A Semidefinite Relaxation Scheme for Multivariate Quartic Polynomial Optimization With Quadratic Constraints

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2009 Lunteran MOR Conference, The Netherland

Talk Outline

- Quartic optimization: motivation
- What is SDP/SOS relaxation?
- Approximation bounds

Quartic Optimization

Maximization form

$$\begin{array}{ll} \text{maximize} & f(x) = \sum_{1\le i,j,k,\ell\le n} a_{ijk\ell} x_i x_j x_k x_\ell \\ \text{subject to} & x^{\mathrm{T}} A_i x \le 1,\ i = 1, \ldots, m, \end{array} \tag{1}$$

or the minimization form

$$\begin{array}{ll} \text{minimize} & f(x) = \sum_{1\le i,j,k,\ell\le n} a_{ijk\ell} x_i x_j x_k x_\ell \\ \text{subject to} & x^{\mathrm{T}} A_i x \ge 1,\ i = 1, \ldots, m, \end{array} \tag{2}$$

where $A_i \in \mathbb{R}^{n\times(n+1)/2}$: positive semidefinite, $i = 1, \ldots, m$.

- $f_{\max}$ and $f_{\min}$ denote the optimal values of (1) and (2) respectively.
- To ensure $f_{\min}$ and $f_{\max}$ exist, we assume throughout that $\sum_i^m A_i \succ 0$.

Quartic Optimization: Motivation

Quartic optimization problems arise in various engineering applications

- **Sensor localization:** let $\mathcal{A}$ and $\mathcal{S}$ denote the anchor nodes and sensor nodes respectively

$$\text{minimize} \sum_{i,j\in\mathcal{S}} \left(\|\mathbf{x}_i - \mathbf{x}_j\|^2 - d_{ij}^2\right)^2 + \sum_{i\in\mathcal{S},j\in\mathcal{A}} \left(\|\mathbf{x}_i - \mathbf{s}_j\|^2 - d_{ij}^2\right)^2$$

$\Rightarrow$ Quartic minimization (Known: **NP-hard; constant factor approximation is also hard**)

- **Digital communication:** blind channel equalization of constant modulus signals

$$\mathbf{x}(t) = \mathbf{H}\mathbf{s}(t) + \mathbf{n}(t)$$

where $\mathbf{H}$ is unknown, the components of $\mathbf{s}(t)$ are constant ($|s_i(t)| = 1,\ \forall i$) A channel equalizer $\mathbf{g}$ can be found by

$$\text{minimize} \sum_t (|\mathbf{g}^T\mathbf{x}(t)|^2 - 1)^2, \quad \Rightarrow \text{ Quartic minimization}$$

- **Signal processing:** independent component analysis (ICA)

$$\mathbf{x} = \mathbf{H}\mathbf{s}, \quad \mathbf{H} \text{ full column rank, unknown}$$

⋆ $\mathbf{s}$ is independent, high 4-th Kurtosis, non-Gaussian sources;

$\mathbf{x}$: measurement, unknown linear mixture of $\mathbf{s}$

⋆ **Goal:** Find $\mathbf{G}$ such that $\mathbf{G}\mathbf{x}$ is a permutation of $\mathbf{s}$

⋆ $\mathbf{G}\mathbf{x}$ is separate, independent $\Leftrightarrow$ the 4-th order Kurtosis of $\mathbf{G}\mathbf{x}$ is high

$\Rightarrow$ maximize the 4-th order Kurtosis of $\mathbf{G}\mathbf{x}$ (fourth order polynomial of $\mathbf{G}$) subject to ball constraint (power constraint)

$\Rightarrow$ ball-constrained homogeneous quartic maximization

Quartic Optimization: Complexity

- The quartic polynomial optimization problems (1)–(2) are **nonconvex**, **NP-hard**

$\Rightarrow$ consider polynomial time relaxation procedures that can deliver provably high quality approximate solutions (for special subclasses of quartic optimization problems).

Approximation Ratio

- $\hat{x}$ is a c-**factor approximation** of quartic minimization problem (2) if

$$f_{\min} \leq f(\hat{x}) \leq c f_{\min}$$

with c independent of problem data. (Therefore, $f_{\min} = 0 \Leftrightarrow f(\hat{x}) = 0$.)

