Spectrum Management: Complexity, Duality and Approximation

Zhi-Quan Luo
Department of Electrical and Computer Engineering
University of Minnesota
Minneapolis, MN 55455

Shuzhong Zhang
Department of Systems Engineering and Engineering Management
The Chinese University of Hong Kong
Outline

- Motivation: Wireless/Wireline Multiuser Systems
- Problem Statement: Continuous and Discrete Versions
- Game Theoretic Approach: Nash equilibriums
- Optimality of FDMA Solutions
- Complexity Analysis: NP-hardness
- Approximation Algorithms
- Numerical Experiments
- Extensions

Role of optimization:

- characterizing problem complexity and the structure of optimal solution;
- providing efficient algorithms for distributed maximization with quality assurance.
Motivation: Spectrum Management

- With the proliferation of various radio devices and services, multiple systems sharing a common spectrum must coexist
  - **Wireline:** unbundled DSL  
  - **Wireless:** 802.11, Bluetooth, cognitive radio, ...

- Static Spectrum Management: **FDMA**
  - **advantage:** orthogonal transmission, zero interference
  - **drawback:** high system overhead and low bandwidth utilization
Motivation: Spectrum Management

- With the proliferation of various radio devices and services, multiple systems sharing a common spectrum must coexist
  - Wireline: unbundled DSL
  - Wireless: 802.11, Bluetooth, cognitive radio, ...

- Static Spectrum Management: FDMA
  - Advantage: orthogonal transmission, zero interference
  - Drawback: high system overhead and low bandwidth utilization
Motivation: Dynamic Spectrum Management

- Dynamic Spectrum Management: users access a common spectrum simultaneously
  - Each user’s performance depends on not only the power allocation (across spectrum) of his own, but also those of other users in the system
  ⇒ Proper spectrum management is needed
Dynamic Spectrum Management - A Dangerous Business

Multi-party Communication
Formulation: Spectrum Management

- \( K \) users sharing a common frequency band \( f \in \Omega \); user \( k \)'s power spectral density
  \[
s_k(f) \geq 0, \quad \int_{\Omega} s_k(f) df \leq P_k
  \]

- User \( k \)'s utility:
  \[
u_k = \int_{\Omega} R_k(s_1(f), \ldots, s_K(f), f) df, \quad R_k(\cdot) : \text{Lesbegue measurable, non-concave}\]

- Social optimum: maximizing total system utility \( H(u_1, \ldots, u_K) \)
  \[
  \max H(u_1, \ldots, u_K) \\
  \text{s.t. } u_1 = \int_{\Omega} R_1(s_1(f), \ldots, s_K(f), f) df \\
  \vdots \\
  u_K = \int_{\Omega} R_K(s_1(f), \ldots, s_K(f), f) df \\
  s_k(f) \geq 0, \quad \int_{\Omega} s_k(f) df \leq P_k, \quad k = 1, \ldots, K
  \]

\((P_c)\) nonconvex infinite dimensional
Formulation: Spectrum Management

- Discretized frequency band $\Omega = \{1, 2, \ldots, N\}$; Lebesque measure $\rightarrow$ discrete uniform measure; user $k$’s power allocation vector

$$s_k^n \geq 0, \quad \frac{1}{N} \sum_{n=1}^{N} s_k^n \leq P_k$$

- User $k$’s utility: $u_k = \frac{1}{N} \sum_{n=1}^{N} R_k(s_1^n, \ldots, s_K^n, n/N)$, $R_k(\cdot)$: non-concave

- Social optimum: maximizing total system utility $H(u_1, \ldots, u_K)$

$$\begin{align*}
\max & \quad H(u_1, \ldots, u_K) \\
\text{s.t.} & \quad u_1 = \frac{1}{N} \sum_{n=1}^{N} R_1(s_1^n, \ldots, s_K^n, n/N) \\
& \quad \vdots \\
& \quad u_K = \frac{1}{N} \sum_{n=1}^{N} R_K(s_1^n, \ldots, s_K^n, n/N) \\
& \quad \frac{1}{N} \sum_{n=1}^{N} s_k^n \leq P_k, \quad s_k^n \geq 0, \quad k = 1, \ldots, K,
\end{align*}$$

