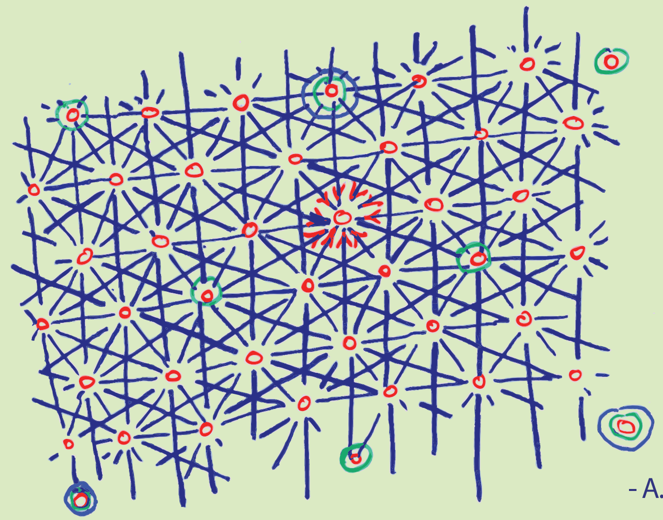
Stability and Asymptotic Optimality of h-MaxWeight Policies



- A. Rybko, 2006

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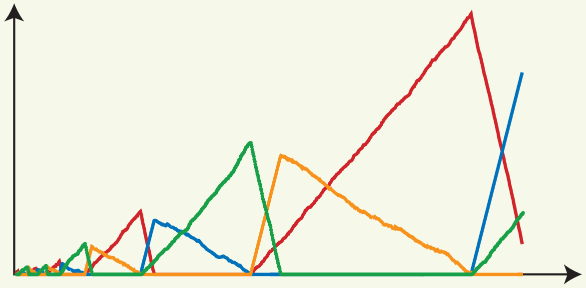
NSF support: ECS 05-23620 and DARPA ITMANET

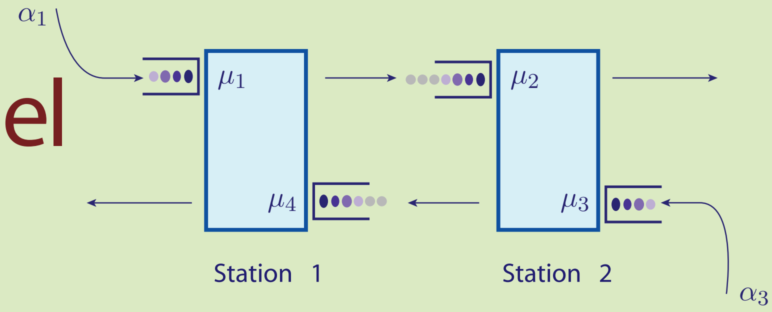
Outline

- I Models & Background
- II h-MaxWeight Policies
- III Heavy Traffic
- Conclusions



I
Models & Background

Controlled Random-Walk Model



$$Q(k+1) = Q(k) + B(k+1)U(k) + A(k+1), \qquad Q(0) = x$$

Statistics & topology:

$$B(k) = \begin{bmatrix} -S_1(k) & 0 & 0 & 0 \\ S_1(k) & -S_2(k) & 0 & 0 \\ 0 & 0 & -S_3(k) & 0 \\ 0 & 0 & S_3(k) & -S_4(k) \end{bmatrix}$$

$$A(k) = \begin{bmatrix} A_1(k) \\ 0 \\ A_3(k) \\ 0 \end{bmatrix}$$

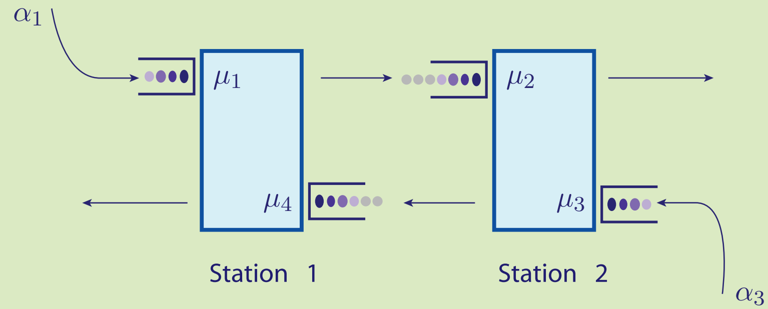
Constituency constraints:

$$C\,U(k) \le \mathbf{1}$$
$$U(k) \ge \mathbf{0}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

- Lippman 1975
- Henderson & M. 1997

Fluid Model & Workload



$$q(t) = x + Bz(t) + \alpha t\,, \qquad t \geq 0 \qquad q(0) = x$$

Fluid model captures mean-flow:

$$B = \mathsf{E}[B(k)] = \begin{bmatrix} -\mu_1 & 0 & 0 & 0 \\ \mu_1 & -\mu_2 & 0 & 0 \\ 0 & 0 & -\mu_3 & 0 \\ 0 & 0 & \mu_3 & -\mu_4 \end{bmatrix}$$

$$\alpha = \mathsf{E}[A(k)] = \begin{bmatrix} \alpha_1 \\ 0 \\ \alpha_3 \\ 0 \end{bmatrix}$$

Workload and load parameters:

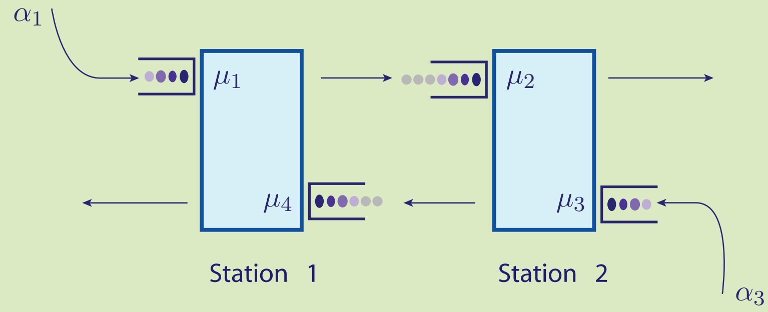
$$\xi^1 = \begin{bmatrix} m_1 \\ 0 \\ m_4 \\ m_4 \end{bmatrix}, \quad \xi^2 = \begin{bmatrix} m_2 \\ m_2 \\ m_3 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \rho_1 &= m_1\alpha_1 + m_4\alpha_3 \\ \rho_2 &= m_2\alpha_1 + m_3\alpha_3 \end{aligned}$$

with $m_i = \mu_i^{-1}$

- Newell 1982, - Vandergraft 1983
- Perkins & Kumar 1989
- Chen & Mandelbaum 1991, - Cruz 1991

Value Functions



$$q(t) = x + Bz(t) + \alpha t \qquad Q(k+1) - Q(k) = B(k+1)U(k) + A(k+1)$$

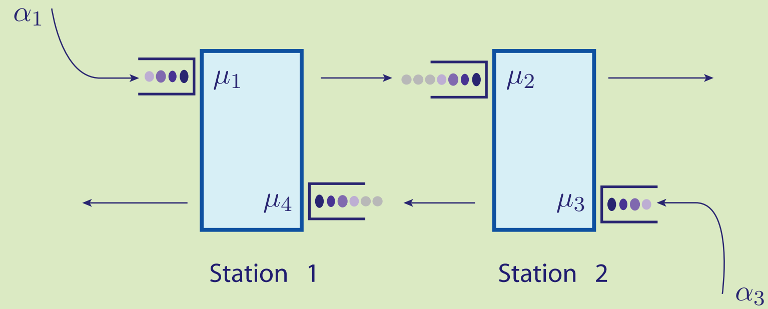
$$J(x) = \int_0^\infty c(q(t;x))\,dt \qquad h(x) = \int_0^\infty \mathsf{E}[c(Q(t;x)) - \eta]\,dt$$