- **Weaker notion:** $(1-\epsilon)$-approximation of quartic minimization problem (2) if

$$f(\hat{x}) - f_{\min} \leq (1-\epsilon)(f_{\max} - f_{\min})$$

with ϵ independent of problem data.

- Similarly for quartic maximization problem.

SDP/SOS Relaxation

- the sum-of-squares (SOS) technique
  - ⋆ represent each nonnegative polynomial as a sum of squares of some other polynomials a given degree
  - ⋆ Alternatively, use matrix lifting

$$X := \begin{pmatrix} 1 \\ x_i \\ x_i x_j \\ x_i x_j x_k \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & x_i & x_i x_j & x_i x_j x_k & \cdots \end{pmatrix}$$

  - ⋆ Under the lifting, each polynomial inequality is relaxed to a **convex**, **linear** matrix inequality
- approximate (arbitrarily well) by a hierarchy of SDPs with increasing size
- **difficulty:** the size of the resulting SDPs in the hierarchy grows exponentially fast

SDP/SOS Relaxation

- The most effective use of SDP relaxation so far has been for the quadratic optimization problems whereby only the first level relaxation in the SOS hierarchy is used.
  - ⋆ **difficulty:** cannot provide arbitrarily tight approximation in general
  - ⋆ does lead to **provably high quality approximate solution** for certain type of quadratic optimization problems (e.g., Max-Cut)
- **Question:** find a **provably good** *first level SOS approximation* of some quartic optimization problems (1)–(2)?

SDP Relaxation of Nonconvex Quadratic Optimization Problem

- focus here on a specific class of problems: general QCQPs
- vast range of applications...

the generic QCQP can be written:

$$\begin{array}{ll} \text{minimize} & x^{\mathrm{T}} A_0 x + r_0 \\ \text{subject to} & x^{\mathrm{T}} A_i x + r_i \leq 0, \quad i = 1, \ldots, m \end{array}$$

- if all A_i are p.s.d., convex problem,
- here, we suppose at least one A_i not p.s.d.

Convex Relaxation

Using a fundamental observation:

$$\boxed{X := xx^{\mathrm{T}} \quad \Leftrightarrow \quad X_{ij} = x_i x_j \quad \Leftrightarrow \quad X \succeq 0,\ \operatorname{rank}(X) = 1,}$$

and noting $x^{\mathrm{T}} A_i x = \operatorname{Tr}(XA_i)$, the original QCQP:

$$\begin{array}{ll} \text{minimize} & f(x) = x^{\mathrm{T}} A_0 x + r_0 \\ \text{subject to} & x^{\mathrm{T}} A_i x + r_i \leq 0, \quad i = 1, \ldots, m \end{array}$$

can be rewritten:

$$\begin{array}{ll} \text{minimize} & g(X) = \operatorname{Tr}(XA_0) + r_0 \\ \text{subject to} & \operatorname{Tr}(XA_i) + r_i \leq 0, \quad i = 1, \ldots, m \\ & X \succeq 0,\ \operatorname{rank}(X) = 1 \end{array}$$

the only nonconvex constraint is now $\operatorname{rank}(X) = 1$...

Convex Relaxation: Semidefinite Relaxation

- we can directly relax this last constraint, i.e. drop the nonconvex $\mathrm{rank}(X) = 1$ to keep only $X \succeq 0$
- the resulting program gives a lower bound on the optimal value

$$\begin{array}{ll} \text{minimize} & g(X) = Tr(XA_0) + r_0 \\ \text{subject to} & \mathrm{Tr}\,(XA_i) + r_i \le 0, \quad i = 1, \ldots, m \\ & X \succeq 0 \end{array} \quad \Rightarrow \quad \textbf{SDP}$$

How to Generate a Feasible Solution?

Let X^* be the optimal solution of

- pick x as a Gaussian variable with $x \sim \mathcal{N}(0, X^*)$
- Since $\mathrm{Tr}\,(X^* A_i) + r_i = \mathrm{E}[x^T A_i x + r_i]$, x will solve the QCQP "on average" over this distribution

Generate a Feasible Solution

In other words, SDP is equivalent to

$$\begin{array}{ll} \text{minimize} & \mathrm{E}[x^T A_0 x + r_0] \\ \text{subject to} & \mathrm{E}[x^T A_i x + r_i] \leq 0, \quad i = 1, \ldots, m \end{array}$$

a good feasible point can then be obtained by sampling enough x. . .

