A bird-eye view of fluid queues in communication network models: heavy tails and long memory. New developments

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Recall: we are considering the deviation from the average, at time scale T, of the cumulative input process to a fluid queue, produced by n "users". The deviation process is defined by

$$D_{n,T}(t) = \sum_{j=1}^{n} \left(I_j(tT) - tT\, EN(0) \right), \quad t \geq 0.$$

- $I_j(t)$ is the total amount of work brought by the jth user up to time t;
- $EN(0)$ is the average number of sessions open by a single user at any given moment of time.

Recall: when the input process by a single user follows either the ON-OFF process or the Infinite Source Poisson model, with heavy tailed session durations, then the properly scaled deviation from the mean process converges, as the number of "users" and the time scale grow to infinity, to

- a Fractional Brownian motion if n grows much faster than T (the fast growth regime);
- a stable Lévy motion if T grows much faster than n (the slow growth regime)

(Mikosch, Resnick, Rootzén and Stegeman (2002)).

- A Fractional Brownian motion, and a Lévy stable motion are two very different processes.
- The ON-OFF model and the Infinite Source Poisson model are very special models.

Question: Two what extent the Fractional Brownian motion and the Lévy stable motion limits are robust?

In other words, would the deviation from the average of the cumulative input process still tend to look like a Fractional Brownian motion or a Lévy stable motion if one departs from specifically the ON-OFF model and the Infinite Source Poisson model?

Suppose that the input to the queue caused a single "user" follows a general stationary marked point process

$$((T_n, Z_n))_{n\in\mathbb{Z}} \,,$$

where

- $\cdots \leq T_{-1} \leq T_0 \leq 0 \leq T_1 \leq T_2 \leq \cdots$ are the starting times for each session;
- the mark $Z_n \in \mathbb{Z}$ is the duration of the session starting at time T_n;
- while each session lasts, the work is added to the queue at the unit rate.

- Stationarity of the marked point process means that for any $t \in \mathbb{R}$, the shift θ_t of the process by t units of time preserves the distributions:

$$\theta_t\big(((T_n, Z_n))_{n\in\mathbb{Z}}\big) = ((T_n - t, Z_n))_{n\in\mathbb{Z}} \stackrel{\mathrm{d}}{=} ((T_n, Z_n))_{n\in\mathbb{Z}};$$

- in general, the marks (Z_n) are NOT independent of the underlying point process (T_n).

The goal: to understand whether the Fractional Brownian limit and a stable Lévy limit occur “often” under a “relatively general” stationary marked point process input model.

Fast and slow growth regimes

We would like to understand the possible limits of the deviation process of the cumulative input to a queue from its mean both in fast growth regimes and slow growth regimes.

- We will understand these notions in a wide sense.
- Supose that a properly normalized deviation from the mean process has a certain limit when $n \gg B(T)$ where $B(T) \uparrow \infty$. Then we say that this limit occurs in the fast growth regime.
- Supose that a properly normalized deviation from the mean process has a certain limit when $n \ll B(T)$ where $B(T) \uparrow \infty$. Then we say that this limit occurs in the slow growth regime.

Many of the properties of the input process generated by a stationary marked point process are effectively described by **the Palm measure** of the process.

- Suppose that the point process $((T_n, Z_n))_{n\in\mathbb{Z}}$ lives on a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the path space: the collection of all realizations of the process $((T_n, Z_n))_{n\in\mathbb{Z}}$ with the cylindrical σ-field on it.
- Let $(\theta_t)_{t\in\mathbb{R}}$ be the group of left shifts on Ω:

$$\theta_t\big(((T_n, Z_n))_{n\in\mathbb{Z}}\big) = ((T_n - t, Z_n))_{n\in\mathbb{Z}},$$

a distribution-preserving transformation.

Let

$$\lambda = E \sum_n \mathbf{1}_{[0,1]}(T_n)$$

be the intensity of the stationary point process; we assume it is positive and finite.

We define the Palm measure of the stationary marked point process $((T_n, Z_n))_{n\in\mathbb{Z}}$ to be a new probability measure on $(\Omega, \mathcal{F})$ given by

$$P_0(A) = \frac{1}{\lambda} \sum_n P\Big(0 \le T_n \le 1,\, \theta_{T_n}((T_m, Z_m)_{m\in\mathbb{Z}}) \in A\Big)\,.$$

- Under the Palm measure the point process always has a point at the origin, and
- the interarrival times and the marks are (jointly) stationary.