Fluid value function Relative value function

$$\eta = \int c(x)\,\pi(dx)$$

$=$ *average cost*

Value Functions



$$q(t) = x + Bz(t) + \alpha t \qquad Q(k{+}1) - Q(k) = B(k{+}1)U(k) + A(k{+}1)$$

$$J(x) = \int_0^\infty c(q(t;x))\,dt \qquad h(x) = \int_0^\infty \mathsf{E}[c(Q(t;x)) - \eta]\,dt$$

Fluid value function

Relative value function

$$\eta = \int c(x)\,\pi(dx)$$

Large-state solidarity

$$\lim_{\|x\|\to\infty} \left[\frac{J(x)}{h(x)}\right] = 1$$

Holds for wide class of stabilizing policies, including average-cost optimal policy

- M 96, 01, ... following
-Dai 95, - Dai & M 95

Myopic Policy: Fluid Model

$$q(t) = x + Bz(t) + \alpha t$$

$$\frac{d^+}{dt} q(t) = B\zeta(t) + \alpha$$

Constraints: X subset of $\mathbb{R}^\ell_+$

U(x) feasible values of $\zeta(t)$

when $x = q(t) \in$ X

Given: Convex monotone cost function,

$$c \colon \mathbb{R}^\ell_+ \to \mathbb{R}_+$$

Myopic Policy: Fluid Model

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Given: Convex monotone cost function,

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$$\arg\min_{u\in\mathsf{U}(x)} \frac{d^+}{dt} c(q(t)) = \arg\min_{u\in\mathsf{U}(x)} \langle \nabla c(x), Bu + \alpha \rangle$$

Myopic Policy: CRW Model

$$Q(k+1) - Q(k) = B(k+1)U(k) + A(k+1)$$

Constraints: $\mathsf{X}_\diamond$ subset of $\mathbb{R}^\ell_+$ (lattice constraints, etc.)

$\mathsf{U}_\diamond(x)$ feasible values of $U(k)$

when $x = Q(k) \in \mathsf{X}_\diamond$

Given: Convex monotone cost function,

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Myopic policy:

$$\arg\min_{u \in \mathsf{U}_\diamond(x)} \mathsf{E}[c(Q(k+1)) \mid Q(k) = x,\ U(k) = u]$$

Myopic Policy: CRW Model

$$Q(k+1) - Q(k) = B(k+1)U(k) + A(k+1)$$

Motivation: Average cost optimal policy is h-myopic,

$h\colon \mathbb{R}_+^\ell \to \mathbb{R}_+$ is the relative value function,

$$h(x) = \inf_{\boldsymbol{U}} \int_0^\infty \mathsf{E}[c(Q(t;x)) - \eta^*]\, dt$$

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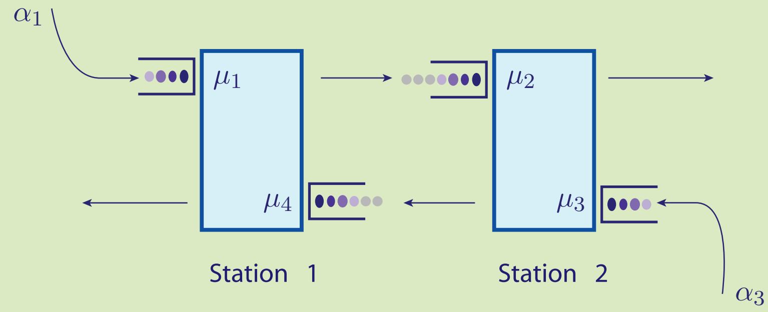
Dynamic programming equation:

$$\min_{u \in \mathsf{U}_\diamond(x)} \mathsf{E}[h(Q(k+1)) \mid Q(k) = x,\ U(k) = u] = h(x) - c(x) + \eta^*$$

Fluid Model & Myopia



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Given: Convex monotone cost function,

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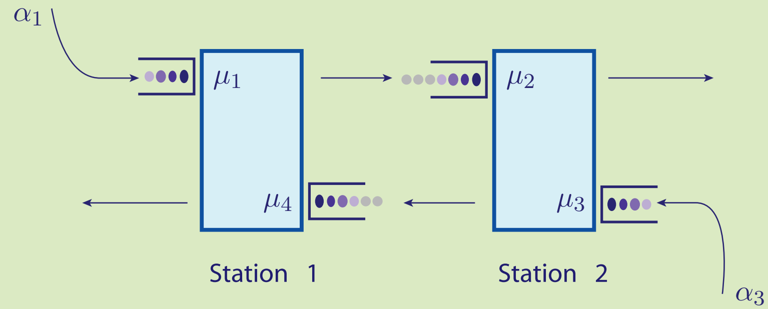
Myopic policy *for fluid model* is stabilizing:

$$q(t) = 0 \qquad t \geq T_0$$

- Chen & Yao 93
- M' 01

Myopia & Instability



Myopic policy may or may not be stabilizing

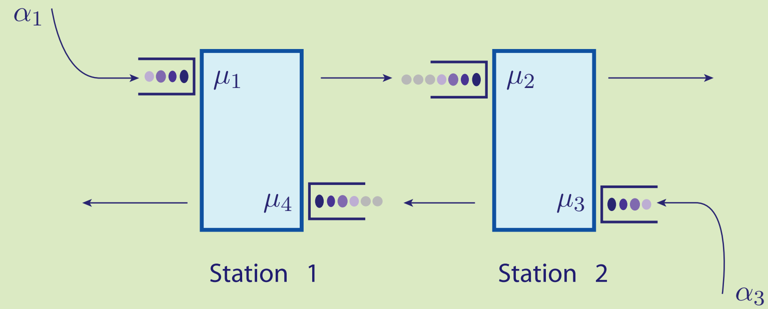
Example: Two station model above with linear cost,

$$c(x) = x_1 + x_2 + x_3 + x_4$$

Myopic policy for CRW model: Priority to exit buffers

- Kumar & Seidman 89
- Rybko & Stolyar 93

Myopia & Instability



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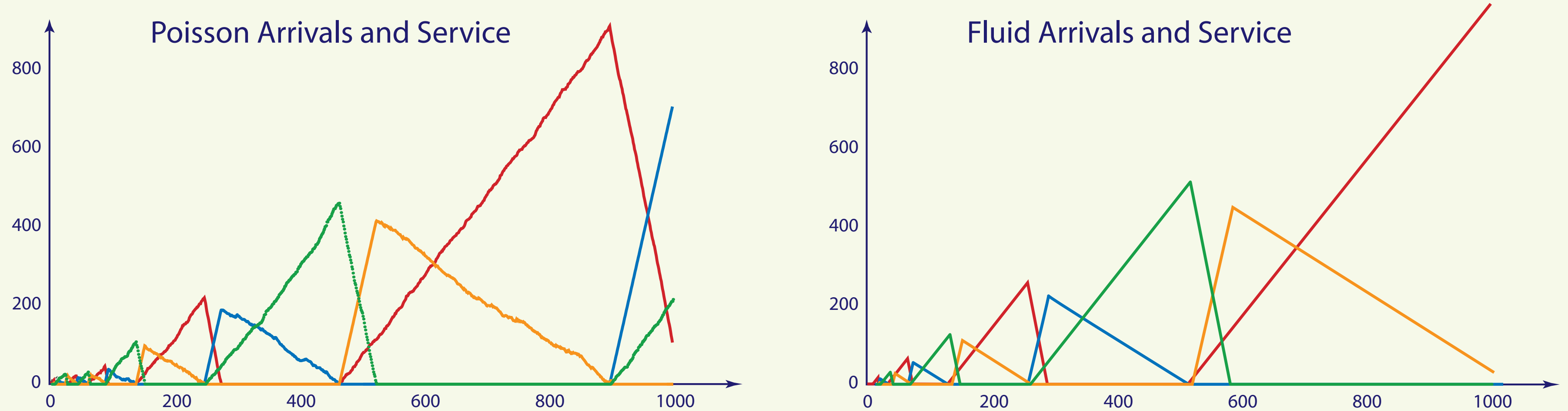
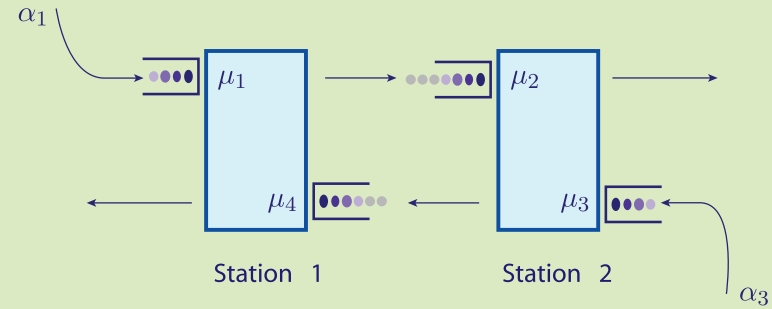