Two observations:

- SDP finds the convariance matrix used in sampling
- The relaxed function $g(X)$ satisfies
  - ⋆ **Consistency:** $g(X) = f(x)$ when $X = xx^T$
  - ⋆ **Compatibility:** $g(X) = E(f(x))$ when $x \sim N(0, X)$

Key question:

- how good is the approximate solution x?
- can we bound $f(x)/f^*$ by a constant?

Summary of Existing Results

Assume

- $\mathbf{A}_i, \bar{\mathbf{A}}_i \succeq \mathbf{0}$, $i = 0, 1, 2, \ldots, m$
- $\mathbf{B}_j \not\succeq \mathbf{0}$ indefinite, $j = 0, 1, 2, \ldots, d$

	$\mathbb{R},\ d = 0$	$\mathbb{R},\ d = 1$ or $\mathbb{C},\ d = 0, 1$	$\mathbb{R}$ or $\mathbb{C},\ d \geq 2$
$\min \mathbf{w}^H \mathbf{A}_0 \mathbf{w}$ s.t. $\mathbf{w}^H \mathbf{A}_i \mathbf{w} \geq 1,\ \mathbf{w}^H \mathbf{B}_j \mathbf{w} \geq 1$	$\Theta(m^2)$	$\Theta(m)$	∞
$\max \mathbf{w}^H \mathbf{B}_0 \mathbf{w}$ s.t. $\mathbf{w}^H \mathbf{A}_i \mathbf{w} \leq 1,\ \mathbf{w}^H \mathbf{B}_j \mathbf{w} \leq 1$	$\Theta(\log^{-1} m)$	$\Theta(\log^{-1} m)$	∞
$\max \min_{1 \leq i \leq m} \frac{\mathbf{w}^H \mathbf{A}_i \mathbf{w}}{\mathbf{w}^H \bar{\mathbf{A}}_i \mathbf{w} + \sigma^2}$ s.t. $\\|\mathbf{w}\\|^2 \leq P$	$\Theta(m^2)$	$\Theta(m)$	N.A.

Blue: **NRT'99**, Red: **LSTZ'06, CLC'07, HLNZ'07**

SDP Relaxation for Quartic Optimization

Consider the first level SOS hierarchy so that

$$x_i x_j \mapsto X_{ij}, \quad X \succeq 0.$$

Under this mapping, each quartic term is mapped, **non-uniquely**, to a quadratic term, e.g.,

$$x_1 x_2 x_3 x_4 \mapsto \begin{cases} X_{12} X_{34} \\ X_{13} X_{24} \\ X_{14} X_{23} \end{cases}$$

- Which one should we use?
- Should we choose a convex combination of the three choices?
- Does it matter?

It Matters!

Consider the following quartic optimization problem in $\mathbb{R}^4$:

$$\begin{array}{ll} \text{minimize} & f(x) = (x_1 x_2)^2 \\ \text{subject to} & x_1^2 \geq 1, \quad x_2^2 \geq 1. \end{array} \tag{3}$$

Under the matrix lifting transformation $X = xx^{\mathrm{T}}$, (3) is relaxed to

$$\begin{array}{ll} \text{minimize} & g(X) = X_{12}^2 \\ \text{subject to} & X_{11} \geq 1, \quad X_{22} \geq 1, \quad X \succeq 0. \end{array}$$

- It can be checked
  - ⋆ $f_{\min} = 1$
  - ⋆ $g_{\min} = g(I) = 0$ since $X = I$ is a feasible solution.
- This shows that the approximation ratio is unbounded!

$$\frac{f_{\min}}{g_{\min}} = \infty. \tag{4}$$

It Matters!

- On the other hand, consider the symmetric mapping

$$x_i x_j x_\ell x_m \mapsto \frac{1}{3}(X_{ij}X_{\ell m} + X_{i\ell}X_{jm} + X_{im}X_{j\ell}).$$

Under this mapping, the quartic objective function

$$f(x) = x_1^2 x_2^2$$

is relaxed to

$$h(x) = \frac{1}{3}(X_{11}X_{22} + 2X_{12}^2).$$

- Let $h_{\min} :=$ minimize $h(X)$ subject to $X_{11} \geq 1, \quad X_{22} \geq 1, \quad X \succeq 0.$
- Notice that $h_{\min} = h(I) = \frac{1}{3}$, implying

$$\frac{f_{\min}}{h_{\min}} = \frac{1}{\frac{1}{3}} = 3,$$

which is indeed finite.