Moment measures

A stationary marked point process with a finite intensity has an intensity measure. This is a shift invariant in the first component measure γ on $\mathbb{R} \times \mathbb{R}_+$ such that for any nonnegative measurable function f

$$E\Big(\sum_{n\in\mathbb{Z}} f(T_n, Z_n)\Big) = \int_{\mathbb{R}\times\mathbb{R}_+} f(t,z)\,\gamma(dt,dz)\,,$$

and the two sides are finite at the same time.

If, for example, the session durations (marks) are independent of the stationary point process, then $\gamma = \lambda\,\mathrm{Leb} \times F_{Z_0}$.

Many stationary marked point processes have the second moment measure m_2. This is a measure on $\mathbb{R}^2 \times \mathbb{R}_+^2$ such that for any nonnegative measurable function f

$$E\Big(\sum_{n\in\mathbb{Z}} f(T_n, Z_n)\Big)^2 = \int_{\mathbb{R}^2\times\mathbb{R}_+^2} f(t_1, z_1)\, f(t_2, z_2)\, m_2(dt_1, dt_2, dz_1, dz_2)\,,$$

and the two sides are finite at the same time.

The covariance measure of such a stationary marked point process is defined by

$$\gamma_2(dt_1, dt_2, dz_1, dz_2) = m_2(dt_1, dt_2, dz_1, dz_2) - \gamma(dt_1, dz_1)\, \gamma(dt_2, dz_2).$$

Let $\big(I(t),\ t \geq 0\big)$ be the total amount of work brought in the system by the time t by a single "user". Then

$$\mathrm{Var}\, I(t) = 2\int_0^t (t-x)\, g(x)\, dx\,,$$

where

$$g(x) = \int_{\mathbb{R}^2\times\mathbb{R}^2_+} I\left\{s_1 \leq 0 < s_1 + u_1, s_2 \leq x < s_2 + u_2\right\}$$

$$\gamma_2(ds_1, ds_2, du_1, du_2)\,.$$

It turns out that it is the function g that, mostly, determines, from the bird-eye view, the deviations of the cumulative input from the average in the fast growth situation.

Fast Growth Regime

Theorem 1 (Mikosch and Samorodnitsky (2007))
Assume that the function g is regularly varying at infinity with an exponent $\beta \leq 0$, and that the stationary number of open sessions satisfies $E[|N(0)|^{2+\delta}] < \infty$ for some $\delta > 0$.

- If $\beta \in (-1, 0]$ or $\beta = -1$ and $\int_0^\infty g(x)dx = \infty$ and for some $\delta' < \delta$

$$n \gg T^{|\beta|(1+2/\delta')}$$

then, as $n, T \to \infty$,

$$[n\,\mathrm{Var}(I(T)]^{-1/2}\, D_{n,T} \Rightarrow B_H$$

in finite-dimensional distributions as $n \to \infty$, with $H = 1 + \beta/2$.

- Suppose that $\beta < -1$ or $\beta = -1$ and $\int_0^\infty |g(x)|\, dx < \infty$. Let $\int_0^\infty g(x)\, dx \neq 0$. If
$$n \gg T^{1+2/\delta},$$
then the above convergence holds with $H = 0.5$.

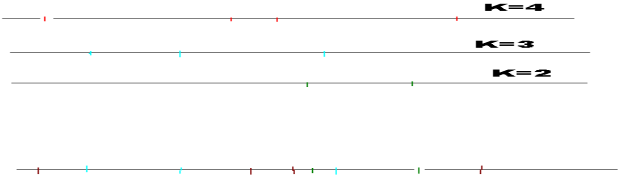
Under certain conditions on λ_n, the convergence can be strengthen to the weak convergence in $C[0,\infty)$.

Therefore: a Fractional Brownian limit of the deviation of the cumulative input from its mean is very robust. A regular variation assumption on a function related to the covariance of the input process is sufficient for a Fractional Brownian limit in a fast growth regime.

Example The cluster Poisson model

Basic ingredients:

- A rate λ_0 Poisson process of arrivals of clusters.
- A generic cluster: a point process N_c stopped at a random renewal K.
- At each point of a Poisson process an independent cluster (stopped point process) starts.
- Clusters are independent of the Poisson process.