- Intuition: $(P_d^N) \rightarrow (P_c)$ as $N \rightarrow \infty$. 7
System Utility Functions

- **Sum-utility (arithmetic mean)**
  \[ H_1(u_1, \ldots, u_K) = \frac{1}{K}(u_1 + \cdots + u_K) \]

- **Proportional fairness (geometric mean)**
  \[ H_2(u_1, \ldots, u_K) = \left( \prod_{k=1}^{K} u_k \right)^{\frac{1}{K}} \iff \frac{1}{K}(\log u_1 + \cdots + \log u_K) \]

- **Harmonic mean utility**
  \[ H_3(u_1, \ldots, u_K) = \frac{K}{\frac{1}{u_1} + \cdots + \frac{1}{u_K}} \]

- **Min-utility**
  \[ H_4(u_1, \ldots, u_K) = \min_{1 \leq k \leq K} u_k \]

- **Ordering of system utility functions:** \( H_1 \geq H_2 \geq H_3 \geq H_4; \) Fairness ranks in reverse order.
Channel Model

- $K$ users, $N$ frequency tones; channels static, frequency selective
- Each user acts both as a transmitter and as a receiver, indexed by $\{1, 2, \ldots, K\}$. In this way, a physical user may act as transmitter $k$ and receiver $l$, with $l \neq k$.

Assume three transmitters. Then transmitter 1’s data rate at the frequency tone $n$ is

$$
\text{information rate} = R_1^n = \log(1 + \text{SNR}_n) = \log \left(1 + \frac{s_1^n}{\sigma_1^n + \alpha_{12}^n s_2^n + \alpha_{13}^n s_3^n}\right)
$$
Social optimum: maximization of sum-rate

Assume two users.

\[
\begin{align*}
\text{maximize} & \quad \frac{1}{N} \sum_{n=1}^{N} \log \left( 1 + \frac{s_n^1}{\sigma_n^1 + \alpha_{12}^n s_n^2} \right) + \frac{1}{N} \sum_{n=1}^{N} \log \left( 1 + \frac{s_n^2}{\sigma_n^2 + \alpha_{21}^n s_n^1} \right) \\
\text{subject to} & \quad \frac{1}{N} \sum_{n=1}^{N} s_n^1 \leq P_1, \quad \frac{1}{N} \sum_{n=1}^{N} s_n^2 \leq P_2, \\
& \quad s_n^1 \geq 0, \quad s_n^2 \geq 0, \quad \forall n = 1, 2, \ldots, N,
\end{align*}
\]

where \(P_i\) is user \(i\)'s total available power.

- The problem is nonconvex.
- Interested in a distributed algorithm which requires little user coordination.
Connection to the Spectrum Management Formulation

This is a special case of

$$\begin{align*}
\text{max} & \quad H(u_1, \ldots, u_K) \\
\text{s.t.} & \quad u_1 = \frac{1}{N} \sum_{n=1}^{N} R_1(s_1^n, \ldots, s_K^n) \\
& \quad \vdots \\
& \quad u_K = \frac{1}{N} \sum_{n=1}^{N} R_K(s_1^n, \ldots, s_K^n) \\
& \quad \frac{1}{N} \sum_{n=1}^{N} s_k^n \leq P_k, \quad s_k^n \geq 0, \quad k = 1, \ldots, K,
\end{align*}$$

($P_d^N$) nonconvex finite dimensional

- Users’ utilities:
  $$u_1 = \frac{1}{N} \sum_{n=1}^{N} \log \left( 1 + \frac{s_1^n}{\sigma_1^n + \alpha_{12} s_2^n} \right), \quad u_2 = \frac{1}{N} \sum_{n=1}^{N} \log \left( 1 + \frac{s_2^n}{\sigma_2^n + \alpha_{21} s_1^n} \right).$$

- System utility: $H(u_1, u_2) = u_1 + u_2$;  
  $$R_k^n(s_1^n, s_2^n) = \log \left( 1 + \frac{s_2^n}{\sigma_1^n + \alpha_{12} s_2^n} \right).$$
### Social Optimum: $K$ User Case

- Upon normalizing the channel coefficients, we obtain

$$R_k^n(s_1^n, \ldots, s_K^n) := \log \left( 1 + \frac{s_k^n}{\sigma_k^n + \sum_{l \neq k} \alpha_{lk}^n s_l^n} \right),$$

where $\sigma_k^n = \frac{N_0}{|h_{k,k}^n|^2}$, $\alpha_{lk}^n = \frac{|h_{l,k}^n|^2}{|h_{k,k}^n|^2}$.