SDP Relaxation for Quartic Optimization

- Suppose $g(X)$ is a **quadratic function** to be used as a relaxation of the quartic function $f(x)$. Then $g(X)$ should satisfy

consistency property: $g(X) = f(x) = \sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}x_i x_j x_k x_\ell,$ whenever $X = xx^{\mathrm{T}}$.

- There are many quadratic functions $g(X)$ satisfying this property, e.g.

$$x_i x_j x_k x_\ell \mapsto \begin{cases} X_{ij}X_{k\ell} \\ X_{ik}X_{j\ell} \\ X_{i\ell}X_{jk} \end{cases}$$

- Which one should we pick?

  Goal: pick one that ensures good approximation of quartic problem (1).

SDP Relaxation for Quartic Optimization

- Let $\hat{X} \succeq 0$ denote the optimal solution of the following **quadratic SDP** relaxation of (1):

$$\begin{array}{ll} \text{maximize} & g(X) \\ \text{subject to} & \mathrm{Tr}(A_i X) \le 1, \quad i = 1, 2, ..., m, \ X \succeq 0. \end{array}$$

- To generate a feasible solution for the original problem (1), we draw random samples x from the Gaussian distribution $N(0, \hat{X})$.
- To ensure approximate quality, we wish to maximize $\mathsf{E}[f(x)]$.
- **Key observation:** $\mathsf{E}[f(x)]$ is a **quadratic** function of X. This motivates the following

compatibility property: $g(X) = c\,\mathsf{E}[f(x)],$ for some $c > 0$, where $X = \mathsf{E}(xx^{\mathrm{T}})$.

- **Question:** Is there a positive constant c satisfying both the **compatibility** and the **consistency** conditions?

SDP Relaxation for Quartic Optimization

- **Fact:** Suppose $x \in \mathbb{R}^n$ is a random vector drawn a Gaussian distribution $N(0, X)$ where $X \succeq 0$. Then for any $1 \le i \ne j \ne k \ne \ell \le n$, we have

$$\begin{aligned}
\mathsf{E}[x_i^4] &= 3X_{ii}^2 \\
\mathsf{E}[x_i^3 x_j] &= 3X_{ii}X_{jj} \\
\mathsf{E}[x_i^2 x_j^2] &= X_{ii}X_{jj} + 2X_{ij}^2 \\
\mathsf{E}[x_i^2 x_j x_k] &= X_{ii}X_{jk} + 2X_{ij}X_{ik} \\
\mathsf{E}[x_i x_j x_k x_\ell] &= X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}.
\end{aligned}$$

- Based on this fact, we propose to relax each quartic term symmetrically as

$$x_i x_j x_k x_\ell \;\mapsto\; \frac{1}{3}\left(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}\right), \quad \forall\, 1 \le i, j, \ell, m \le n.$$

- It can be easily checked that the **consistency property** and the **compatibility property** is satisfied with $c = 1/3$!

- Under the above symmetric mapping, the quartic polynomial maximization problem (1) is relaxed to

$$\begin{array}{ll} \text{maximize} & g(X) = \frac{1}{3} \sum_{1 \le i,j,k,\ell \le n} a_{ijk\ell} \left(X_{ij} X_{k\ell} + X_{ik} X_{j\ell} + X_{i\ell} X_{jk} \right) \\ \text{subject to} & \mathrm{Tr}(A_i X) \le 1, \; i = 1, \ldots, m \\ & X \succeq 0, \end{array} \tag{5}$$

and the quartic polynomial minimization problem (2) can be relaxed as

$$\begin{array}{ll} \text{minimize} & g(X) = \frac{1}{3} \sum_{1 \le i,j,k,\ell \le n} a_{ijk\ell} \left(X_{ij} X_{k\ell} + X_{ik} X_{j\ell} + X_{i\ell} X_{jk} \right) \\ \text{subject to} & \mathrm{Tr}(A_i X) \ge 1, \; i = 1, \ldots, m \\ & X \succeq 0. \end{array} \tag{6}$$

- **Property:**

$$\mathsf{E}(f(x)) = \mathsf{E}\left(\sum_{1 \le i,j,k,\ell \le n} a_{ijk\ell} x_i x_j x_k x_\ell \right) = 3g(X)$$

- Are these good approximations?