Periodic starvation creates instability

- Kumar & Seidman 89
- Rybko & Stolyar 93

Myopia & Instability



Quadratic Cost

Myopic policy stabilizing for *diagonal* quadratic

Example: Two station model above with,

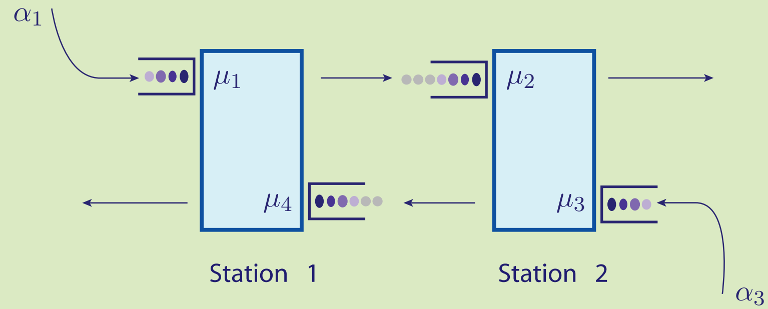
$$c(x) = \frac{1}{2}[x_1^2 + x_2^2 + x_3^2 + x_4^2]$$

Myopic policy: Approximated by linear switching curves

- Tassiulas & Ephremides 92

Myopia & Instability

Quadratic Cost



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Myopic policy: Approximated by linear switching curves

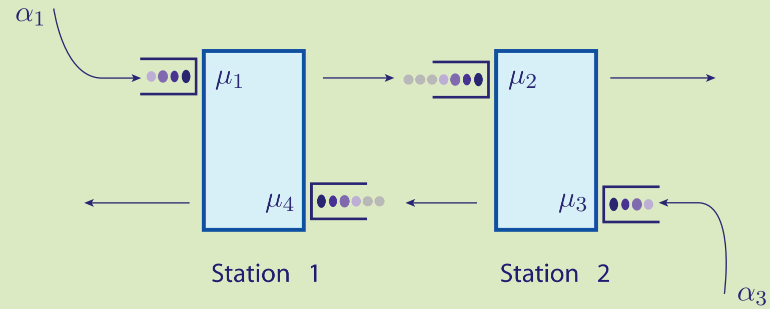
Condition (V3) holds with Lyapunov function $V = c$

For positive constants ε and $\bar{\eta}$

$$PV(x) := \mathsf{E}[V(Q(k+1)) | Q(k) = x] \leq V(x) - \varepsilon \|x\| + \bar{\eta}$$

- Tassiulas & Ephremides 92

MaxWeight Policy



Tassiulas considers myopic policy *for fluid model*

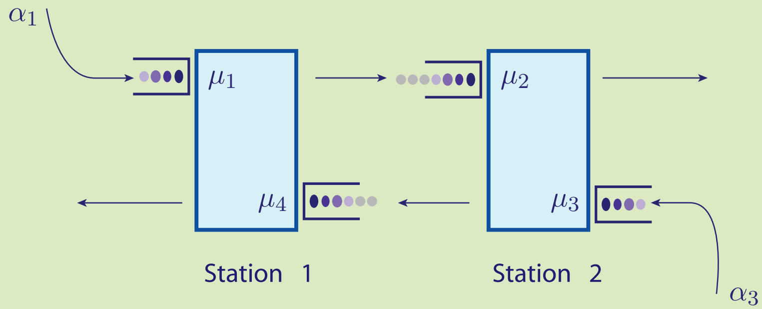
$$\mathop{\arg\min}_{u \in \mathsf{U}_\diamond(x)} \langle \nabla c(x), Bu + \alpha \rangle$$

subject to lattice constraints

where $c(x) = \frac{1}{2} x^T D x$, $D = \mathrm{diag}\,(d_1, \ldots, d_\ell)$

- Tassiulas & Ephremides 92

MaxWeight Policy



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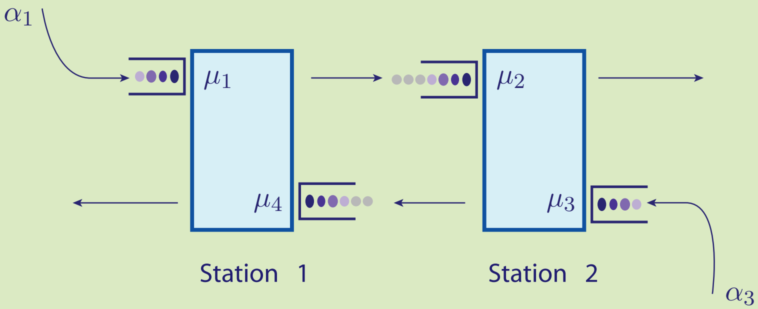
Obtains negative drift: For non-zero x,

$$\langle \nabla c(x), Bu + \alpha \rangle \quad \le \quad -\varepsilon \|x\|$$

Implies (V3) for MaxWeight policy

MaxWeight Policy



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Implies (V3) for MaxWeight policy

Implies (V3) for myopic policy

since myopic has minimum drift

Questions Since 1996

$$\lim_{\|x\|\to\infty} \left[\frac{J(x)}{h(x)}\right] = 1$$

Value functions for fluid and stochastic models:

Quadratic growth for linear cost with similar asymptotes;

Policies are similar for large state-values

Questions Since 1996

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Value functions for fluid and stochastic models:
Quadratic growth for linear cost with similar asymptotes;
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- *What is the gap between policies?*
- *What is the gap between value functions?*
- *How to translate policy for fluid model to cope with volatility?*
- *Connections with heavy traffic theory?*

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- *What is the gap between value functions?*
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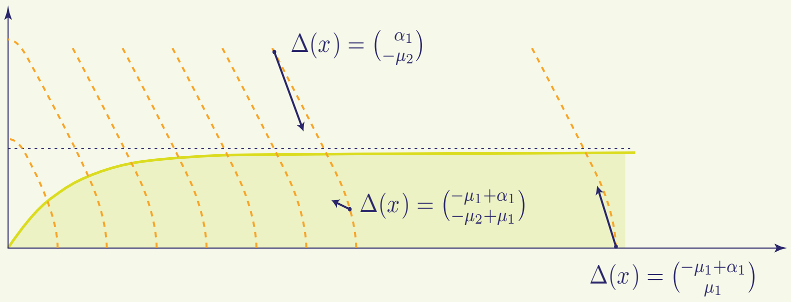
- *Connections with heavy traffic theory?*

Many positive answers in new monograph, as well as new applications for value function approximation

Today's lecture focuses on third and fourth topics



II
h-MaxWeight Policies

Why Does MW Work?

