Spectrum Management: Complexity, Duality and Approximation

Zhi-Quan Luo

Department of Electrical and Computer Engineering
University of Minnesota
Minneapolis, MN 55455

Shuzhong Zhang

Department of Systems Engineering and Engineering Management
The Chinese University of Hong Kong

Outline

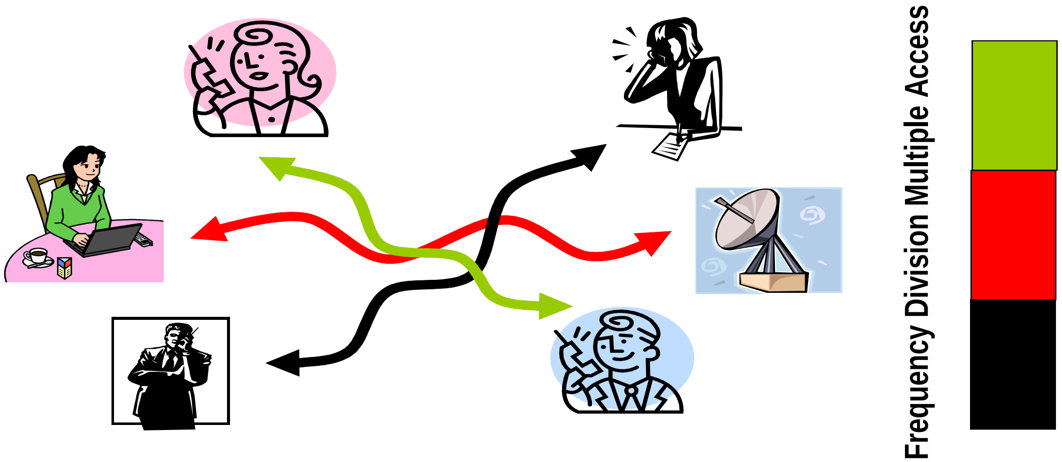
- Motivation: Wireless/Wireline Multiuser Systems
- Problem Statement: Continuous and Discrete Versions
- Game Theoretic Approach: Nash equilibriums
- Optimality of FDMA Solutions
- Complexity Analysis: NP-hardness
- Approximation Algorithms
- Numerical Experiments
- Extensions

Role of optimization:

- characterizing problem complexity and the structure of optimal solution;
- providing efficient algorithms for distributed maximization with quality assurance.

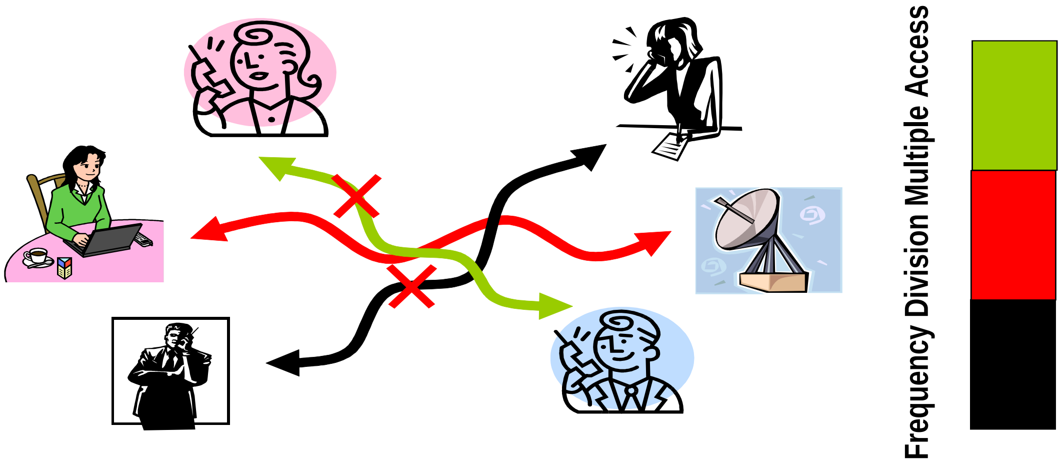
Motivation: Spectrum Management

- With the proliferation of various radio devices and services, multiple systems sharing a common spectrum must coexist
  - Wireline: unbundled DSL
  - Wireless: 802.11, Bluetooth, cognitive radio, ...
- Static Spectrum Management: FDMA
  - advantage: orthogonal transmission, zero interference
  - drawback: high system overhead and low bandwidth utilization



Motivation: Spectrum Management

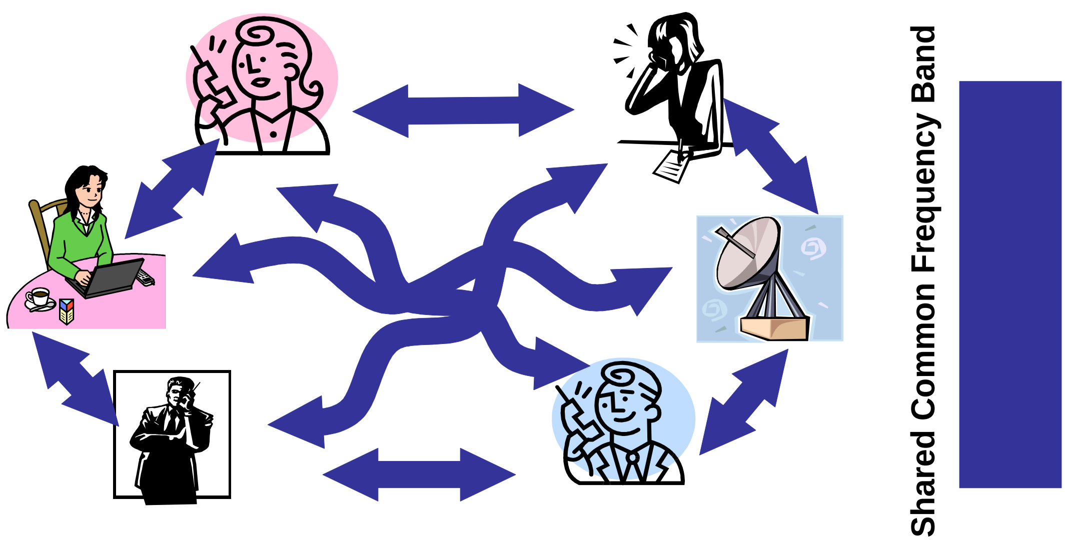
- With the proliferation of various radio devices and services, multiple systems sharing a common spectrum must coexist
  - Wireline: unbundled DSL
  - Wireless: 802.11, Bluetooth, cognitive radio, ...
- Static Spectrum Management: FDMA
  - advantage: orthogonal transmission, zero interference
  - drawback: high system overhead and low bandwidth utilization