Several Issues

- **Bad news:** the relaxed quadratic SDPs (5)–(6) are **NP-hard**!
- **Good news:** Let $\hat{X}$ be an α-approximate solution of (5). Suppose we randomly generate a sample x from Gaussian distribution $N(0, \hat{X})$. Let $\hat{x} = x / \max_{1 \le i \le m} x^{\mathrm{T}} A_i x$. Then
  - ⋆ $\hat{x}$ is a feasible solution of (1)
  - ⋆ the probability that
  $$f_{\max} \ge f(\hat{x}) \ge \frac{3\alpha}{4\left(\ln \frac{2mn}{\theta}\right)^2} f_{\max}$$
  is at least $\theta/2$ with $\theta := 1.443 \times 10^{-7}$, where $f_{\max}$ denotes the optimal value of (1).
- In other words, good approximation of the relaxed quadratic SDPs (5)–(6) leads to good approximation of (1)–(2).

Note: A feasible $\hat{X} \succeq 0$ is said to be an α-approximate solution of (5) if $g(\hat{X})/g_{\max} \ge \alpha$.

Ideas in the Proof: feasibility

- **Observation:** the relaxed quadratic SDP (5) can be viewed as picking a covariance matrix $X \succeq 0$ for $x \sim N(0, X)$ according to

$$\begin{array}{ll} \text{maximize} & \mathsf{E}(f(x)) \\ \text{subject to} & \mathsf{E}(x^{\mathrm{T}} A_i x) \le 1, \; i = 1, \ldots, m \end{array}$$

- Suppose $\hat{X} \succeq 0$ is an α-approximate solution: $g(\hat{X}) \ge \alpha g_{\max}$.
- For random samples $x \sim N(0, \hat{X})$, the constraint $x^{\mathrm{T}} A_i x \le 1$ is satisfied in expectation.
- Since $A_i \succeq 0$, it can be shown that $\mathsf{P}(x^{\mathrm{T}} A_i x > \gamma^2 \mathsf{E}(x^{\mathrm{T}} A_i x)) = O(n\gamma^{-1} e^{-\gamma^2/2})$, for all $\gamma > 0$. So the probability of getting a x such that

$$\mathsf{P}(x^{\mathrm{T}} A_i x \le \gamma^2 \mathsf{E}(x^{\mathrm{T}} A_i x) \le \gamma^2) = 1 - O(mn\gamma^{-1} e^{-\gamma^2/2}), \qquad \forall\, i = 1, 2, \ldots, m.$$

- Choosing $\gamma = O(\ln nm) \Rightarrow x/O(\ln(nm))$ is feasible with a positive probability.

Ideas in the Proof: objective value

- **Observation:**

$$\mathsf{E}(f(x)) = 3g(\hat{X}) \geq 3\alpha g_{\max} \geq 3\alpha f_{\max}$$

where

  - ⋆ the first step is due to the definition of g (**compatibility property**)
  - ⋆ the second step is due to the definition of α
  - ⋆ the last step is due to $g(xx^{\mathrm{T}}) = f(x)$ (**consistency property**)

- **Question:** Is there a positive (and independent of data) probability of getting a x from $N(0, \hat{X})$ such that

$$f(x) \geq \mathsf{E}(f(x))?$$

- The answer is **YES**!