Geometric explanation

Define drift vector field (for given policy)

$$\Delta(x) = \mathsf{E}[Q(k+1) - Q(k) \mid Q(k) = x\,] = Bu + \alpha$$

MaxWeight policy:

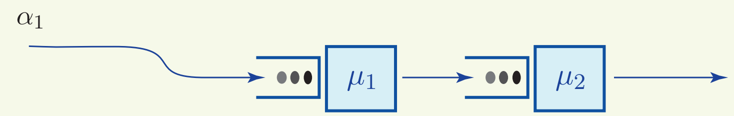
$$\mathop{\arg\min}_{u \in \mathsf{U}_\diamond(x)} \langle \nabla c(x),\, \Delta(x)\,\rangle$$

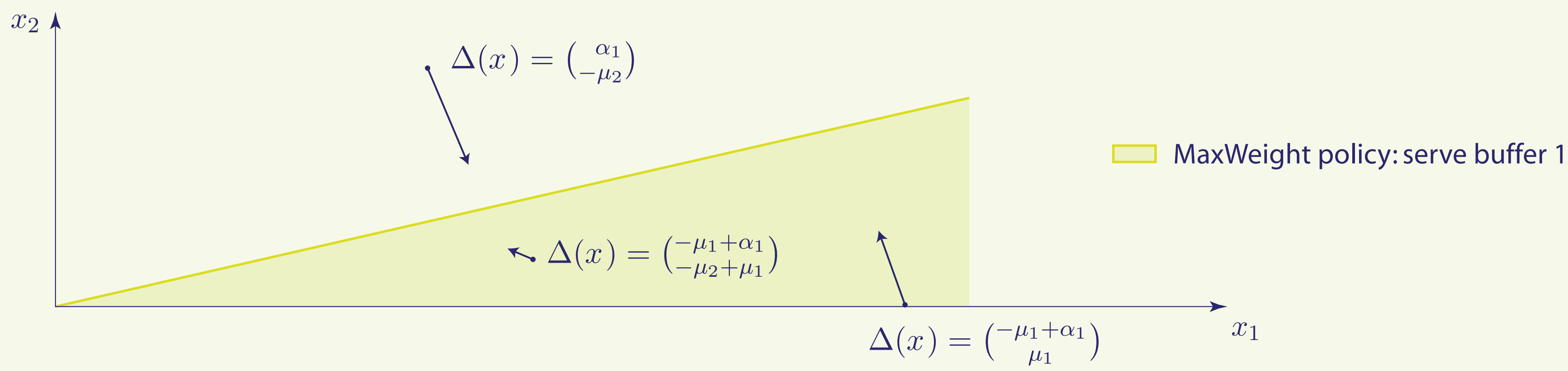
with c diagonal quadratic

Why Does MW Work?

$$\Delta(x) = \mathsf{E}[Q(k+1) - Q(k) \mid Q(k) = x\,]$$

Example: Queues in tandem





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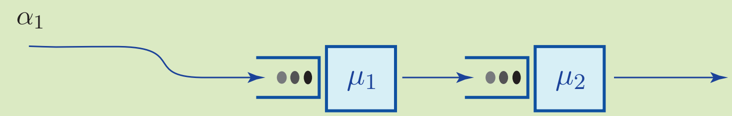

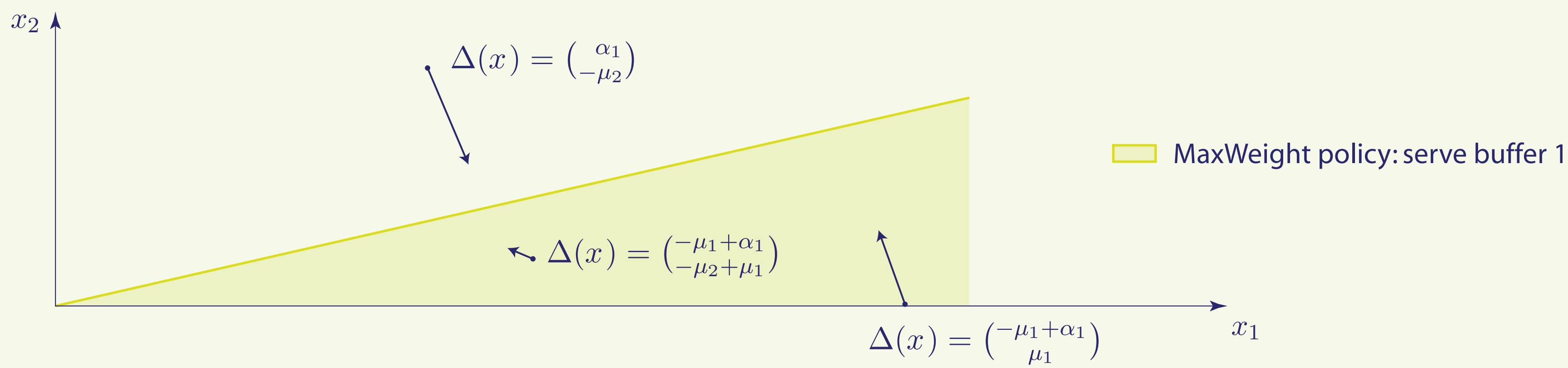

Key observation: Boundaries of the state space are *repelling*

Why Does MW Work?

$$\Delta(x) = \mathsf{E}[Q(k+1) - Q(k) \mid Q(k) = x]$$

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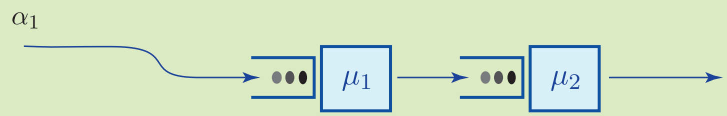
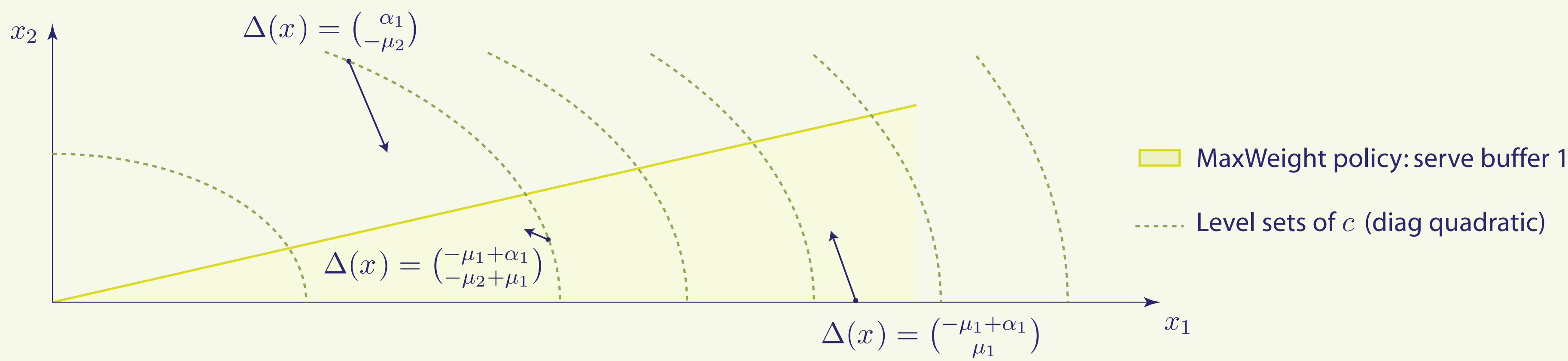
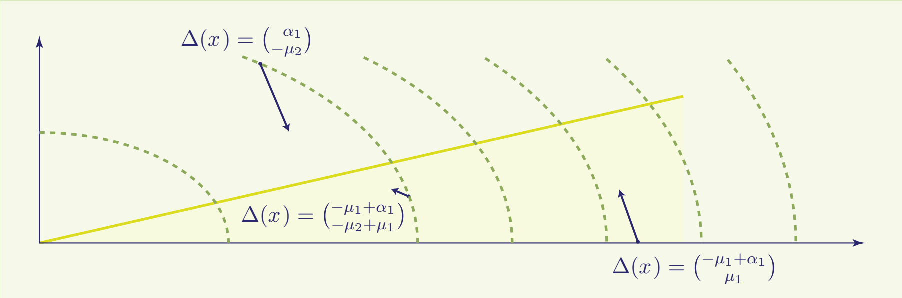




