On the Impact of Combinatorial Structure on Congestion Games

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Congestion Games - Formal Definition

A *congestion game* is a tuple $\Gamma = (\mathcal{N}, \mathcal{R}, (\Sigma_i)_{i \in \mathcal{N}}, (d_r)_{r \in \mathcal{R}})$ with

- $\mathcal{N} = \{1, \ldots, n\}$, set of players
- $\mathcal{R} = \{1, \ldots, m\}$, set of resources
- $\Sigma_i \subseteq 2^{[m]}$, strategy space of player i
- $d_r : \{1, \ldots, n\} \to \mathbb{R}$, delay function or resource r

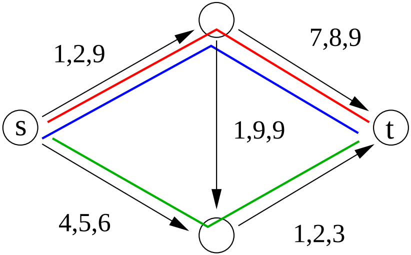
For any state $S = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \Sigma_n$

- n_r = number of players with $r \in S_i$
- $d_r(n_r)$ = delay of resource r
- $\delta_i(S) = \sum_{r \in S_i} d_r(n_r)$ = delay of player i

S is Nash equilibrium if no player can unilaterally decrease its delay.

Example: Network (Path) Congestion Games

- Given a directed graph $G = (V, E)$ with delay functions $d_e : \{1, \dots, n\} \rightarrow \mathbb{N}$, $e \in E$.
- Player i wants to allocate a path of minimal delay between a source s_i and a target t_i.



- A game is called *symmetric* if all players have the same source/target pair.

The transition graph

Definition

- The *transition graph* of a congestion game Γ contains a node for every state S and a directed edge (S, S') if S' can be reached from S by an improvement step of a single player.
- The *best reply transiton graph* contains only edges for best reply improvement steps.

The sinks of the (best reply) transition graph corresponds to the Nash equilibria of Γ.

Questions

- Does every congestion game posses a Nash equilibrium in pure strategies?
- Does any sequence of improvement steps lead to a Nash equilibrium?
- How many steps are needed to reach a Nash equilibrium?
- What is the complexity of computing Nash equilibria in congestion games?

Finite Improvement Property

Proposition (Rosenthal 1973)

For every congestion game, every sequence of improvement steps is finite.

Proof: For every state S, define

$$\phi(S) = \sum_{r \in \mathcal{R}} \sum_{i=1}^{n_r(S)} d_r(i) \quad .$$

ϕ is an exact potential, i.e., if a single player decreases its latency by a value of Δ, then ϕ decreases by Δ as well. $\square$

Congestion Games vs Potential Games

Corollary

Every congestion game is a potential game.

Theorem (Monderer and Shapley, 1996)

Every potential game is isomorphic to a congestion game.

Fast convergence for singleton congestion games

Theorem (Ieong, McGrew, Nudelman, Shoham, Sun, 2005)

In singleton congestion games, all improvement sequences have length $O(n^2m)$.

Question:

Which combinatorial property of the players' strategy spaces guarantees a polynomial upper bound on the length of improvement sequences?

Matroid Congestion Games

Def: Matroid congestion games

- A game Γ is called *matroid congestion game* if, for every $i \in \mathcal{N}$, Σ_i is the bases of a matroid over $\mathcal{R}$.
- All strategies of a player have the same cardinaility, which corresponds to the *rank* of the player's matroid.
- The *rank of the game*, $\mathrm{rk}(\Gamma)$, is defined to be the maximum matroid rank over all players.

Theorem (Ackermann, Röglin, V., 2006)

In a matroid game Γ, all best response improvement sequences have length $O(n^2\, m\, rk(\Gamma))$.

Matroid Congestion Games: Proof of Fast Convergence

- Sort delay values $d_r(i)$, for $r \in \mathcal{R}$ and $1 \leq k \leq n$, in non-decreasing order.
- Define alternative delay functions:

$$\bar{d}_r(k) := \text{rank of } d_r(k) \text{ in sorted list.}$$

Lemma:

Let S be a state of the game. Let S' be the state obtained from S after a best response of player i. Then $\bar{\delta}_i(S') < \bar{\delta}_i(S)$.