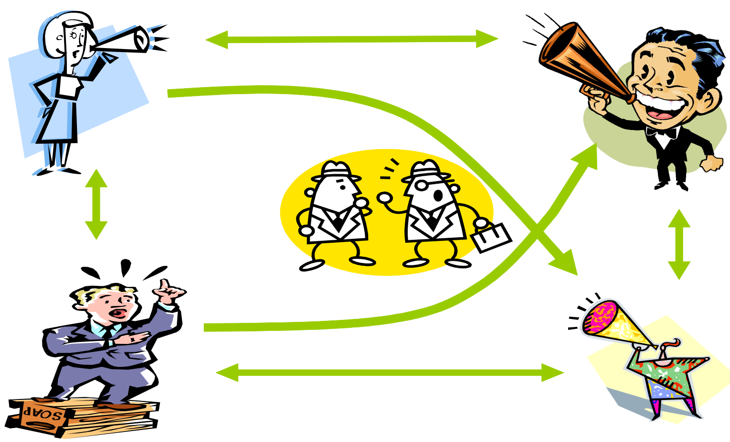
Motivation: Dynamic Spectrum Management

- Dynamic Spectrum Management: users access a common spectrum simultaneously
  - Each user's performance depends on not only the power allocation (across spectrum) of his own, but also those of other users in the system

    $\Rightarrow$ Proper spectrum management is needed



Dynamic Spectrum Management - A Dangerous Business



Multi-party Communication

Formulation: Spectrum Management

- K users sharing a common frequency band $f \in \Omega$; user k's power spectral density

$$s_k(f) \geq 0, \quad \int_\Omega s_k(f) df \leq P_k$$

- **User k's utility:**

$$u_k = \int_\Omega R_k(s_1(f), \ldots, s_K(f), f) df, \quad R_k(\cdot) : \text{ Lesbegue measurable, non-concave}$$

- **Social optimum:** maximizing total system utility $H(u_1, \ldots, u_K)$

$$\begin{array}{ll} \text{max} & H(u_1, \cdots, u_K) \\ \text{s.t.} & u_1 = \int_\Omega R_1(s_1(f), \ldots, s_K(f), f) df \\ & \quad \vdots \\ & u_K = \int_\Omega R_K(s_1(f), \ldots, s_K(f), f) df \\ & s_k(f) \geq 0, \int_\Omega s_k(f) df \leq P_k, \ k = 1, \ldots, K, \end{array}$$

(P_c)

nonconvex

infinite dimensional

Formulation: Spectrum Management

- Discretized frequency band $\Omega = \{1, 2, \ldots, N\}$; **Lebesque measure** $\longrightarrow$ **discrete uniform measure**; user k's power allocation vector

$$s_k^n \geq 0, \quad \frac{1}{N}\sum_{n=1}^{N} s_k^n \leq P_k$$

- **User k's utility:** $u_k = \dfrac{1}{N}\displaystyle\sum_{n=1}^{N} R_k(s_1^n, \ldots, s_K^n, n/N), \quad R_k(\cdot)$: non-concave

- **Social optimum:** maximizing total system utility $H(u_1, \ldots, u_K)$

$$\begin{array}{ll} \text{max} & H(u_1, \cdots, u_K) \\ \text{s.t.} & u_1 = \frac{1}{N}\sum_{n=1}^{N} R_1(s_1^n, \ldots, s_K^n, n/N) \\ & \vdots \\ & u_K = \frac{1}{N}\sum_{n=1}^{N} R_K(s_1^n, \ldots, s_K^n, n/N) \\ & \frac{1}{N}\sum_{n=1}^{N} s_k^n \leq P_k,\ s_k^n \geq 0,\ k = 1, \ldots, K, \end{array}$$

(P_d^N)

nonconvex

finite dimensional

- **Intuition:** $(P_d^N) \longrightarrow (P_c)$ as $N \longrightarrow \infty$.

System Utility Functions

- **Sum-utility (arithmetic mean)**

$$H_1(u_1, \ldots, u_K) = \frac{1}{K}(u_1 + \cdots + u_K)$$

- **Proportional fairness (geometric mean)**

$$H_2(u_1, \ldots, u_K) = \left(\prod_{k=1}^{K} u_k\right)^{\frac{1}{K}} \quad \Leftrightarrow \quad \frac{1}{K}(\log u_1 + \cdots + \log u_K)$$

- **Harmonic mean utility**

$$H_3(u_1, \ldots, u_K) = \frac{K}{u_1^{-1} + \cdots + u_K^{-1}}$$

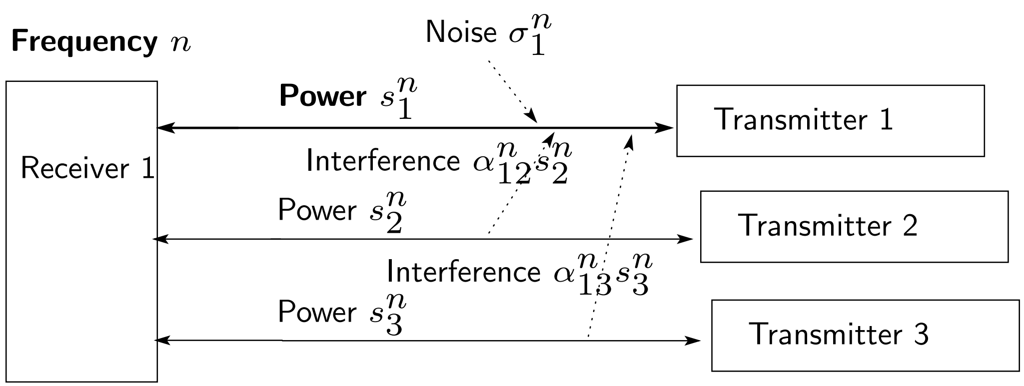
- **Min-utility**

$$H_4(u_1, \ldots, u_K) = \min_{1 \le k \le K} u_k$$

- **Ordering of system utility functions:** $H_1 \geq H_2 \geq H_3 \geq H_4$;
  Fairness ranks in reverse order.