Key observation: Boundaries of the state space are *repelling*

Consequence of vanishing partial derivatives on boundary

h-MaxWeight Policy

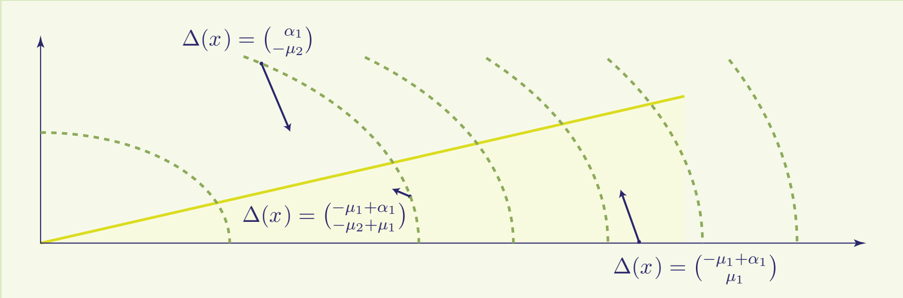


Given: Convex monotone function h

Boundary conditions

$$\frac{\partial}{\partial x_j} h(x) = 0 \quad \text{when } x_j = 0.$$

h-MaxWeight Policy



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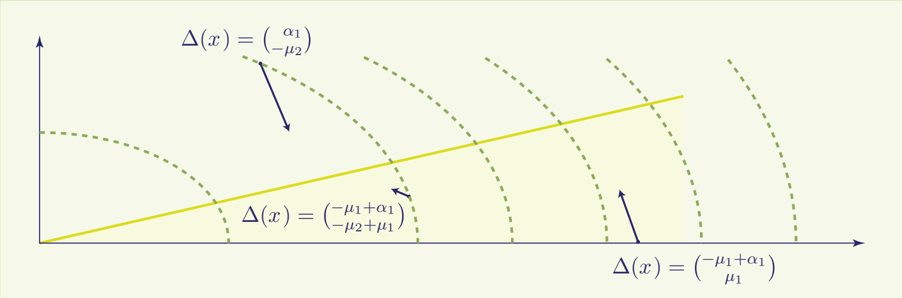
Economic interpretation:

Marginal disutility vanishes for vanishingly small inventory

h-MaxWeight Policy



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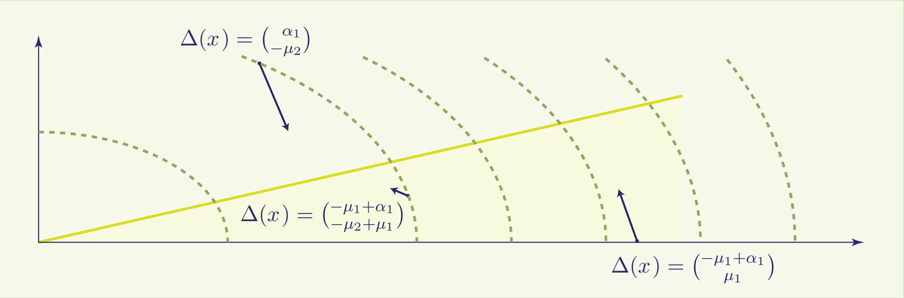
$$\frac{\partial}{\partial x_j} h(x) = 0 \quad \text{when } x_j = 0.$$

Economic interpretation:

Marginal disutility vanishes for vanishingly small inventory

Condition rarely holds, but we can fix that ...

h-MaxWeight Policy

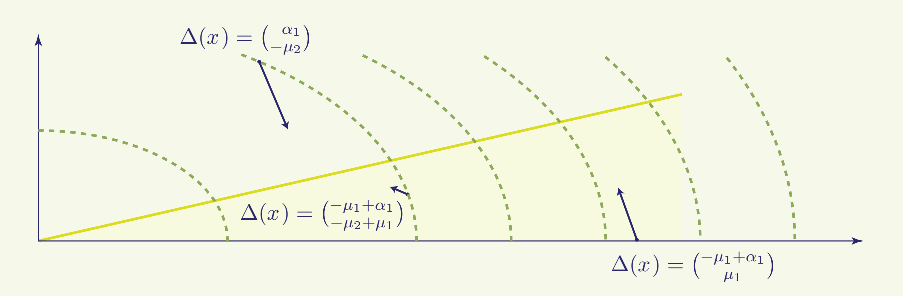


Given: Convex monotone function h_0 (perhaps violating ∂ condition)

Introduce perturbation: For fixed $\theta \geq 1$ and any $x \in \mathbb{R}_+^\ell$

$$\tilde{x}_i := x_i + \theta(e^{-x_i/\theta} - 1), \quad \text{and } \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_\ell)^T \in \mathbb{R}_+^\ell$$

h-MaxWeight Policy



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$$\tilde{x}_i := x_i + \theta(e^{-x_i/\theta} - 1), \quad \text{and } \tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_\ell)^T \in \mathbb{R}^\ell_+$$

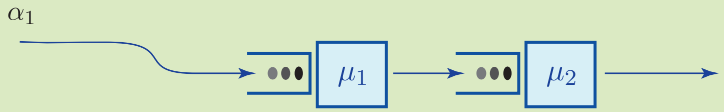
Perturbed function:

$$h(x) = h_0(\tilde{x}), \qquad x \in \mathbb{R}^\ell_+$$

Convex, monotone, and boundary conditions are satisfied

h-MaxWeight Policy

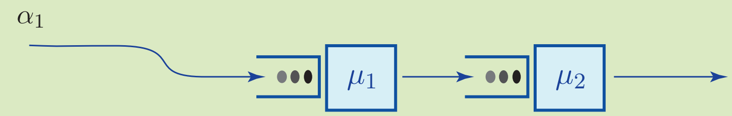
Perturbed linear function



h_0 linear: *never* satisfies ∂ condition

h-myopic and h-MaxWeight polices stabilizing provided $\theta \geq 1$ is sufficiently large

h-MaxWeight Policy

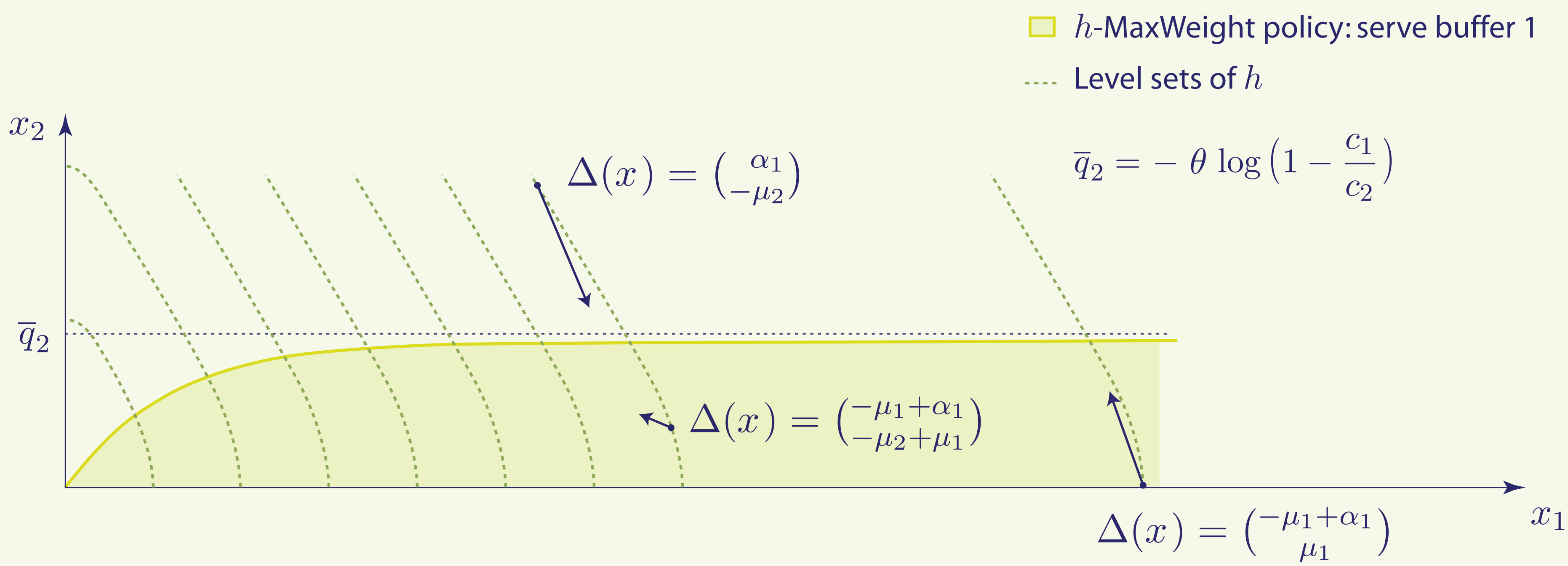


Perturbed linear function

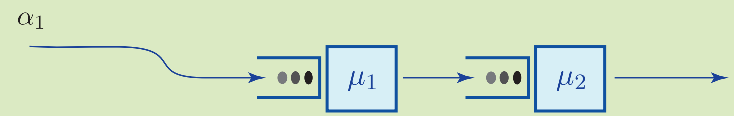
h_0 linear: *never* satisfies ∂ condition

h-myopic and h-MaxWeight polices stabilizing provided $\theta \geq 1$ is sufficiently large