- **Frequency flat**: $h_{l,k}^n$ independent of $n$.

- The sum-rate maximization problem can be written as follows:

$$\begin{align*}
\text{maximize} & \quad \frac{1}{NK} \sum_{k=1}^K \sum_{n=1}^N \log \left( 1 + \frac{s_k^n}{\sigma_k^n + \sum_{l \neq k} \alpha_{lk}^n s_l^n} \right) \\
\text{subject to} & \quad \frac{1}{N} \sum_{n=1}^N s_k^n \leq P_k, \quad s_k^n \geq 0 \quad n \in \mathcal{N}, \ k \in \mathcal{K}.
\end{align*}$$

where $\mathcal{N} := \{1, 2, \ldots, N\}$, $\mathcal{K} := \{1, 2, \ldots, K\}$. 
Key Issues in Spectrum Management

Main challenges:

- Nonconvexity
- Problem size ($N \geq 4000$, $K \geq 50$)
- Distributed optimization

Main goals:

- Structural property of optimal solutions
- Complexity of optimal spectrum management
- Approximation algorithms (i.e., finding $\epsilon$-optimal solution)
- Game theoretic formulations
Existing Work

Mostly studied in the engineering literature

- **Nash equilibrium formulation, convergence analysis**
  - Huang-Berry-Honig (2006)
  - Cendrillon-Huang-Chiang-Moonen (2007)
  - ...

- **Sum-rate maximization**
  - ...

- **Characterizing optimal solutions, complexity analysis**
  - Etkin-Parekh-Tse (2006)
FDMA solution set:

\[
S = \begin{cases} 
\{ s \geq 0 \mid s_k^n s_l^n = 0, \forall k \neq l, \forall n \} & \text{discrete} \\
\{ s(f) \geq 0 \mid s_k(f)s_l(f) = 0, \forall k \neq l, \forall f \} & \text{continuous}.
\end{cases}
\]

- FDMA solutions are not necessarily the vertex solutions.
When is FDMA Optimal?

**Theorem 1 (Hayashi-L. (2007))** Suppose that $K = 2$, and each user uses at least $C \geq 2$ tones. If

$$\alpha_{12}^n \alpha_{21}^n > \frac{1}{4} \left( 1 + \frac{1}{C - 1} \right)^2$$

for all $n \in \mathcal{N}$, then the global maximum of sum-rate maximization problem (2) is FDMA.

- The proof relies on the strict quasi-concavity of the sum-rate function at each tone.
- **Etkin-Parekh-Tse (2006)** showed that in the frequency flat case ($\alpha_{ij}^n = \alpha_{ij}$, independent of $n$), FDMA is optimal when

$$\alpha_{12} \alpha_{21} > 1.$$
When is FDMA Optimal?

**Theorem 2 (Hayashi-L. (2007))** Any global maximum of problem (2) must be FDMA, provided that

\[
\alpha^n_{lk} > \frac{1}{2} \quad \text{and} \quad \alpha^n_{lk} \alpha^n_{kl} > \frac{1}{4} \left( 1 + \frac{1}{C - 1} \right)^2
\]

for all \( n \in \mathcal{N} \) and \((k, l) \in \mathcal{K} \times \mathcal{K} \) with \( k \neq l \).

**Main message:** strong interference leads to FDMA.
When is FDMA Optimal?

**Theorem 3 (Hayashi-L. (2007))** Let us denote

\[ P_0 := \min_{k \in \mathcal{K}} P_k, \quad \sigma_M := \max_{(n,k) \in \mathcal{N} \times \mathcal{K}} \sigma_k^n, \]

\[ A_0 := \min_{(n,k,l) \in \mathcal{N} \times \mathcal{K} \times \mathcal{K}, k \neq l} \alpha_{nlk}^n \alpha_{kl}^n. \]

If

\[ P_0 \geq \left( N - (K - 1)C \right) \left( \frac{1}{A_0} + \frac{1}{\sqrt{A_0}} + 1 \right) \sigma_M, \] (3)

then there exists a local maximum of sum-rate maximization problem (2) that is FDMA.