A Key Step in the Proof

- **Fact:** Suppose $X \succeq 0$ and let $x \sim N(0, X)$. Suppose $f(x)$ be any homogeneous quartic polynomial in $\mathbb{R}^n$. Then

$$\mathsf{P}\left\{f(x) \geq \mathsf{E}[f(x)]\right\} \geq 1.443 \times 10^{-7}$$

  and

$$\mathsf{P}\left\{f(x) \leq \mathsf{E}[f(x)]\right\} \geq 1.443 \times 10^{-7}.$$

- The proof (brute force) relies on the following bound

$$\mathsf{E}\left[(f(x) - \mathsf{E}[f(x)])^4\right] \leq 1732500\, \mathsf{Var}^2(f(x))$$

  and the following fact (**HLNZ'07**)

  ⋆ Let ξ be a random variable with bounded fourth order moment. Suppose

$$\mathsf{E}[(\xi - \mathsf{E}(\xi))^4] \leq \tau\, \mathsf{Var}^2(\xi), \quad \text{for some } \tau > 0.$$

  Then $\mathsf{P}\left\{\xi \geq \mathsf{E}(\xi)\right\} \geq 0.25\tau^{-1}$ and $\mathsf{P}\left\{\xi \leq \mathsf{E}(\xi)\right\} \geq 0.25\tau^{-1}$.

SDP Approximation Ratio for Quartic Minimization

- Consider the following SDP relaxation of (2)

$$\begin{array}{rl} g_{\min} := \text{minimize} & g(X) = \dfrac{1}{3} \displaystyle\sum_{1 \le i,j,k,\ell \le n} a_{ijk\ell} \left(X_{ij} X_{k\ell} + X_{ik} X_{j\ell} + X_{i\ell} X_{jk} \right) \\ \text{subject to} & \mathrm{Tr}(A_i X) \ge 1,\ i = 1, \ldots, m,\ X \succeq 0. \end{array} \quad (7)$$

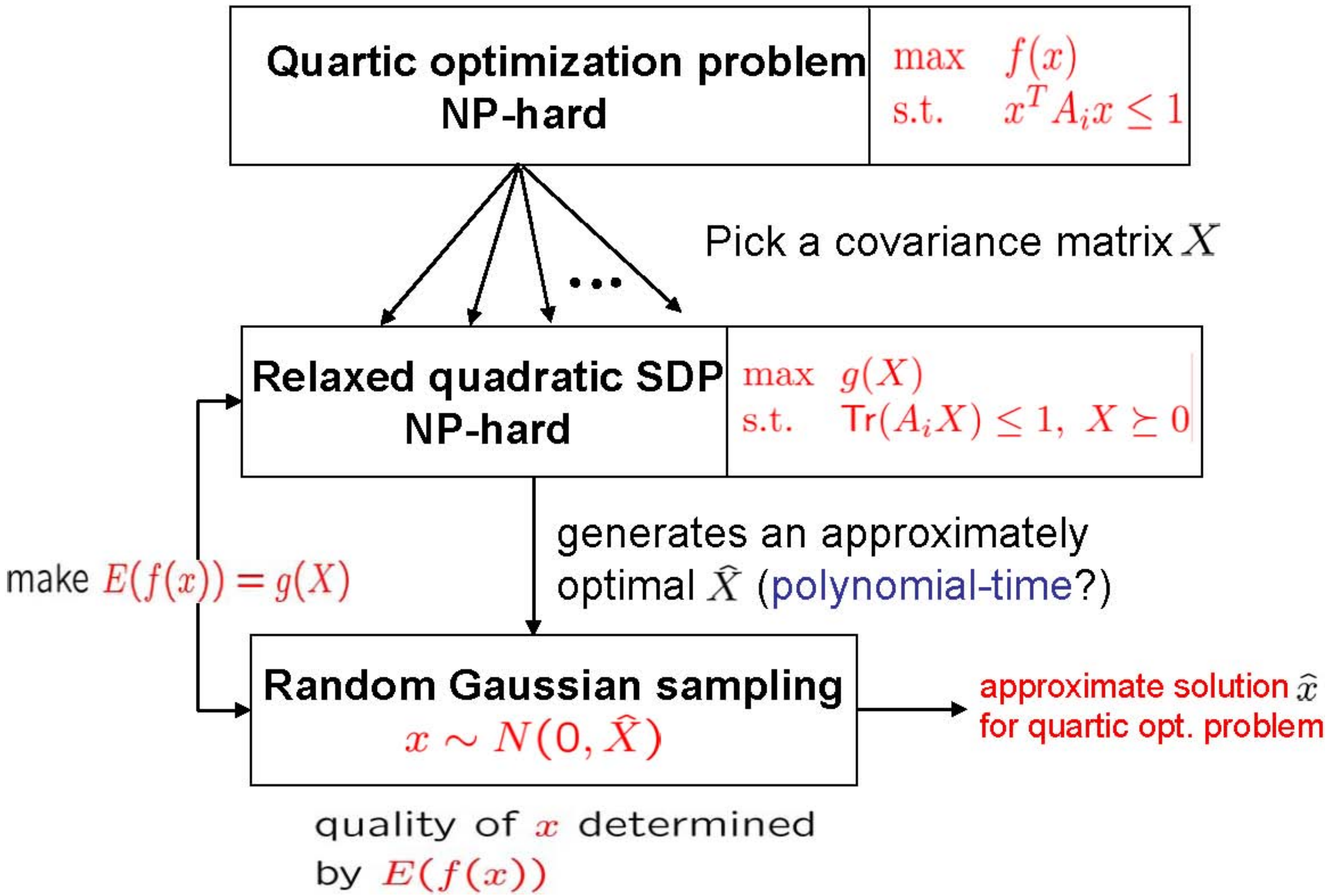
  Let $\hat{X}$ be an β-approximate solution of (7).