Example: Tandem queues



h-MaxWeight Policy



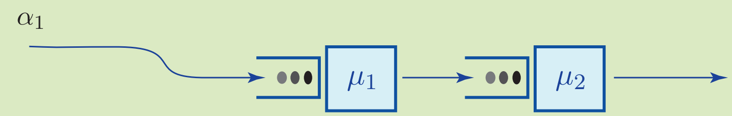
Perturbed value function

h_0 minimal fluid value function, $$J(x) = \inf \int_0^\infty c(q(t;x))\,dt$$

h-myopic and h-MaxWeight polices stabilizing

provided $\theta \geq 1$ is sufficiently large

h-MaxWeight Policy



Perturbed value function

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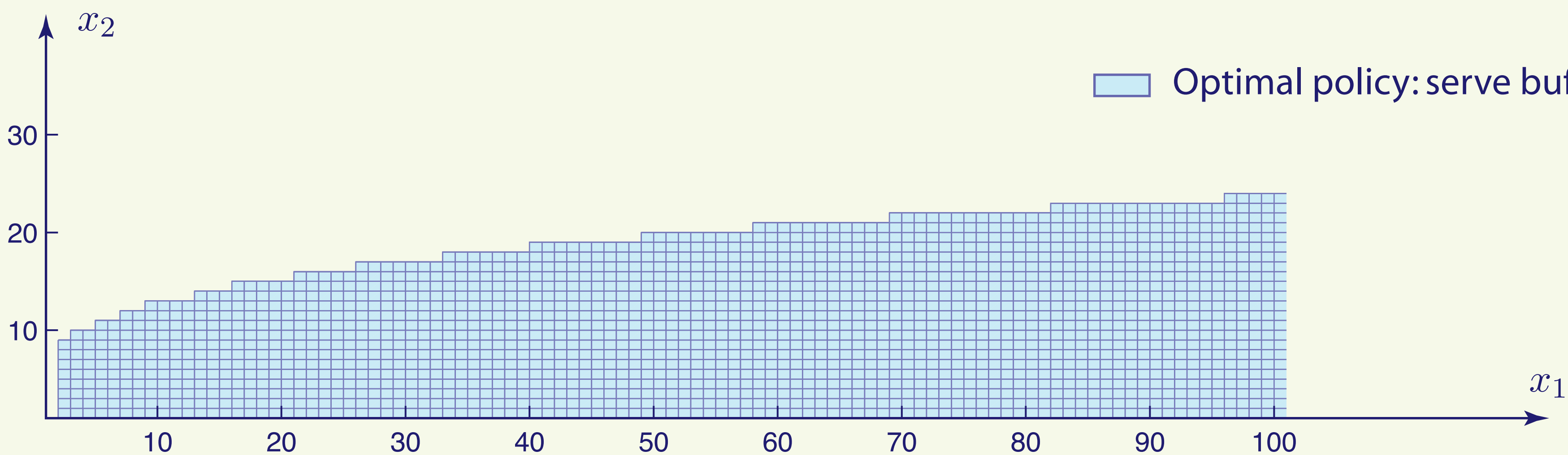
h-myopic and h-MaxWeight polices stabilizing provided $\theta \geq 1$ is sufficiently large

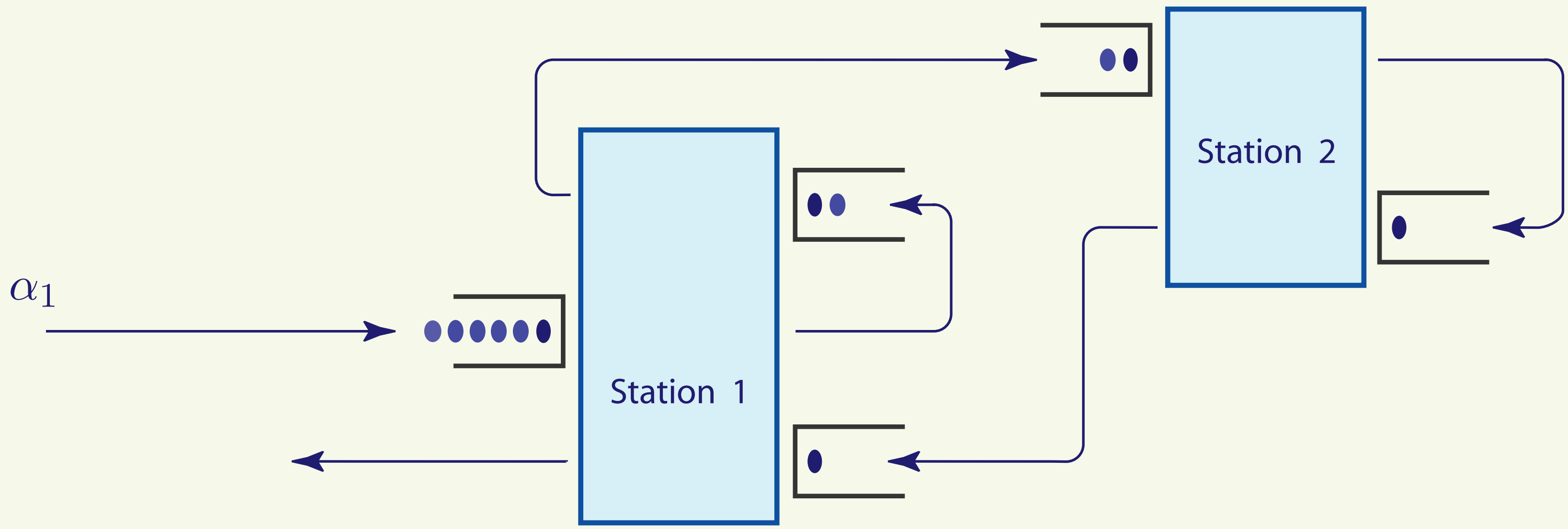
Resulting policy very similar to average-cost optimal policy:



III
Heavy Traffic

Relaxations & Asymptotic Optimality

Single example for sake of illustration:



Model of Dai & Wang

Relaxations & Asymptotic Optimality

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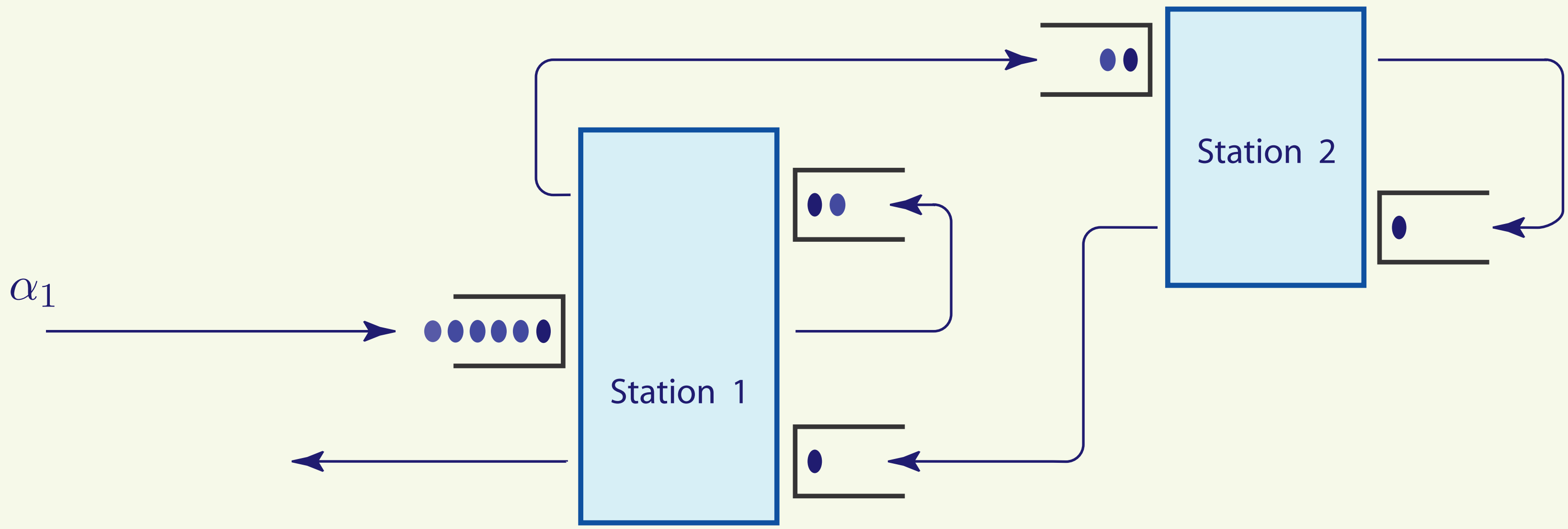
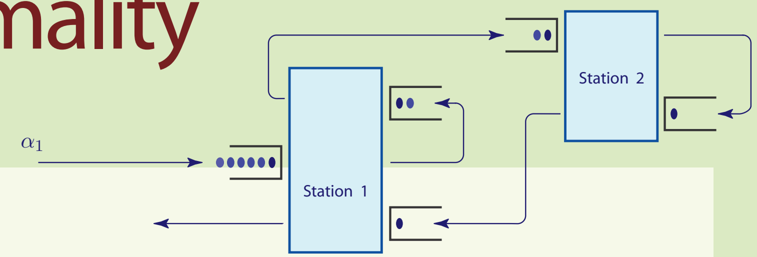


Assume: *Homogeneous model*

Service rate at Station i is μ_i

Relaxations & Asymptotic Optimality



Homogeneous CRW model:

$$Q_1(k+1) - Q_1(k) = -S_1(k+1)U_1(k) + A_1(k+1)$$

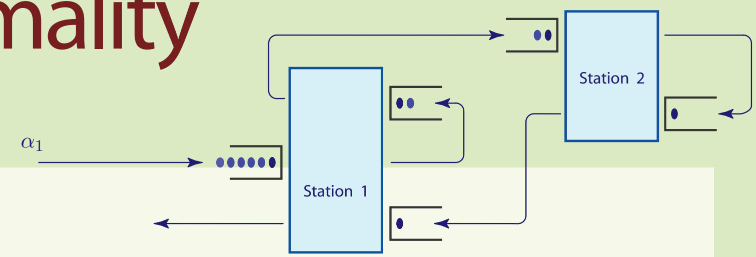
$$Q_2(k+1) - Q_2(k) = -S_1(k+1)U_2(k) + S_1(k+1)U_1(k)$$

$$Q_3(k+1) - Q_3(k) = -S_2(k+1)U_3(k) + S_2(k+1)U_2(k)$$

$$Q_4(k+1) - Q_4(k) = -S_2(k+1)U_4(k) + S_2(k+1)U_3(k)$$

$$Q_5(k+1) - Q_5(k) = -S_1(k+1)U_5(k) + S_2(k+1)U_4(k)$$

Relaxations & Asymptotic Optimality



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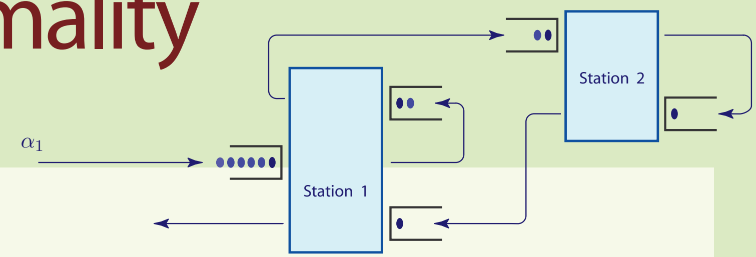
$$Q_5(k+1) - Q_5(k) = -S_1(k+1)U_5(k) + S_2(k+1)U_4(k)$$

Constituency constraints: $U_i(k) \in \{0,1\}$

$$U_1(k) + U_2(k) + U_5(k) \leq 1 \qquad U_3(k) + U_4(k) \leq 1$$

Relaxations & Asymptotic Optimality

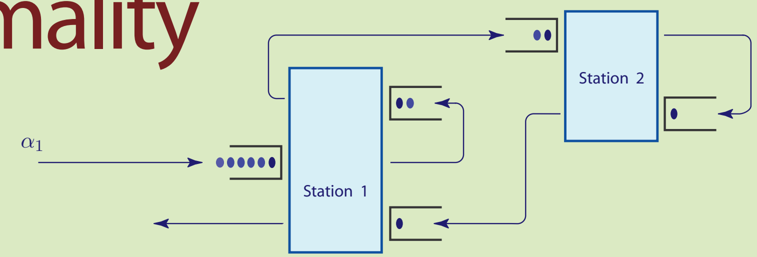


Workload (units of inventory)

$$Y_1(k) = 3Q_1(k) + 2Q_2(k) + Q_3(k) + Q_4(k) + Q_5(k)$$

$$Y_2(k) = 2(Q_1(k) + Q_2(k) + Q_3(k)) + Q_4(k)$$

Relaxations & Asymptotic Optimality



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Idleness processes:

$$\iota_1(k) = 1 - (U_1(k) + U_2(k) + U_5(k))$$

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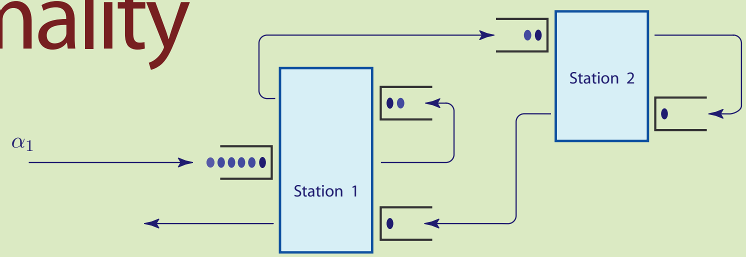
Relaxations & Asymptotic Optimality