* Sufficient power budget also leads to FDMA.
FDMA Optimality

\[
\begin{align*}
\text{max} & \quad H_1(u_1, \ldots, u_K) \\
\text{s.t.} & \quad u_1 = \int_{\Omega} R_1(s_1(f), \ldots, s_K(f))df \\
& \quad \vdots \\
& \quad u_K = \int_{\Omega} R_K(s_1(f), \ldots, s_K(f))df \\
& \quad s_k(f) \geq 0, \quad \int_{\Omega} s_k(f)df \leq P_k, \quad k = 1, \ldots, K,
\end{align*}
\]
Finding an Optimal FDMA Solution?

• Let us denote the set of FDMA solutions by

\[ S = \{ s \geq 0 \mid s_k^n s_l^n = 0, \forall k \neq l, \forall n \} . \]

• Then, the optimal FDMA frequency allocation problem can be described as follows:

\[
\begin{align*}
\text{maximize} & \quad \frac{1}{NK} \sum_{k=1}^{K} \sum_{n=1}^{N} \log \left( 1 + \frac{s_k^n}{\sigma_k^n} \right) \\
\text{subject to} & \quad s \in S, \quad \frac{1}{N} \sum_{n=1}^{N} s_k^n \leq P_k, \quad k = 1, \ldots, K.
\end{align*}
\]

where \( s \) denotes the \( NK \)-dimensional vector with entries equal to \( s_l^n \).

• Note that there is no interference in the sum-rate function (5).
Complexity Analysis: NP-hardness

Theorem 4 (Hayashi-L. (2007)) For $K = 2$, the optimal bandwidth allocation problem (5) is NP-hard. Thus, the general sum-rate maximization problem (2) is also NP-hard, even in the two-user case.

- The proof consists of reducing the so-called equipartition problem to (5).
- Specifically, given a set of $N$ (even) positive integers, $a_1, a_2, \ldots, a_N$, the equipartition problem asks: does there exist a subset $T \subset \{1, 2, \ldots, N\}$ of size $|T| = N/2$ such that

$$\sum_{n \in T} a_n = \sum_{n \notin T} a_n = \frac{1}{2} \sum_{n=1}^{N} a_n ?$$

- The equipartition problem is known to be NP-complete.
- Finding optimal FDMA solution is hard. What do we do now?
## Further Complexity Results

Complexity of the discrete resource management problem ($P_d^N$) (L.-Zhang (2007))

<table>
<thead>
<tr>
<th>Utility Function</th>
<th>Problem Class</th>
<th>Sum-Rate $H_1$ (FDMA Soln)</th>
<th>Sum-Rate $H_1$ (arithmetic mean)</th>
<th>Proportional Fairness $H_2$ (geometric mean)</th>
<th>Harmonic mean $H_3$</th>
<th>Min-Rate $H_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>K≥2 and fixed, N arbitrary</td>
<td>NP-hard</td>
<td>NP-hard</td>
<td>NP-hard</td>
<td>NP-hard</td>
<td>NP-hard</td>
<td></td>
</tr>
<tr>
<td>N&gt;2 and fixed, K arbitrary</td>
<td>Strongly NP-hard</td>
<td>Strongly NP-hard</td>
<td>NP-hard</td>
<td>Strongly NP-hard</td>
<td>Strongly NP-hard</td>
<td></td>
</tr>
<tr>
<td>N=1, K arbitrary</td>
<td>Linear time solvable</td>
<td>Strongly NP-hard</td>
<td>Convex Opt</td>
<td>Convex Opt</td>
<td>LP</td>
<td></td>
</tr>
</tbody>
</table>

- Reduction from partition problem and 3-coloring problem
- The status of $N = 2$ not resolved yet.
Optimal FDMA Solution?

- Let us denote the set of FDMA solutions by

\[ S = \{ s \geq 0 \mid s_k^n s_l^n = 0, \ \forall \ k \neq l, \ \forall \ n \}. \]

- Then, the optimal FDMA frequency allocation problem can be described as follows:

\[
\begin{aligned}
\text{maximize} \quad & \frac{1}{NK} \sum_{k=1}^{K} \sum_{n=1}^{N} \log \left( 1 + \frac{s_k^n}{\sigma_k^n} \right) \\
\text{subject to} \quad & s \in S, \ \sum_{n=1}^{N} s_k^n \leq P_k, \ k = 1, \ldots, K.
\end{aligned}
\]

where \( s \) denotes the \((NK)\)-dimensional vector with entries equal to \( s_l^n \).