Channel Model

- K users, N frequency tones; channels static, frequency selective
- Each user acts both as a transmitter and as a receiver, indexed by $\{1, 2, ..., K\}$. In this way, a physical user may act as transmitter k and receiver l, with $l \neq k$.



Assume three transmitters. Then transmitter 1's data rate at the frequency tone n is

$$\text{information rate} = R_1^n = \log(1 + \mathsf{SNR}_n) = \log\left(1 + \frac{s_1^n}{\sigma_1^n + \alpha_{12}^n s_2^n + \alpha_{13}^n s_3^n}\right)$$

Social optimum: maximization of sum-rate

Assume two users.

$$\begin{array}{ll} \text{maximize} & \frac{1}{N}\sum_{n=1}^{N} \log\left(1 + \frac{s_1^n}{\sigma_1^n + \alpha_{12}^n s_2^n}\right) + \frac{1}{N}\sum_{n=1}^{N} \log\left(1 + \frac{s_2^n}{\sigma_2^n + \alpha_{21}^n s_1^n}\right) \\ \text{subject to} & \frac{1}{N}\sum_{n=1}^{N} s_1^n \le P_1, \ \frac{1}{N}\sum_{n=1}^{N} s_2^n \le P_2, \\ & s_1^n \ge 0, \ \ s_2^n \ge 0, \ \ \forall\, n = 1, 2, ..., N, \end{array}$$

where P_i is user i's total available power.

- The problem is nonconvex.
- Interested in a distributed algorithm which requires little user coordination.

Connection to the Spectrum Management Formulation

This is a special case of

$$\begin{aligned} \text{max} \quad & H(u_1, \cdots, u_K) \\ \text{s.t.} \quad & u_1 = \frac{1}{N}\sum_{n=1}^{N} R_1(s_1^n, \ldots, s_K^n) \\ & \vdots \\ & u_K = \frac{1}{N}\sum_{n=1}^{N} R_K(s_1^n, \ldots, s_K^n) \\ & \frac{1}{N}\sum_{n=1}^{N} s_k^n \le P_k, \; s_k^n \ge 0, \; k = 1, \ldots, K, \end{aligned}$$

(P_d^N)
nonconvex
finite dimensional

- **Users' utilities:**

$$u_1 = \frac{1}{N}\sum_{n=1}^{N} \log\left(1 + \frac{s_1^n}{\sigma_1^n + \alpha_{12}^n s_2^n}\right), \quad u_2 = \frac{1}{N}\sum_{n=1}^{N} \log\left(1 + \frac{s_2^n}{\sigma_2^n + \alpha_{21}^n s_1^n}\right).$$

- **System utility:** $H(u_1, u_2) = u_1 + u_2$;
- $R_k^n(s_1^n, s_2^n) = \log\left(1 + \frac{s_1^n}{\sigma_1^n + \alpha_{12}^n s_2^n}\right)$

Social Optimum: K User Case

- Upon normalizing the channel coefficients, we obtain

$$R_k^n(s_1^n, \ldots, s_K^n) \quad := \quad \log\left(1 + \frac{s_k^n}{\sigma_k^n + \sum_{l \neq k} \alpha_{lk}^n s_l^n}\right), \tag{1}$$

where $\sigma_k^n = N_0/|h_{k,k}^n|^2$, $\alpha_{lk}^n = |h_{l,k}^n|^2/|h_{k,k}^n|^2$.

- **Frequency flat:** $h_{l,k}^n$ independent of n.
- The sum-rate maximization problem can be written as follows:

$$\begin{array}{ll} \text{maximize} & \displaystyle\frac{1}{NK}\sum_{k=1}^{K}\sum_{n=1}^{N}\log\left(1+\frac{s_k^n}{\sigma_k^n+\sum_{l\neq k}\alpha_{lk}^n s_l^n}\right) \\ \text{subject to} & \displaystyle\frac{1}{N}\sum_{n=1}^{N}s_k^n \le P_k, \quad s_k^n \ge 0 \quad n \in \mathcal{N},\ k \in \mathcal{K}. \end{array} \tag{2}$$

where $\mathcal{N} := \{1, 2, \ldots, N\}$, $\mathcal{K} := \{1, 2, \ldots, K\}$.

Key Issues in Spectrum Management

Main challenges:

- Nonconvexity
- Problem size ($N \geq 4000$, $K \geq 50$)
- Distributed optimization

Main goals:

- Structural property of optimal solutions
- Complexity of optimal spectrum management
- Approximation algorithms (i.e., finding ϵ-optimal solution)
- Game theoretic formulations

Existing Work

Mostly studied in the engineering literature

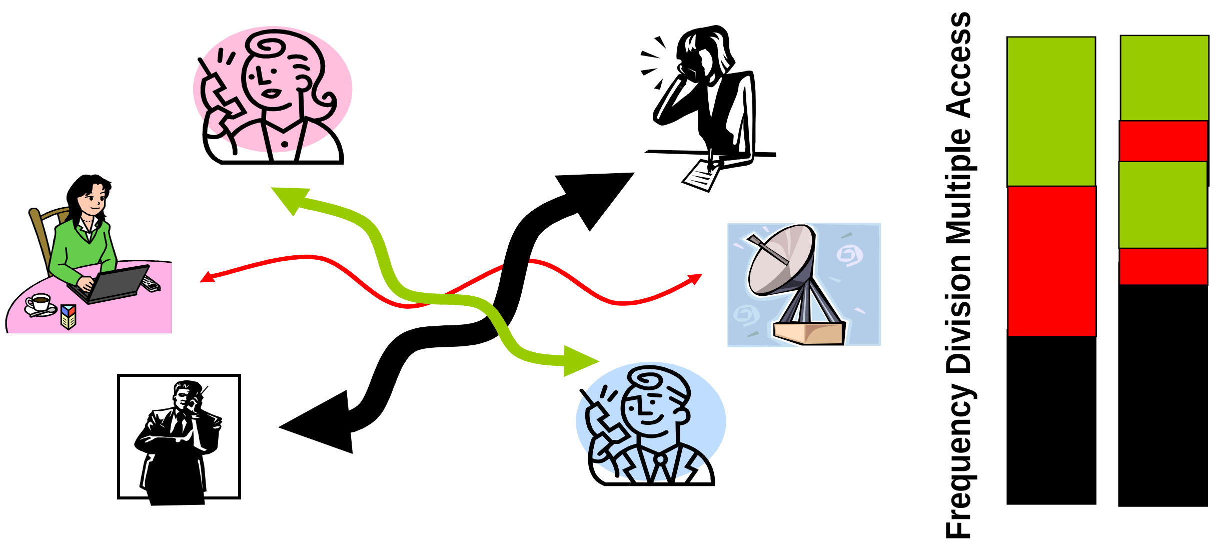
- **Nash equilibrium formulation, convergence analysis**
  - Yu-Ginis-Cioffi (2002)
  - Yamashita-L. (2004)
  - L.-Pang (2006)
  - Huang-Berry-Honig (2006)
  - Cendrillon-Huang-Chiang-Moonen (2007)
  - ...
- **Sum-rate maximization**
  - Yu-Lui-Cendrillon, Chan-Yu (2004/2006)
  - Cendrillon-Yu-Moonen-Verliden-Bostoen (2006)
  - ...
- **Characterizing optimal solutions, complexity analysis**
  - Etkin-Parekh-Tse (2006)
  - Hayashi-L. (2007)