- Suppose we randomly generate a sample x from Gaussian distribution $N(0, \hat{X})$. Let $\hat{x} = x / \min_{1 \le i \le m} x^{\mathrm{T}} A_i x$. Then
  - ⋆ $\hat{x}$ is a feasible solution of (2)
  - ⋆ the probability that

$$f_{\min} \le f(\hat{x}) \le 12\beta \max \left\{ \frac{m^2}{\theta^2}, \frac{m(n-1)}{\theta(\pi - 2)} \right\} f_{\min}$$

    is at least $\theta/2$ with $\theta := 1.443 \times 10^{-7}$, where $f_{\min}$ denotes the optimal value of (2).

Where do we stand?



We reduce NP-hard quartic optimization problem to a quadratic SDP problem.

How to Approximate the Relaxed Quadratic SDP?

- Consider the quartic maximization problem over a ball:

$$\begin{array}{ll} \text{maximize} & \sum_{1\le i,j,k,\ell\le n} a_{ijk\ell} x_i x_j x_k x_\ell \\ \text{subject to} & \|x\|^2 \le 1. \end{array}$$

- The relaxed SDP problem is

$$\begin{array}{ll} \text{maximize} & \frac{1}{3} \sum_{1\le i,j,k,\ell\le n} a_{ijk\ell} \left(X_{ij} X_{k\ell} + X_{ik} X_{j\ell} + X_{i\ell} X_{jk} \right) \\ \text{subject to} & \mathrm{Tr}(X) \le 1 \\ & X \succeq 0. \end{array} \tag{8}$$

How to Approximate the Relaxed Quadratic SDP?

- We provide a polynomial time algorithm for the relaxed quadratic SDP problem to find an $1/n^2$ approximate solution
  - **Idea:** approximate (and replace) the SDP simplex constraint by a ball constraint:

$$\{X \in \mathcal{S}^{n\times n} \mid \sqrt{n-1}\,\|X\|_F \le \mathrm{Tr}(X)\} \subseteq \mathcal{S}_+^{n\times n} \subseteq \{X \in \mathcal{S}^{n\times n} \mid \|X\|_F \le \mathrm{Tr}(X)\}$$

  - Ball constrained (nonconvex) QP is solvable in polynomial time
  - If $g(I) \ge 0$, then the optimal solution of the ball constrained QP is a $1/n^2$-approximate solution of (8).
- Combined with an appropriate probabilistic rounding procedure, we can find a feasible $\hat{x}$ for the original quartic optimization problem (1) satisfying

$$\frac{f(\hat{x})}{f_{\max}} \ge \Omega\left(\frac{1}{(n\ln n)^2}\right)$$

for the quartic maximization problem (1), provided $A_1 \succ 0$ and $m = 1$.

Polynomial-Time Approximation of Quartic Minimization

- Consider the quartic maximization problem over a ball:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}x_i x_j x_k x_\ell \\ \text{subject to} & \|x\|^2 \ge 1. \end{array}$$

- The relaxed SDP problem is

$$\begin{array}{ll} \text{minimize} & \displaystyle\frac{1}{3}\sum_{1\le i,j,k,\ell\le n} a_{ijk\ell}\left(X_{ij}X_{k\ell} + X_{ik}X_{j\ell} + X_{i\ell}X_{jk}\right) \\ \text{subject to} & \mathrm{Tr}(X) \ge 1 \\ & X \succeq 0. \end{array} \quad (9)$$

How to Approximate the Relaxed Quadratic SDP?