- Note that there is no interference in the sum-rate function (5).
Finding an Approximate FDMA Solution

- Dual problem is convex and decomposes across tones
- Dual function

\[
\begin{align*}
    d(\lambda) & := \max_{s \in \mathcal{S}} \left( \sum_{k=1}^{K} \sum_{n=1}^{N} \log \left( 1 + \frac{s_k^n}{\sigma_k^n} \right) - \sum_{k=1}^{K} \lambda_k \left( \sum_{n=1}^{N} s_k^n - P_k \right) \right) \\
    &= \sum_{k=1}^{K} \lambda_k P_k + \sum_{n=1}^{N} \max_{0 \leq s_i^n \leq P_i, s_i^n s_j^n = 0, i \neq j} \sum_{k=1}^{K} \left( \log \left( 1 + \frac{s_k^n}{\sigma_k^n} \right) - \lambda_k s_k^n \right)
\end{align*}
\]

- The inner maximization in (5) can be solved by allocating each tone to the user which can provide the maximum *shadow rate* \( \log \left( 1 + \frac{s_k^n}{\sigma_k^n} \right) - \lambda_k s_k^n \) on that tone.
Finding an Approximate FDMA Solution

• Maximum shadow rate for user $k$ at tone $n$ is given by

$$\max_{0 \leq s_k^n \leq P_k} \left( \log \left( 1 + \frac{s_k^n}{\sigma_k^n} \right) - \lambda_k s_k^n \right) = \begin{cases} \lambda_k \sigma_k^n - \log(\lambda_k \sigma_k^n) - 1, & 0 < \lambda_k \sigma_k^n \leq 1, \\ 0, & \lambda_k \sigma_k^n > 1, \\ \infty, & \lambda_k < 0, \end{cases}$$

where the optimal power level is

$$s_k^n = P_k (\lambda_k^{-1} - \sigma_k^n).$$  \hspace{1cm} (6)

• Thus, the dual function (5) can be written analytically as

$$d(\lambda) = \sum_{k=1}^{K} \lambda_k P_k + \sum_{n=1}^{N} \max_{k: \lambda_k \sigma_k^n \leq 1} (\lambda_k \sigma_k^n - \log(\lambda_k \sigma_k^n) - 1).$$  \hspace{1cm} (7)
Finding an Approximate FDMA Solution

• For each $n$, the maximum in (7) is attained at the user $k$ for which $\lambda_k \sigma_k^n$ is smallest.

• Then a subgradient of $d(\lambda)$ is given by

$$
\nabla d(\lambda) = \left( P_1 - \sum_{n \in \mathcal{N}_1(\lambda)} s_1^n, P_2 - \sum_{n \in \mathcal{N}_2(\lambda)} s_2^n, \ldots, P_K - \sum_{n \in \mathcal{N}_K(\lambda)} s_K^n \right)^T
$$

where we denote the set of tones assigned to user $k$ by $\mathcal{N}_k(\lambda)$. Notice that the components of subgradient $\nabla d(\lambda)$ correspond to each user’s unused power (or deficit power if negative).

• The dual minimization problem is given by

$$
\begin{array}{l}
\text{minimize} \quad d(\lambda) \\
\text{subject to} \quad \lambda \geq 0
\end{array}
$$

(D_d^N)

which is convex and solvable in polynomial time (e.g., using ellipsoid algorithm).
Duality Gap

- Recall the primal sum-rate maximization problem is NP-hard, implying there is a positive duality gap.

- **Dual optimality:** \( 0 \in \partial d(\lambda) \).

**Theorem 5** Let \( \lambda^* \geq 0 \) and \( s^* \geq 0 \) be the limit points generated by the dual decomposition algorithm. If there holds

\[
\frac{1}{N} \sum_{n=1}^{N} s^n_k \leq P_k, \quad \lambda^*_k \left( \frac{1}{N} \sum_{n=1}^{N} s^n_k - P_k \right) = 0, \quad \forall k \in K,
\]

then the duality gap is zero and \( s^* \) is a global optimal solution of the bandwidth allocation problem.