Social Optimum: FDMA Solutions

FDMA solution set:

$$\mathcal{S} = \begin{cases} \{\mathbf{s} \geq 0 \mid s_k^n s_l^n = 0, \ \forall \ k \neq l, \ \forall \ n\} & \text{discrete} \\ \{\mathbf{s}(f) \geq 0 \mid s_k(f) s_l(f) = 0, \ \forall \ k \neq l, \ \forall \ f\} & \text{continuous.} \end{cases}$$

- FDMA solutions are *not* necessarily the vertex solutions.



When is FDMA Optimal?

Theorem 1 (Hayashi-L. (2007)) Suppose that $K = 2$, and each user uses at least $C \geq 2$ tones. If

$$\alpha_{12}^{n}\alpha_{21}^{n} \quad > \quad \frac{1}{4}\left(1 + \frac{1}{C-1}\right)^{2}$$

for all $n \in \mathcal{N}$, then the global maximum of sum-rate maximization problem (2) is FDMA.

- The proof relies on the strict quasi-concavity of the sum-rate function at each tone.
- **Etkin-Parekh-Tse (2006)** showed that in the frequency flat case ($\alpha_{ij}^{n} = \alpha_{ij}$, independent of n), FDMA is optimal when

$$\alpha_{12}\alpha_{21} > 1.$$

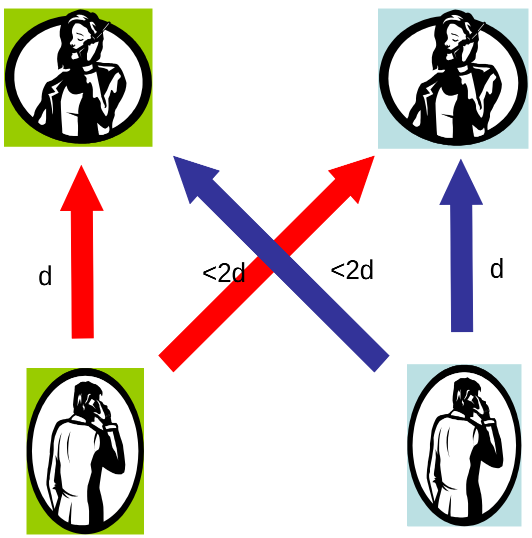
When is FDMA Optimal?

Theorem 2 (Hayashi-L. (2007)) Any global maximum of problem (2) must be FDMA, provided that

$$\alpha_{lk}^{n} > \frac{1}{2} \quad \text{and} \quad \alpha_{lk}^{n}\alpha_{kl}^{n} > \frac{1}{4}\left(1+\frac{1}{C-1}\right)^{2}$$

for all $n \in \mathcal{N}$ and $(k,l) \in \mathcal{K} \times \mathcal{K}$ with $k \neq l$.

Main message: strong interference leads to FDMA.



When is FDMA Optimal?

Theorem 3 (Hayashi-L. (2007)) Let us denote

$$\mathbf{P}_0 := \min_{k \in \mathcal{K}} P_k, \quad \sigma_M := \max_{(n,k) \in \mathcal{N} \times \mathcal{K}} \sigma_k^n,$$
$$A_0 := \min_{\substack{(n,k,l) \in \mathcal{N} \times \mathcal{K} \times \mathcal{K} \\ k \neq l}} \alpha_{lk}^n \alpha_{kl}^n.$$

If

$$\mathbf{P}_0 \geq \Big(N - (K-1)C\Big) \left(\frac{1}{A_0} + \frac{1}{\sqrt{A_0}} + 1 \right) \sigma_M, \tag{3}$$

then there exists a local maximum of sum-rate maximization problem (2) that is FDMA.

- Sufficient power budget also leads to FDMA.

FDMA Optimality

$$
\begin{aligned}
\text{max} \quad & H_1(u_1, \cdots, u_K) \\
\text{s.t.} \quad & u_1 = \int_\Omega R_1(s_1(f), \ldots, s_K(f))df \\
& \vdots \\
& u_K = \int_\Omega R_K(s_1(f), \ldots, s_K(f))df \\
& s_k(f) \geq 0, \int_\Omega s_k(f)df \leq P_k, \ k = 1, \ldots, K,
\end{aligned}
$$

$$
\begin{aligned}
\text{max} \quad & H_1(u_1, \cdots, u_K) \\
\text{s.t.} \quad & u_1 = \int_\Omega R_1(s_1(f), \ldots, s_K(f))df \\
& \vdots \\
& u_K = \int_\Omega R_K(s_1(f), \ldots, s_K(f))df \\
& s_k(f)s_l(f) = 0, \ s_k(f) \geq 0, \int_\Omega s_k(f)df \leq P_k, \ \forall\ k,
\end{aligned}
$$

Finding an Optimal FDMA Solution?