Consequence: Rosenthal's potential function yields an upper bound of $n^2\, m\, \mathrm{rk}(\Gamma)$ on the length of a best response sequence since

$$\bar{\phi}(S) = \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} \bar{d}_r(k) \leq \sum_{r \in \mathcal{R}} \sum_{k=1}^{n_r(S)} n\, m \leq n^2\, m\, \mathrm{rk}(\Gamma) \ . \qquad \square$$

Fast Convergence beyond the Matroid Property?

Theorem (Ackermann, Röglin, V., 2006)

Let $\mathcal{S}$ be any inclusion-free non-matroid set system. Then, for every n, there exists a $4n$-player congestion game with the following properties:

- *the strategy space of each player is isomorph to $\mathcal{S}$, and*
- *there is a best response sequence of length 2^n.*

Corollary

The matroid property is the maximal property *on the individual players' strategy spaces that guarantees polynomial convergence.*

Proof Idea for Exponential Convergence

Every inclusion-free, non-matroid set system $\mathcal{S}$ satisfies the following property:

1-2-exchange property

There exist three resources a, b, and c with the property that, an optimal solution for $\mathcal{S}$ contains

- a but not b and c if $w_a < w_b + w_c$, and
- b and c but not a if $w_a > w_b + w_c$.

Using this property one can interweave the strategy spaces in form of a counter such that there is a best response sequence of length 2^n. $\square$

Further negative results about convergence

Fabrikant, Papadimitriou, Talwar, 2004

There are instances of network congestion games that have initial states for which all improvement sequences have exponential length.

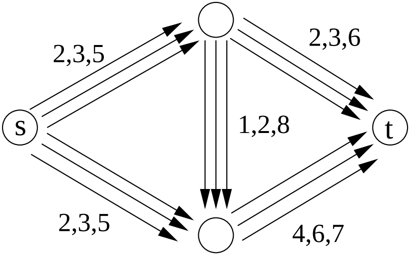
Ackermann, Röglin, V., 2006

Dito for symmetric network congestion games, although Nash equilibria can be found in polynomial time.

Complexity of symmetric network congestion games

Poly-time algorithm via a reduction to min-cost flow:
(Fabrikant, Papadimitriou, Talwar 2004)

- Each edge is replaced by n parallel edges of capacity 1 each.
- The ith copy of edge e has cost $d_e(i)$, $1 \leq i \leq n$.



- Optimal solution minimizes Rosenthal's potential function and, hence, is a Nash equilibrium.

The relationship to local search

Rosenthal's potential function allows us to interprete congestion games as local search problems:

Nash equilibria are local optima wrt potential function.

How difficult is it to compute local optima? ... PLS ...

The complexity class PLS

PLS (Polynomial Local Search)

PLS contains optimization problems with a specified neighborhood relationship Γ. It is required that there is a poly-time algorithm that, given any solution s,

- computes a solution in $\Gamma(s)$ with better objective value, or
- certifies that s is a local optimum.

Examples:

- FLIP (circuit evaluation with Flip-neighborhood)
- Max-Sat with Flip-neighborhood
- Max-Cut with Flip-neighborhood
- TSP with 2-Opt-neighborbood
- Congestion games wrt improvement steps

The complexity class PLS

PLS reductions

Given two PLS problems Π_1 and Π_2 find a mapping from the instances of Π_1 to the instances of Π_2 such that

- the mapping can be computed in polynomial time,
- the local optima of Π_1 are mapped to local optima of Π_2, and
- given any local optimum of Π_2, one can construct a local optimum of Π_1 in polynomial time.

Examples for PLS-complete problem:

- FLIP (via a master reduction)
- Max-Sat and POS-NAE-SAT
- Max-Cut

Complexity of congestion games

Results from [Fabrikant, Papadimitriou, Talwar 2004]

	network games	general games
symmetric	$\exists$ poly-time Algo	PLS-complete
asymmetric	PLS-complete	PLS-complete

Simplified hardness proofs (Ackermann, Röglin, V., 2006)

The Party Affiliation Game (Max-Cut)

Players correspond to nodes in a weighted graph $G = (V, E)$.