In other words, **primal feasibility ensures zero duality gap.**

- This holds true if \( \partial d(\lambda) \) is singleton.
Constructing an Approximate Primal Optimal Solution

- When $\partial d(\lambda)$ is not singleton, primal feasibility cannot be attained and there is a positive duality gap.

- However, we can further “split the tones” and construct a primal feasible solution for a more refined discretized primal problem with zero duality gap.

Zero duality gap, primal FDMA feasibility achieved via LP
Approximation Quality

Main Consequences (L.-Zhang, 2007):

- The nonconvex continuous optimal FDMA spectrum allocation problem and its dual are equivalent. The duality gap is zero.
- For each $\epsilon > 0$, we can find an $\epsilon$-optimal solution in $\text{Poly}(K, \epsilon)$ time.

\[
\begin{align*}
\max & \quad H_1(u_1, \ldots, u_K) \\
\text{s.t.} & \quad u_1 = \int_{\Omega} R_1(s_1(f), \ldots, s_K(f)) df \\
& \quad \vdots \\
& \quad u_K = \int_{\Omega} R_K(s_1(f), \ldots, s_K(f)) df \\
& \quad s_k(f)s_l(f) = 0, \quad s_k(f) \geq 0, \quad \int_{\Omega} s_k(f) df \leq P_k, \quad \forall k,
\end{align*}
\]

$(P_c)$

\[
\begin{align*}
\min_{\lambda \geq 0} \max_{s_k(f) \geq 0, \ s_k(f)s_l(f)=0,} \sum_{k=1}^{K} \int_{\Omega} \left( \left( 1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j \neq k} \alpha_{kj} s_j(f)} \right) - \lambda_k s_k(f) \right) df - \lambda_k P_k
\end{align*}
\]

$(D_c)$
Key Step

• For a Lebesgue integrable vector function $R(s(f), f)$, we have in general

$$\frac{1}{N} \sum_{n=1}^{N} R(s(n/N), n/N) \not\to \int_{\Omega} R(s(f), f)df, \quad N \to \infty.$$  

• However, we show there exists some piecewise constant function $s^n$ (not necessarily equal to $s(n/N)$), such that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} R(s^n, n/N) - \int_{\Omega} R(s(f), f)df \right\| = O \left( \frac{1}{\sqrt{N}} \right).$$

• This implies that the gap between $(P_c)$ and $(P_d^N)$ is $O(1/\sqrt{N})$.

• $O(1/\sqrt{N})$ can be improved to $O(1/N)$ for frequency flat case.
Key Observation

\[
\begin{align*}
\max & \quad H(u_1, \ldots, u_K) \\
\text{s.t.} & \quad u_1 = \frac{1}{N} \sum_{n=1}^{N} R_1(s^n_1, \ldots, s^n_K, n/N) \\
& \quad \vdots \\
& \quad u_K = \frac{1}{N} \sum_{n=1}^{N} R_K(s^n_1, \ldots, s^n_K, n/N) \\
& \quad \frac{1}{N} \sum_{n=1}^{N} s^n_k \leq P_k, \quad s^n_k \geq 0, \quad k = 1, \ldots, K 
\end{align*}
\]

\[
\downarrow
\]

\[
\begin{align*}
\max & \quad H(u_1, \ldots, u_K) \\
\text{s.t.} & \quad u_1 = \int_{\Omega} R_1(s_1(f), \ldots, s_K(f), f) df \\
& \quad \vdots \\
& \quad u_K = \int_{\Omega} R_K(s_1(f), \ldots, s_K(f), f) df \\
& \quad s_k(f) \geq 0, \quad \int_{\Omega} s_k(f) df \leq P_k, \quad k = 1, \ldots, K 
\end{align*}
\]
Extensions