Proposition Assume that a generic cluster is a stopped renewal process, and that the marks are independent of the point process, and are iid. Let the cluster size law $F_K \in \mathrm{Reg}(\alpha)$ for some $\alpha \in (1, 2)$, and the session duration law $\overline{F}_Z(t) = o(\overline{F}_K(t))$ as $t \to \infty$.

Assume that the within-cluster interarrival time X has a non-arithmetic distribution and $EX < \infty$.

Then in a fast growth regime $n \gg T^{\alpha-1+\epsilon}$ for some $\epsilon > 0$ the convergence to a Fractional Brownian motion holds with $H = (3 - \alpha)/2$.

Slow Growth Regime

Under slow growth regimes, two competing factors can affect the limiting behaviour of the deviation $\left(D_{n,T}(t)\right)$ of the cumulative input process from its mean:

- the deviations of the partial sums of the marks (session durations) from their mean, $\sum_{k=1}^{m} S_k - m\, E_0 S_0$ under the Palm probability;
- the deviations of the arrival numbers from their mean, number of $\left\{k : 0 < T_k \le T\right\} - \lambda T$, under the stationary probability.

Note: Under the Palm probability the marks form an arbitrary stationary process with a finite mean.

The available limit theorems for the partial sums of stationary stochastic processes show that the deviations of these partial sums from their mean can converge to a large variety of different self-similar processes with stationary increments, such as

- Brownian motion;
- Fractional Brownian motion;
- processes in a higher order Gaussian chaos, such as the Rosenblat process;
- stable Lévy motion;
- various Fractional stable motions;
- processes in a higher order stable chaos.

The stable Lévy motion is only one of the many possible limits, and it arises under only very specific set of circumstances.

Therefore: in the slow growth regime the Lévy stable limit is not robust if the variability of the session durations exceedds that of the session beginnings.

It was not known, however, whether the Lévy stable limit was robust in the slow growth regime when the variability of the session beginnings dominated.

Example The renewal cluster Poisson model, revisited

Proposition (Mikosch and Samorodnitsky (2007)) Assume that a generic cluster is a stopped renewal process, and that the marks are independent of the point process, and are iid. Let the cluster size law $F_K \in \mathrm{Reg}(\alpha)$ for some $\alpha \in (1, 2)$, and the session duration law $\overline{F}_Z(t) = o(\overline{F}_K(t))$ as $t \to \infty$.

Assume that the within-cluster interarrival time X satisfies $EX < \infty$.

Then in a slow growth regime $n \ll T^{\alpha-1-\epsilon}$ for some $\epsilon > 0$ the convergence to a spectrally positive α-stable Lévy motion holds.

However: if the within-cluster variability is higher than that of a finite mean renewal process, the limit may well be different.

Suppose that for some $\beta > 1$, the within-cluster interarrival time law satisfies $F_X \in \mathrm{Reg}(1/\beta)$. **The mean within-cluster interarrival time is infnite**.

Let the cluster size law satisfy $F_K \in \mathrm{Reg}(\alpha)$ for some $\alpha \in \big(1, \min(2, \beta)\big)$.

Suppose that the session durations have a finite second moment.

Theorem (Fasen and Samorodnitsky (2008)) In the slow growth regime

$$n \ll \Big(\big(\bar{F}_X(T)\big)^{-2} \bar{F}_K\big(\bar{F}_X(T)^{-1}\big) \Big)^{-\big((\alpha-1)/(\alpha+\beta-1)\big)+\epsilon}$$

for some $\epsilon > 0$, under Technical Assumption A below,

$$[n \operatorname{Var}(I(T))]^{-1/2} D_{n,T} \Rightarrow B_H$$

in finite-dimensional distributions as $n \to \infty$, with $H = (2 + \beta - \alpha)/2\beta$.

Technical Assumption A (needed for renewal-theoretical arguments)

Assume either of the following two conditions:

- $\beta < 2$ and

$$\limsup_{x\to\infty} x \frac{\bar{F}_X(x) - \bar{F}_X(x+1)}{\bar{F}_X(x)} < \infty;$$

- F_X is arithmetic with step Δ, and

$$\sup_{n\geq 0} n \frac{F_X\big(\{n\Delta\}\big)}{\bar{F}_X(n\Delta)} < \infty.$$

Conclusions

The Lévy stable limit of the deviation of the cumulative input to a fluid queue from its mean is very non-robust under deviations from very special input models.

The Fractional Browniam limit is very robust, and occurs under very weak assumptions in the fast growth regime, and may occur in the slow growth regime as well.