- Every player has to strategies: *left* or *right*.
- Thus the outcome of the game corresponds to a *cut*, i.e., a partition of V into left and right nodes.
- Edge weights represent antisympathy.
- Players aim at maximimizing the sum of the weights of their incident edges crossing the cut.

Nash equilibria for the party affiliation game correspond to local optima of the Max-Cut problem with the flip neighborhood.

Simplified hardness proofs (Ackermann, Röglin, V., 2006)

Max-Cut as minimization problem:

The strategies of a node are

- left: choose the left hand side of the cut
- right: choose the right hand side of the cut

The costs for these strategies are

- left: sum of the weights of the incident edges to the left
- right: sum of the weights of the incident edges to the right

Formulation as congestion game:

Represent each edge e by two resources e_{left}, e_{right} with delay functions $d(1) = 0$ and $d(2) = w_e$.

Finding Nash equilibria in general congestion is thus PLS-hard.

Simplified hardness proofs (Ackermann, Röglin, V., 2006)

Threshold congestion games:

Every player i comes with two parameters: a subset $S_i \subseteq \mathcal{R}$ of the resources and a treshold $t_i \geq 0$. The player has only two strategies

- in: allocate resources in S_i (cost as in a congestion game)
- out: do not allocate any resources (cost $= t_i$)

Quadratic threshold games (QTG): A threshold game is called QTG if each resource is contained in the subset of exactly two players.

Simplified hardness proofs (Ackermann, Röglin, V., 2006)

Max-Cut as QTG

The strategies of a node are

- in: choose the left hand side of the cut
- out: choose the right hand side of the cut

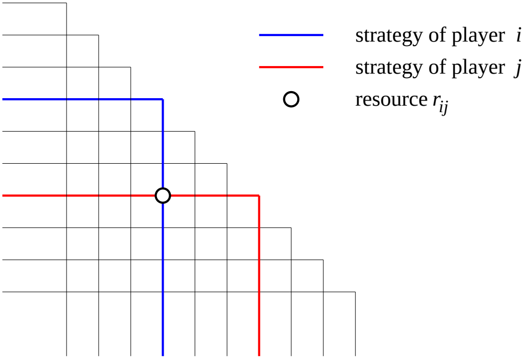
The costs of these strategies are:

- in: sum of the weights of the incident edges to the left
- out: half of the weight of all incident edges

Finding Nash equilibria in QTG is thus PLS-hard.

Simplified hardness proofs (Ackermann, Röglin, V., 2006)

QTG correspond to a path allocation game in a grid:



This is the starting point for the PLS-hardness proofs for network congestion games, network design games, market sharing games etc.

Beyond PLS-hardness

All reductions that we used are *tight* so that

- there are games from all of these classes for which there exist an initial state from which all better response sequences have exponential length, and
- it is PSPACE-hard to compute a Nash equilibrium reachable from a given state.

Conclusions

- The length of best reply improvement sequences in matroid congestion games is polynomially bounded because of the (1,1)-exchange property.
- Every inclusion-free non-matroid set system can be used to construct a congestion game with exponentially long best reply improvement paths because of the (1,2)-exchange property.
- A reduction from threshold games yields PLS-completeness. The strategy spaces of threshold games correspond to $(1, k)$-exchanges with $k = \Omega(n)$.
- Remaing question: What is the compexity of congestion games constructed from $(1, k)$-exchanges for $k > 2$?

Further Directions

- FPTAS for approximating local optima for any PLS-problem (Orlin, Punnen, Abraham, Schulz 2004)
- In contrast: Computing *approximate equilibria* in congestion games is PLS-hard (Skopalik, V., 2008)
- Learning-based approaches minimizing regret (e.g., Blum, Even-Dar, Ligett, 2005)
- Evolutionary approaches achieving bicriteria approximations (e.g., Fischer, Räcke, V., 2007 / Fischer, Kammenhuber, Feldmann, 2006)
- Convergence result with respect to the price of anarchy (e.g., Fanelli, Flammini, Moscardelli, 2008 / Epstein, Awerbuch, Azar, Mirrokni, Skopalik, 2008)