- Let us denote the set of FDMA solutions by

$$\mathcal{S} = \{ \mathbf{s} \geq 0 \mid s_k^n s_l^n = 0, \; \forall \, k \neq l, \; \forall \, n \} \, .$$

- Then, the optimal FDMA frequency allocation problem can be described as follows:

$$\begin{aligned} \underset{\mathbf{s}}{\text{maximize}} \quad & \frac{1}{NK} \sum_{k=1}^{K} \sum_{n=1}^{N} \log \left(1 + \frac{s_k^n}{\sigma_k^n} \right) \\ \text{subject to} \quad & \mathbf{s} \in \mathcal{S}, \quad \frac{1}{N} \sum_{n=1}^{N} s_k^n \leq P_k, \quad k = 1, \ldots, K. \end{aligned} \tag{4}$$

where $\mathbf{s}$ denotes the (NK)-dimensional vector with entries equal to s_l^n.

- Note that there is no interference in the sum-rate function (5).

Complexity Analysis: NP-hardness

Theorem 4 (Hayashi-L. (2007)) For $K = 2$, the optimal bandwidth allocation problem (5) is NP-hard. Thus, the general sum-rate maximization problem (2) is also NP-hard, even in the two-user case.

- The proof consists of reducing the so-called **equipartition** problem to (5).
- Specifically, given a set of N (even) positive integers, a_1, a_2, ..., a_N, the equipartition problem asks: does there exist a subset $T \subset \{1, 2, \ldots, N\}$ of size $|T| = N/2$ such that

$$\sum_{n \in T} a_n = \sum_{n \notin T} a_n = \frac{1}{2} \sum_{n=1}^{N} a_n \ ?$$

- The **equipartition** problem is known to be NP-complete.
- Finding optimal FDMA solution is hard. What do we do now?

Further Complexity Results

Complexity of the discrete resource management problem (P_d^N) (**L.-Zhang (2007)**)

Utility Function / Problem Class	Sum-Rate H_1 FDMA Soln	Sum-Rate H_1 (arithmetic mean)	Proportional Fairness H_2 (geometric mean)	Harmonic mean H_3	Min-Rate H_4
K=1, N arbitrary	Convex Opt (Waterfilling)	Convex Opt (Waterfilling)	Convex Opt (Waterfilling)	Convex Opt (Waterfilling)	Convex Opt (Waterfilling)
K≥2 and fixed, N arbitrary	NP-hard	NP-hard	NP-hard	NP-hard	NP-hard
N>2 and fixed, K arbitrary	Strongly NP-hard	Strongly NP-hard	NP-hard	Strongly NP-hard	Strongly NP-hard
N=1, K arbitrary	Linear time solvable	Strongly NP-hard	Convex Opt	Convex Opt	LP

- Reduction from partition problem and 3-coloring problem
- The status of $N = 2$ not resolved yet.

Optimal FDMA Solution?

- Let us denote the set of FDMA solutions by

$$\mathcal{S} = \{\mathbf{s} \geq 0 \mid s_k^n s_l^n = 0,\ \forall\, k \neq l,\ \forall\, n\}\,.$$

- Then, the optimal FDMA frequency allocation problem can be described as follows:

$$\begin{array}{ll} \underset{\mathbf{s}}{\text{maximize}} & \dfrac{1}{NK}\displaystyle\sum_{k=1}^{K}\sum_{n=1}^{N}\log\left(1+\frac{s_k^n}{\sigma_k^n}\right) \\ \text{subject to} & \mathbf{s}\in\mathcal{S},\ \displaystyle\sum_{n=1}^{N} s_k^n \leq P_k,\ \ k=1,\ldots,K. \end{array} \qquad (P_d^N)$$

where $\mathbf{s}$ denotes the (NK)-dimensional vector with entries equal to s_l^n.

- Note that there is no interference in the sum-rate function (5).

Finding an Approximate FDMA Solution

- Dual problem is convex and decomposes across tones
- Dual function

$$
\begin{aligned}
d(\boldsymbol{\lambda}) \quad &:= \quad \max_{\mathbf{s}\in\mathcal{S}} \left(\sum_{k=1}^{K} \sum_{n=1}^{N} \log\left(1 + \frac{s_k^n}{\sigma_k^n}\right) - \sum_{k=1}^{K} \lambda_k \left(\sum_{n=1}^{N} s_k^n - P_k \right) \right) \\
&= \quad \sum_{k=1}^{K} \lambda_k P_k + \sum_{n=1}^{N} \max_{\substack{0\le s_i^n \le P_i \\ s_i^n s_j^n = 0,\ i\ne j}} \sum_{k=1}^{K} \left(\log\left(1+\frac{s_k^n}{\sigma_k^n}\right) - \lambda_k s_k^n \right) \qquad (5)
\end{aligned}
$$

- The inner maximization in (5) can be solved by allocating each tone to the user which can provide the maximum *shadow rate* $\log\left(1 + s_k^n/\sigma_k^n\right) - \lambda_k s_k^n$ on that tone.

Finding an Approximate FDMA Solution

- Maximum shadow rate for user k at tone n is given by

$$\max_{0 \le s_k^n \le P_k} \left(\log\left(1 + \frac{s_k^n}{\sigma_k^n}\right) - \lambda_k s_k^n \right) = \begin{cases} \lambda_k \sigma_k^n - \log(\lambda_k \sigma_k^n) - 1, & 0 < \lambda_k \sigma_k^n \le 1, \\ 0, & \lambda_k \sigma_k^n > 1 \\ \infty, & \lambda_k < 0, \end{cases}$$

where the optimal power level is

$$s_k^n = \mathcal{P}_k(\lambda_k^{-1} - \sigma_k^n). \tag{6}$$

- Thus, the dual function (5) can be written analytically as

$$d(\boldsymbol{\lambda}) = \sum_{k=1}^{K} \lambda_k P_k + \sum_{n=1}^{N} \max_{k:\lambda_k \sigma_k^n \le 1} \left(\lambda_k \sigma_k^n - \log(\lambda_k \sigma_k^n) - 1\right). \tag{7}$$

Finding an Approximate FDMA Solution

- For each n, the maximum in (7) is attained at the user k for which $\lambda_k \sigma_k^n$ is smallest.
- Then a subgradient of $d(\boldsymbol{\lambda})$ is given by

$$\nabla d(\boldsymbol{\lambda}) = \left(P_1 - \sum_{n \in \mathcal{N}_1(\boldsymbol{\lambda})} s_1^n,\ P_2 - \sum_{n \in \mathcal{N}_2(\boldsymbol{\lambda})} s_2^n,\ \ldots,\ P_K - \sum_{n \in \mathcal{N}_K(\boldsymbol{\lambda})} s_K^n \right)^T$$

  where we denote the set of tones assigned to user k by $\mathcal{N}_k(\boldsymbol{\lambda})$. Notice that the components of subgradient $\nabla d(\boldsymbol{\lambda})$ correspond to each user's unused power (or deficit power if negative).

