \\ x, y_t(b_1, \ldots, b_t) &\in \mathbb{R}^n_+ \times \mathbb{Z}^p_+ \end{aligned}$$

(Uncertainty Set in stage t)

Decision in t^{th} stage depend on uncertain parameters in the first t stages

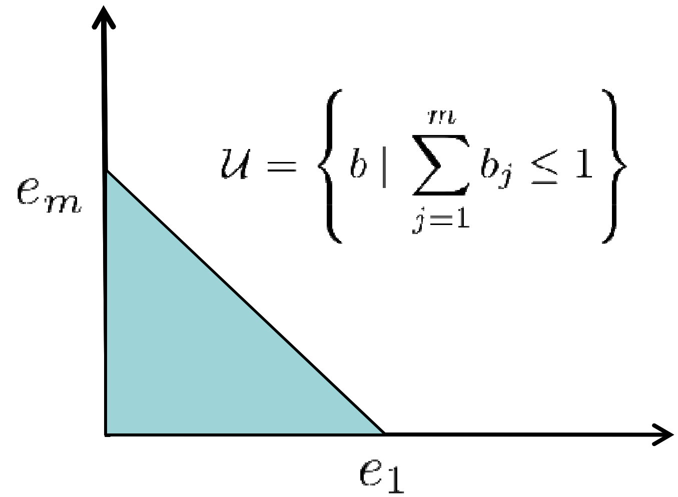
Stochasticity Gap (zRob/zStoch) ≤ 2

for symmetric and positive uncertainty sets for RHS uncertainty

General Convex Uncertainty: Bad Example

$$\min_{y(b)\geq 0} \max_b \, (1,\ldots,1)^T y(b)$$

$$\begin{bmatrix} 1 & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ 0 & & & & 1 \end{bmatrix} y(b) \geq b, \; \forall\, b \in \mathcal{U}$$

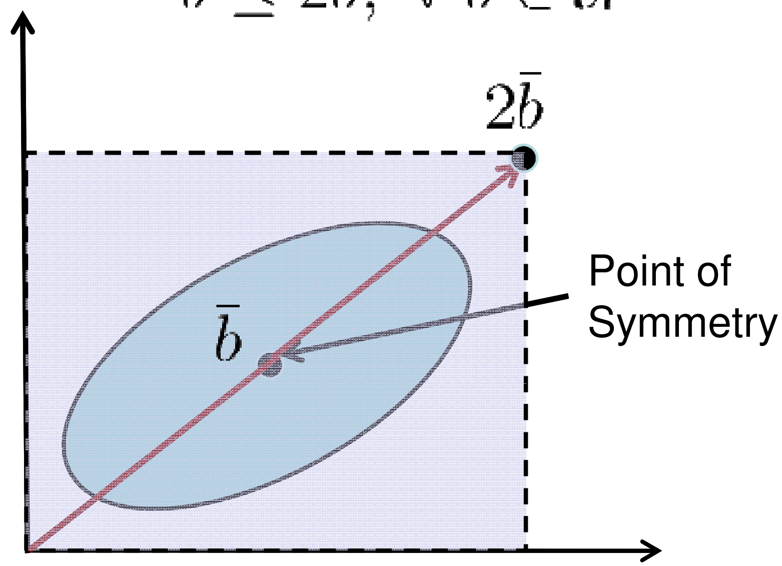


$$\mathcal{U} = \left\{ b \mid \sum_{j=1}^{m} b_j \leq 1 \right\}$$

- **Optimal Static Solution: y = (1,1,…, 1)** and **zRob = m**
- **Optimal fully-adaptable Solution**: **y*(b) = b,** and **zAdapt = 1**

Geometric Intuition

$$b \leq 2\bar{b}, \; \forall \, b \in \mathcal{U}$$



zStoch $\geq$ Optimal cost for covering $\bar{b}$