- We provide a polynomial time algorithm for the relaxed quadratic SDP problem (9) to find an $1/n^2$ approximate solution
  - ⋆ **Idea:** approximate (and replace) the SDP simplex constraint by a ball constraint:

$$\{X \in \mathcal{S}^{n\times n} \mid \sqrt{n-1}\,\|X\|_F \le \mathrm{Tr}(X)\} \subseteq \mathcal{S}_+^{n\times n} \subseteq \{X \in \mathcal{S}^{n\times n} \mid \|X\|_F \le \mathrm{Tr}(X)\}$$

  - ⋆ Ball constrained (nonconvex) QP is solvable in polynomial time
  - ⋆ If $g(I) \ge 0$, then the optimal solution of the ball constrained QP is a $1/n^2$-approximate solution of (8).
- Combined with an appropriate probabilistic rounding procedure, we can find a feasible $\hat{x}$ for the original quartic optimization problem (2) satisfying

$$\frac{f(\hat{x}) - f_{\min}}{f_{\max} - f_{\min}} \le 1 - \Omega\left(\frac{1}{n^2 m \max\{m,n\}}\right)$$

for the quartic minimization problem (1), provided $A_1 \succ 0$ and $m = 1$.

Extensions

- **Fact:** if $x \in N(0, X)$, then

$$
\begin{aligned}
& \mathsf{E}[x_1 x_2 x_3 x_4 x_5 x_6] \\
= \quad & X_{12}X_{34}X_{56} + X_{12}X_{35}X_{46} + X_{12}X_{36}X_{45} + X_{13}X_{24}X_{56} + X_{13}X_{25}X_{46} \\
& + X_{13}X_{26}X_{45} + X_{14}X_{23}X_{56} + X_{14}X_{25}X_{36} + X_{14}X_{26}X_{35} + X_{15}X_{23}X_{46} \\
& + X_{15}X_{24}X_{36} + X_{15}X_{26}X_{34} + X_{16}X_{23}X_{45} + X_{16}X_{24}X_{35} + X_{16}X_{25}X_{34}.
\end{aligned}
$$

- If one wishes to solve the following $2d$-th order polynomial maximization problem

$$
\begin{array}{ll}
\text{maximize} & f_{2d}(x) = \displaystyle\sum_{1 \le i_1, \cdots, i_{2d} \le n} a_{i_1 \cdots i_{2d}} x_{i_1} \cdots x_{i_{2d}} \\
\text{subject to} & x^{\mathrm{T}} A_i x \le 1, \ i = 1, \ldots, m,
\end{array}
\tag{10}
$$

then the corresponding (non-convex) SDP relaxation problem is

$$\begin{array}{ll} \text{maximize} & p_d(X) \\ \text{subject to} & \mathrm{Tr}(A_i X) \leq 1,\ i = 1, \ldots, m \\ & X \succeq 0, \end{array} \tag{11}$$

where $p_d(X)$ is a d-th order polynomial in X.

- Suppose that (11) has an α-approximation solution, then (10) admits an overall $O\left(\frac{\alpha}{(\ln(mn))^d}\right)$ approximation solution.
- **Technical tool:** the hyper-contractive property of Gaussian distributions:
  - ⋆ Suppose that f is a multivariate polynomial with degree r. Let $x \in N(0, I)$. Suppose that $p > q > 0$. Then
  $$(\mathsf{E}|f(x)|^p)^{1/p} \leq \kappa_r c_{pq}^r (\mathsf{E}|f(x)|^q)^{1/q}$$
  where κ_r is a constant depending only on r, and $c_{pq} = \sqrt{(p-1)(q-1)}$.
  - ⋆ Proof was based on the **Paley-Zygmund inequality** and was non-constructive

Concluding Remarks

- An on-going research
- Provided a SDP relaxation scheme for quartic optimization, allowing approximation quality to be data-independent
- Effectively reduced the quartic optimization problem to quadratic SDP problem
- Many issues remaining: efficient algorithms to approximate nonconvex quadratic SDP over simplex? over box? etc

Thank You!