- The dual minimization problem is given by

$$\begin{array}{ll} \text{minimize} & d(\boldsymbol{\lambda}) \\ \text{subject to} & \boldsymbol{\lambda} \geq 0 \end{array} \qquad (D_d^N)$$

  which is convex and solvable in polynomial time (e.g., using ellipsoid algorithm).

Duality Gap

- Recall the primal sum-rate maximization problem is NP-hard, implying there is a **positive duality gap.**
- **Dual optimality:** $\mathbf{0} \in \partial d(\boldsymbol{\lambda})$.

  Theorem 5 Let $\boldsymbol{\lambda}^* \geq 0$ and $\mathbf{s}^* \geq 0$ be the limit points generated by the dual decomposition algorithm. If there holds

$$\frac{1}{N}\sum_{n=1}^{N} s_k^n \leq P_k, \quad \boldsymbol{\lambda}_k^* \left(\frac{1}{N}\sum_{n=1}^{N} s_k^n - P_k \right) = 0, \quad \forall\, k \in \mathcal{K},$$

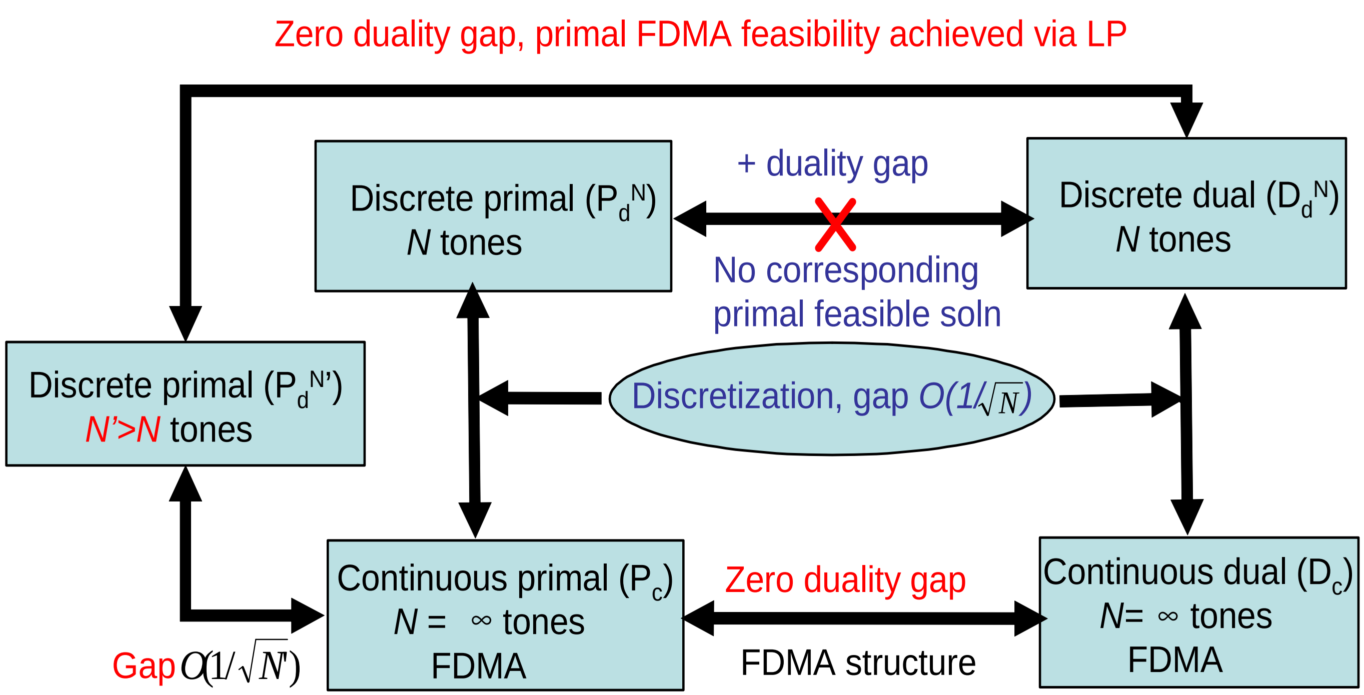
  then the duality gap is zero and $\mathbf{s}^*$ is a global optimal solution of the bandwidth allocation problem.

  In other words, primal feasibility ensures zero duality gap.

- This holds true if $\partial d(\boldsymbol{\lambda})$ is singleton.

Constructing an Approximate Primal Optimal Solution

- When $\partial d(\boldsymbol{\lambda})$ is **not** singleton, primal feasibility cannot be attained and there is a positive duality gap.
- However, we can further "split the tones" and construct a primal feasible solution for a more refined discretized primal problem with zero duality gap.



Approximation Quality

Main Consequences (L.-Zhang, (2007)):

- The nonconvex continuous optimal FDMA spectrum allocation problem and its dual are equivalent. The duality gap is zero.
- For each $\epsilon > 0$, we can find an ϵ-optimal solution in Poly(K, ϵ) time.

$$
\begin{array}{ll}
\max & H_1(u_1, \cdots, u_K) \\
\text{s.t.} & u_1 = \int_\Omega R_1(s_1(f), \ldots, s_K(f))df \\
& \quad \vdots \\
& u_K = \int_\Omega R_K(s_1(f), \ldots, s_K(f))df \\
& s_k(f)s_l(f) = 0,\ s_k(f) \geq 0,\ \int_\Omega s_k(f)df \leq P_k,\ \forall\ k,
\end{array}
\qquad (P_c)
$$

$$\Updownarrow$$

$$
\min_{\boldsymbol{\lambda}\geq 0} \max_{\substack{s_k(f)\geq 0 \\ s_k(f)s_l(f)=0,}} \sum_{k=1}^{K} \int_\Omega \left(\left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j\neq k} \alpha_{kj} s_j(f)} \right) - \lambda_k s_k(f) \right) df - \lambda_k P_k \qquad (D_c)
$$

Key Step

- For a Lebesgue integrable vector function $R(s(f), f)$, we have in general

$$\frac{1}{N}\sum_{n=1}^{N} R(s(n/N), n/N) \not\to \int_{\Omega} R(s(f), f)df, \quad N \to \infty.$$

- However, we show there exists some piecewise constant function s^n (not necessarily equal to $s(n/N)$), such that

$$\left\| \frac{1}{N}\sum_{n=1}^{N} R(s^n, n/N) - \int_{\Omega} R(s(f), f)df \right\| = O\left(\frac{1}{\sqrt{N}}\right).$$

- This implies that the gap between (P_c) and (P_d^N) is $O(1/\sqrt{N})$.
- $O(1/\sqrt{N})$ can be improved to $O(1/N)$ for frequency flat case.