- It is possible to show that, without FDMA constraint,

\[
\text{maximize } \sum_{k=1}^{K} \int_{0}^{1} \left( 1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j \neq k} \alpha_{kj} s_j(f)} \right) \, df
\]

\[
\text{subject to } \int_{0}^{1} s_k(f) \, df \leq P_k, \quad s_k(f) \geq 0, \quad \forall k \in \mathcal{K}.
\]

has the same optimal value as its dual

\[
\min_{\lambda \geq 0} \max_{s_k(f) \geq 0} \sum_{k=1}^{K} \int_{0}^{1} \left( \frac{s_k(f)}{\sigma_k(f) + \sum_{j \neq k} \alpha_{kj} s_j(f)} \right) - \lambda \cdot s_k(f) \, df - \lambda P_k
\]

- **Asymptotic strong duality:** This suggests, for the finite tone case, the duality gap decreases to zero.
Duality Gap $\rightarrow 0$
Lyapunov Theorem

Let $u$ be a non-atomic measure on a Borel field $\mathcal{B}$ generated from subsets of a space $\Omega$. Let $g_i(s(\cdot), \cdot) : \mathbb{R}^2 \to \mathbb{R}_+$ be compatible with $\mathcal{B}$-measurable function (i.e., if $s(\cdot)$ is $\mathcal{B}$-measurable then $g_i(s(\cdot), \cdot)$ is $\mathcal{B}$-measurable), $i = 1, \ldots, m$. Then,

$$\left\{ \left( \begin{array}{c} \int_{\Omega} g_1(s(\cdot), \cdot)du \\ \vdots \\ \int_{\Omega} g_m(s(\cdot), \cdot)du \end{array} \right) \left| x \text{ is } \mathcal{B}\text{-measurable} \right. \right\}$$

is a convex set.
Implications of Lyapunov Theorem

\[
v(P) = \max \frac{1}{K}(u_1 + \cdots + u_K)
\]
\[
s.t. \quad u_1 = \int_{\Omega} R_1(s_1(f), \ldots, s_K(f))df \\
\vdots \\
u_K = \int_{\Omega} R_K(s_1(f), \ldots, s_K(f))df \\
s_k(f) \geq 0, \int_{\Omega} s_k(f)df \leq P_k, \quad k = 1, \ldots, K,
\]

- \(v(P)\) is a concave function of \(P = (P_1, \ldots, P_K)\).
- This implies zero duality gap, re-establishing the result of Yu, Lui and Cendrillon (2006).
Implications of Lyapunov Theorem

\[ v(P) = \max \ H(u_1, \ldots, u_K) \]

s.t. \[ u_1 = \int_{\Omega} R_1(s_1(f), \ldots, s_K(f)) df \]
\[ \vdots \]
\[ u_K = \int_{\Omega} R_K(s_1(f), \ldots, s_K(f)) df \]
\[ s_k(f) \geq 0, \int_{\Omega} s_k(f) df \leq P_k, k = 1, \ldots, K, \]

- \( v(P) \) is a concave function of \( P = (P_1, \ldots, P_K) \) if
  - \( H(u_1, \ldots, u_K) \) is jointly concave.
  - \( H(u_1, \ldots, u_K) \) is monotonically increasing wrt each argument.

- Under these assumptions, the duality gap is zero \( (L.-Zhang (2007)) \).
- \( H_1, H_2, H_3 \) and \( H_4 \) all satisfy the above two assumptions.
Further Implication of Lyapunov Theorem

Given a powerful adversary (user 0) in the system, let us maximize the worst-case performance

$$\max_{s_k(f) \geq 0,} \min_{s_0(f) \geq 0,} \int_{\Omega} s_k(f) df \leq P_k \int_{\Omega} s_0(f) df \leq P_0$$

where $H = H_1, H_2, H_3$ or $H_4$, and

$$u_k = \int_{\Omega} \log \left( 1 + \frac{s_k(f)}{\sum_{l \neq k} \alpha_{kl} s_l(f) + \sigma_k(f)} \right) df, \quad k = 1, 2, \ldots, K.$$

- $H$ is convex in $s_0(f)$. However, $H$ is not concave in $s_k(f)$, $k = 1, 2, \ldots, K$.
- Luckily, Lyapunov Theorem ensures a hidden concavity, which together with the convexity and compactness of the feasible sets, implies

$$\max_{s_k(f) \geq 0,} \min_{s_0(f) \geq 0,} H(u_1, \ldots, u_K) = \min_{s_0(f) \geq 0,} \max_{s_k(f) \geq 0,} H(u_1, \ldots, u_K)$$
Thank You