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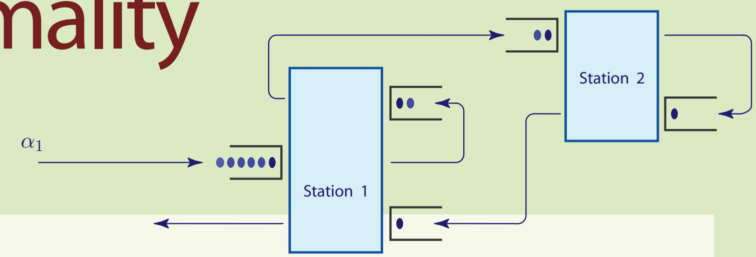
Dynamics:

$$Y_1(k+1) - Y_1(k) = -S_1(k+1) + 3A_1(k+1) \quad + \quad S_1(k+1)\iota_1(k)$$

$$Y_2(k+1) - Y_2(k) = -S_2(k+1) + 2A_1(k+1) \quad + \quad S_2(k+1)\iota_2(k)$$

Relaxations & Asymptotic Optimality



Workload Relaxation of N. Laws

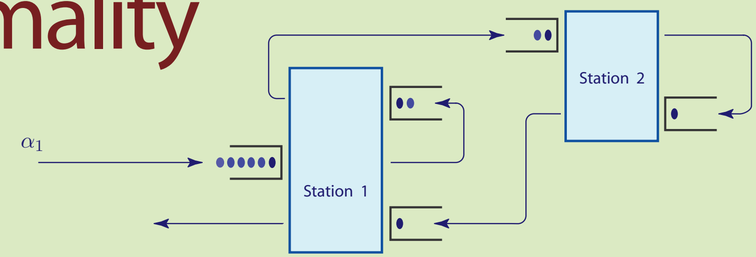
$$Y_1(k+1) - Y_1(k) = -S_1(k+1) + 3A_1(k+1) \quad + \quad S_1(k+1)\iota_1(k)$$

with constraints on idleness process relaxed,

$$\iota_1(k) \in \{0, 1, 2, \dots\}$$

- Laws 90
- Kelly & Laws 93

Relaxations & Asymptotic Optimality



Workload Relaxation of N. Laws

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with constraints on idleness process relaxed,

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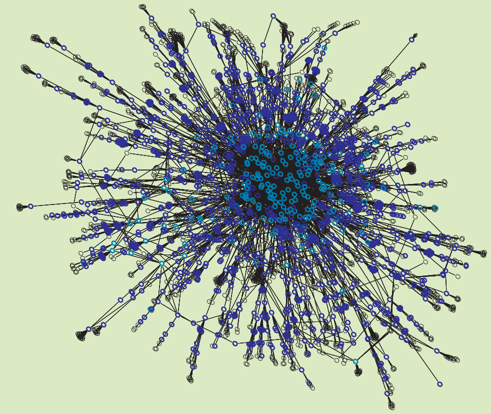
Optimization based on the effective cost,

$$\begin{aligned} \overline{c}(y) = \mathbf{min} \quad & c(x) \\ \mathbf{s.\,t.} \quad & 3x_1 + 2x_2 + x_3 + x_4 + x_5 = y \\ & x \in \mathbb{Z}_+^5 \qquad \text{(+ buffer constraints)} \end{aligned}$$

- Laws 90
- Kelly & Laws 93
- Harrison, Kushner, Reiman, Williams, Dai, Bramson, ...

Asymptotic Optimality

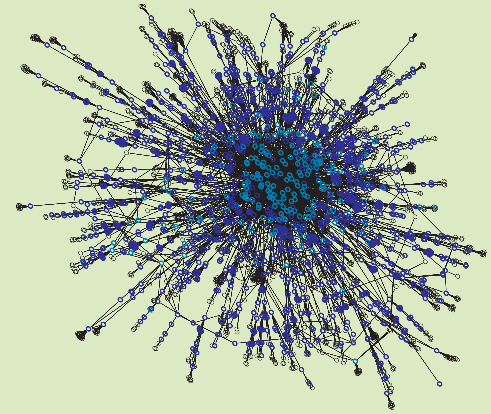


Optimal policy is non-idling for one-dimensional relaxation

Dynamic programing equation solved
via *Pollaczek-Khintchine* formula

Asymptotic Optimality

Heavy traffic assumptions



Load is unity for nominal model

Single bottleneck to define relaxation

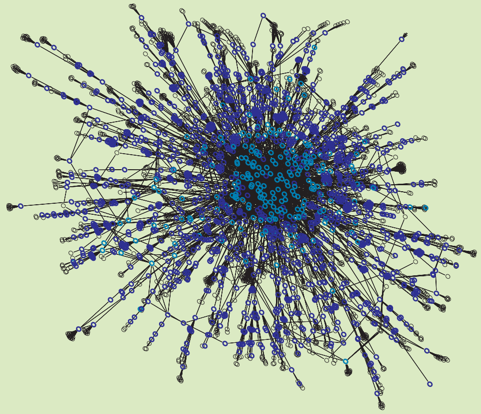
Cost is linear, and effective cost has a unique optimizer

Model sequence:

$$A^{(n)}(k) = \begin{cases} A(k) & \text{with probability } 1 - n^{-1} \\ 0 & \text{with probability } n^{-1} \end{cases}$$

Load less than unity for each *n*

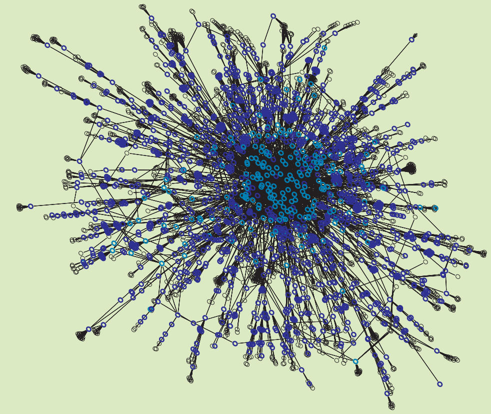
Asymptotic Optimality



$$h_0(x) = \hat{h}^*(y) + \frac{b}{2}\big(c(x) - \overline{c}(y)\big)^2$$

h-MaxWeight policy asymptotically optimal,
with logarithmic regret

Asymptotic Optimality



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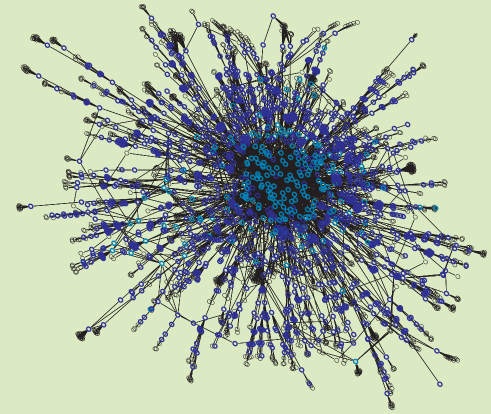
$\hat{\eta}^* = O(n)$ optimal average cost for relaxation

η average cost under h-MW policy

Asymptotic Optimality



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η average cost under h-MW policy

$$\hat{\eta}^* \le \eta \le \hat{\eta}^* + O(\log(n))$$

Conclusions

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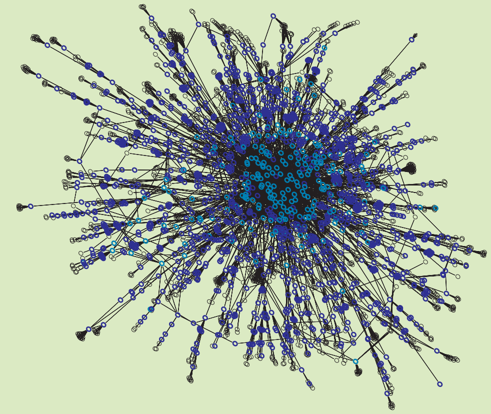
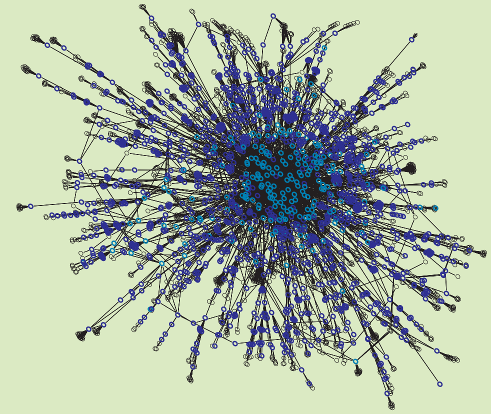


h-MaxWeight policy stabilizing under very general conds.

General approach to policy translation. Resulting policy mirrors optimal policy in examples

Asymptotically optimal, with logarithmic regret for model with single bottleneck

Conclusions



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General approach to policy translation. Resulting policy mirrors optimal policy in examples

Asymptotically optimal, with logarithmic regret for model with single bottleneck

Future work

Models with multiple bottlenecks?

On-line learning for policy improvement?

References

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