Key Observation

$$
\begin{aligned}
\text{max} \quad & H(u_1, \cdots, u_K) \\
\text{s.t.} \quad & u_1 = \frac{1}{N} \sum_{n=1}^{N} R_1(s_1^n, \ldots, s_K^n, n/N) \\
& \vdots \\
& u_K = \frac{1}{N} \sum_{n=1}^{N} R_K(s_1^n, \ldots, s_K^n, n/N) \\
& \frac{1}{N} \sum_{n=1}^{N} s_k^n \leq P_k,\ s_k^n \geq 0,\ k = 1, \ldots, K,
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
\text{max} \quad & H(u_1, \cdots, u_K) \\
\text{s.t.} \quad & u_1 = \int_{\Omega} R_1(s_1(f), \ldots, s_K(f), f)df \\
& \vdots \\
& u_K = \int_{\Omega} R_K(s_1(f), \ldots, s_K(f), f)df \\
& s_k(f) \geq 0,\ \int_{\Omega} s_k(f)df \leq P_k,\ k = 1, \ldots, K,
\end{aligned}
$$

Extensions

- It is possible to show that, without FDMA constraint,

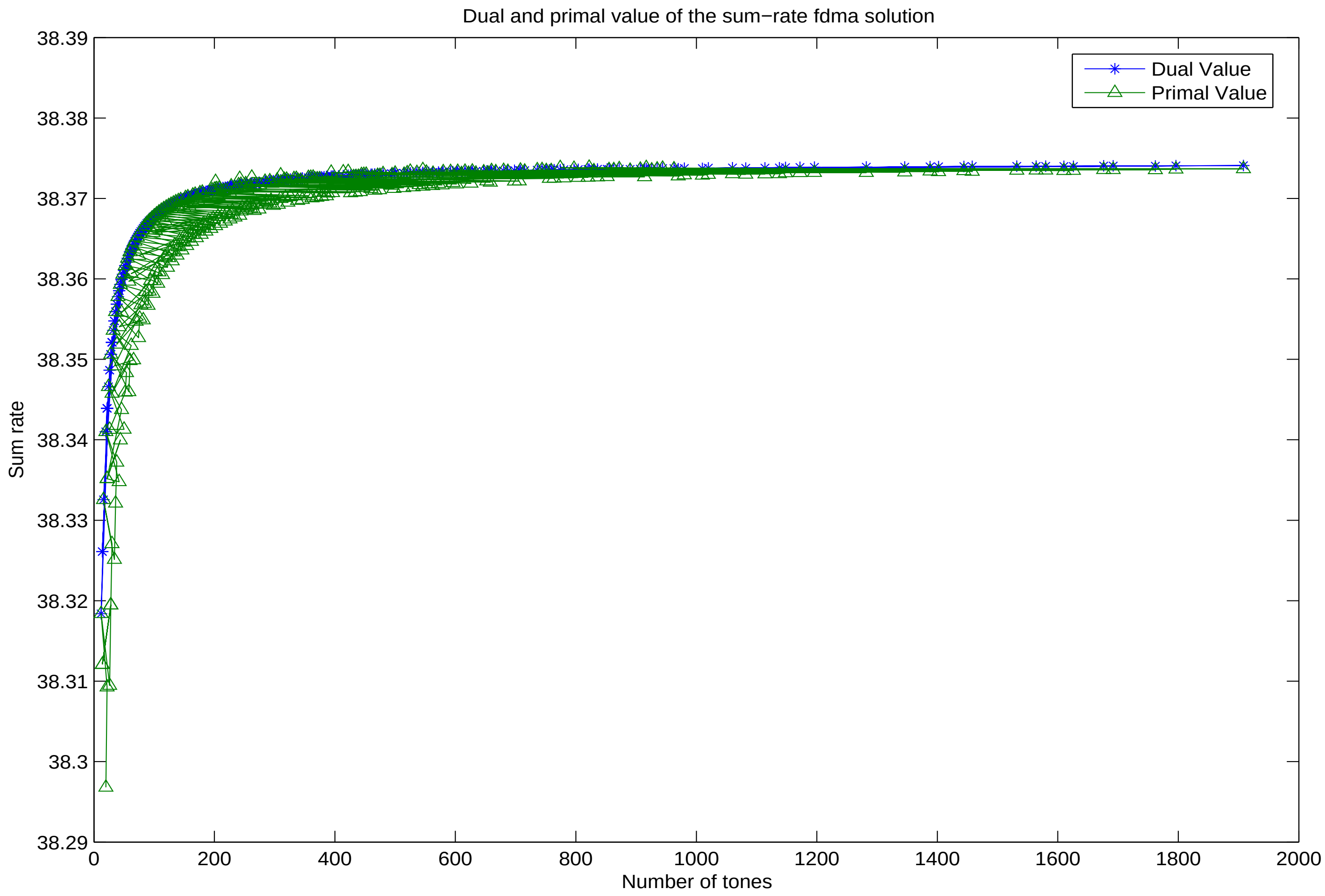
$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{k=1}^{K} \int_0^1 \left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j \neq k} \alpha_{kj} s_j(f)}\right) df \\ \text{subject to} & \displaystyle\int_0^1 s_k(f) df \leq P_k, \quad s_k(f) \geq 0, \quad \forall k \in \mathcal{K}. \end{array}$$

has the same optimal value as its dual

$$\min_{\boldsymbol{\lambda} \geq 0} \max_{s_k(f) \geq 0} \sum_{k=1}^{K} \int_0^1 \left(\left(1 + \frac{s_k(f)}{\sigma_k(f) + \sum_{j \neq k} \alpha_{kj} s_j(f)}\right) - \lambda_k s_k(f) \right) df - \lambda_k P_k$$

- **Asymptotic strong duality:** This suggests, for the finite tone case, the **duality gap decreases to zero**.

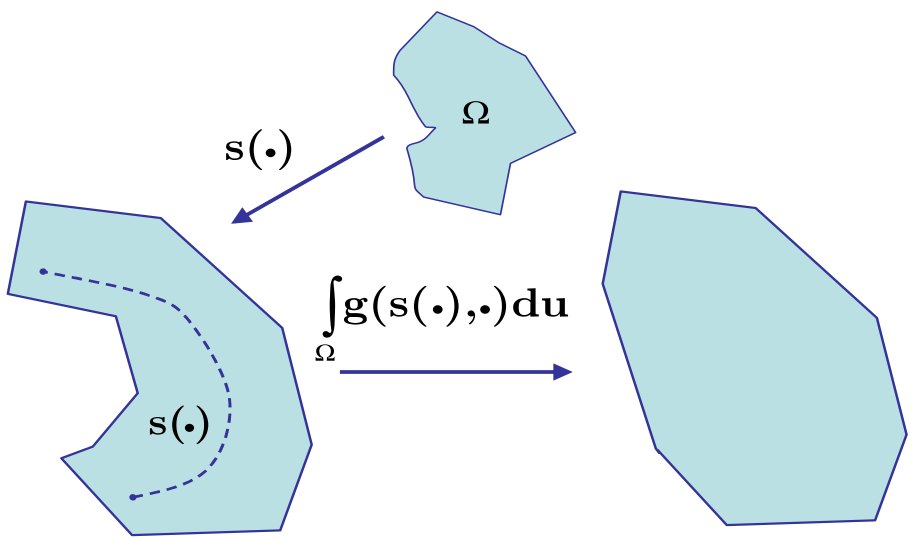
Duality Gap $\rightarrow 0$



Lyapunov Theorem

Let u be a non-atomic measure on a Borel field $\mathcal{B}$ generated from subsets of a space Ω. Let $g_i(s(\cdot),\cdot) : \Re^2 \longrightarrow \Re_+$ be compatible with $\mathcal{B}$-measurable function (i.e., if $s(\cdot)$ is $\mathcal{B}$-measurable then $g_i(s(\cdot),\cdot)$ is $\mathcal{B}$-measurable), $i = 1, \ldots, m$. Then,

$$\left\{ \begin{pmatrix} \int_\Omega g_1(s(\cdot),\cdot)du \\ \vdots \\ \int_\Omega g_m(s(\cdot),\cdot)du \end{pmatrix} \middle| \; x \text{ is } \mathcal{B}\text{-measurable} \right\} \quad \text{is a convex set.}$$



Implications of Lyapunov Theorem

$$
\begin{aligned}
v(\mathbf{P}) = \quad & \text{max} \quad \tfrac{1}{K}(u_1 + \cdots + u_K) \\
& \text{s.t.} \quad u_1 = \int_\Omega R_1(s_1(f), \ldots, s_K(f))df \\
& \qquad\quad \vdots \\
& \qquad\quad u_K = \int_\Omega R_K(s_1(f), \ldots, s_K(f))df \\
& \qquad\quad s_k(f) \geq 0, \int_\Omega s_k(f)df \leq P_k, \; k = 1, \ldots, K,
\end{aligned}
$$

- $v(\mathbf{P})$ is a concave function of $\mathbf{P} = (P_1, \ldots, P_K)$.
- This implies zero duality gap, re-establishing the result of **Yu, Lui and Cendrillon (2006)**.

Implications of Lyapunov Theorem

$$
\begin{aligned}
v(\mathbf{P}) = \quad & \text{max} \quad H(u_1, \ldots, u_K) \\
& \text{s.t.} \quad u_1 = \int_\Omega R_1(s_1(f), \ldots, s_K(f))df \\
& \qquad \vdots \\
& \qquad u_K = \int_\Omega R_K(s_1(f), \ldots, s_K(f))df \\
& \qquad s_k(f) \geq 0, \int_\Omega s_k(f)df \leq P_k, \; k = 1, \ldots, K,
\end{aligned}
$$

- $v(\mathbf{P})$ is a concave function of $\mathbf{P} = (P_1, \ldots, P_K)$ if
  - $H(u_1, \ldots, u_K)$ is jointly concave.
  - $H(u_1, \ldots, u_K)$ is monotonically increasing wrt each argument.
- Under these assumptions, the duality gap is zero **(L.-Zhang (2007))**.
- H_1, H_2, H_3 and H_4 all satisfy the above two assumptions.

Further Implication of Lyapunov Theorem

Given a powerful adversary (**user 0**) in the system, let us maximize the worst-case performance

$$\max_{\substack{s_k(f)\geq 0,\\ \int_\Omega s_k(f)df\leq P_k}} \min_{\substack{s_0(f)\geq 0,\\ \int_\Omega s_0(f)df\leq P_0}} H(u_1, u_2, ..., u_K)$$

where $H = H_1, H_2, H_3$ or H_4, and

$$u_k = \int_\Omega \log\left(1 + \frac{s_k(f)}{\sum_{l\neq k}\alpha_{kl}s_l(f) + \sigma_k(f)}\right) df, \quad k = 1, 2, ..., K.$$

- H is convex in $s_0(f)$. However, H is **not concave** in $s_k(f)$, $k = 1, 2, ..., K$.
- Luckily, Lyapunov Theorem ensures a **hidden concavity**, which together with the convexity and compactness of the feasible sets, implies

$$\max_{\substack{s_k(f)\geq 0,\\ \int_\Omega s_k(f)df\leq P_k}} \min_{\substack{s_0(f)\geq 0,\\ \int_\Omega s_0(f)df\leq P_0}} H(u_1, ..., u_K) = \min_{\substack{s_0(f)\geq 0,\\ \int_\Omega s_0(f)df\leq P_0}} \max_{\substack{s_k(f)\geq 0,\\ \int_\Omega s_k(f)df\leq P_k}} H(u_1, ..., u_K)$$

Thank You