Approximating Submodular Functions

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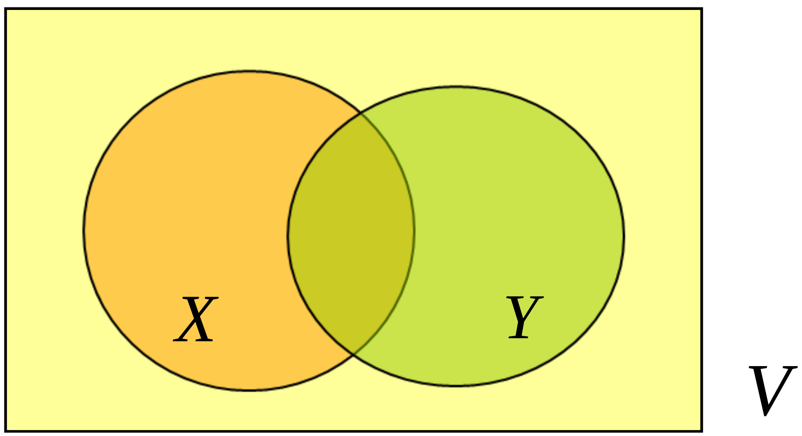
Submodular Functions

V : Finite Set

$f : 2^V \to \mathrm{R}$ $\forall X, Y \subseteq V$

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

- Cut Capacity Functions
- Matroid Rank Functions
- Entropy Functions



Optimizing Submodurlar Functions

- Minimization --- Polynomial Algorithms

  Grötschel,Lovász,Schrijver (1981)

  Schrijver (2000),

  Iwata, Fleischer, Fujishige (2000)

  Iwata, Orlin (SODA 2009)

- Maximization --- Approximation Algorithms

  Nemhauser, Wolsey, Fisher (1978)

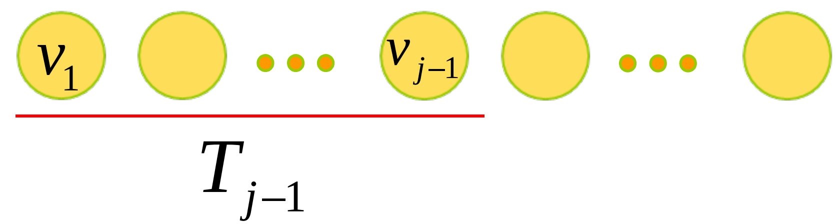
  Feige, Mirrokni, Vondrák (FOCS 2007)

  Vondrák (STOC 2008)

Approximate Maximization

Maximize $f(S)$ subject to $|S| \leq k$

Greedy Algorithm



$$T_0 := \phi$$

$$\left.\begin{aligned} v_j &:= \arg\max f(T_{j-1} \cup \{v\}) \\ T_j &:= T_{j-1} \cup \{v_j\} \\ \rho_j &:= f(T_j) - f(T_{j-1}) \end{aligned}\right\} \quad j = 1, \ldots, k$$

Approximate Maximization

S^* : Optimal Solution $\qquad \eta := f(S^*)$

$$\eta \le k\rho_j + \sum_{i=1}^{j-1} \rho_i \quad (j = 1, \ldots, k)$$

$$f(T_k) = \sum_{i=1}^{k} \rho_i$$

$$\because)\ f(S^*) \le f(S^* \cup T_{j-1})$$

$$\le f(T_{j-1}) + \sum_{u \in S^* \setminus T_{j-1}} \left[f(T_{j-1} \cup \{u\}) - f(T_{j-1}) \right]$$

$$\le \underline{f(T_{j-1})} + k\rho_j$$

$$f(T_{j-1}) \rightarrow \sum_{i=1}^{j-1} \rho_i$$

Approximate Maximization

Minimize $\sum_{i=1}^{k} \rho_i$

subject to $\begin{bmatrix} k & 0 & \cdots & 0 \\ 1 & k & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 1 & \cdots & 1 & k \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \vdots \\ \rho_k \end{bmatrix} \geq \begin{bmatrix} \eta \\ \eta \\ \vdots \\ \eta \end{bmatrix}$

$\rho_j \geq 0 \qquad (j = 1, \ldots, k)$

$$\sum_{i=1}^{k} \rho_i = f(T_k)$$

$$\eta := f(S^*)$$

$$\hat{\rho}_j := \frac{\eta}{k}\left(1 - \frac{1}{k}\right)^j$$

$$(j = 1, \ldots, k)$$

$$\sum_{i=1}^{k} \hat{\rho}_i = \eta\left[1 - \left(1 - \frac{1}{k}\right)^k\right] \geq \eta\left(1 - \frac{1}{\mathrm{e}}\right)$$

Approximating Submodular Functions

Assumption $f(\phi) = 0, \quad f(X) \geq 0, \forall X \subseteq V.$

Problem

Construct a set function $\hat{f}$ such that

$$\hat{f}(X) \leq f(X) \leq \alpha(n)\hat{f}(X), \quad \forall X \subseteq V.$$

For what function α is this possible?

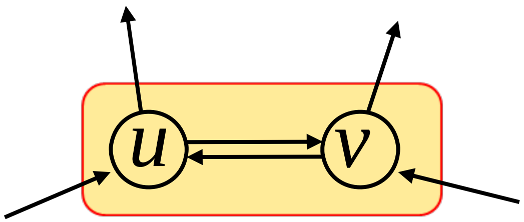
Remarks

$\alpha(n) = 1$ for cut capacity functions

$\alpha(n) = n$ for general monotone submodular functions

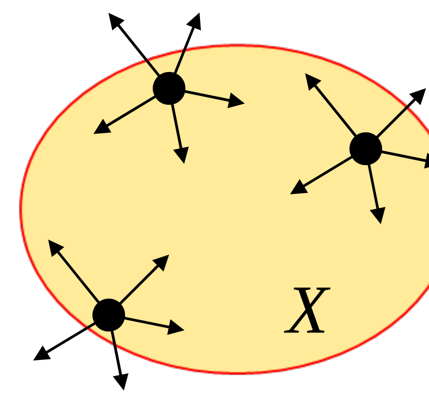
Cut Capacity Functions

$$\kappa(X) = \sum\{c(a) \mid a : \text{leaving } X\}$$



$$d(u,v) := c(u,v) + c(v,u)$$
$$= \kappa(\{u\}) + \kappa(\{v\}) - \kappa(\{u,v\})$$

$$\kappa(X) = \sum_{v \in X} \kappa(v) - \sum_{\{u,v\} \subseteq X} d(u,v)$$



Monotone Submodular Functions

$$\hat{f}(X) := \frac{1}{n}\sum_{v \in X} f(\{v\})$$

$$\alpha(n) = n$$

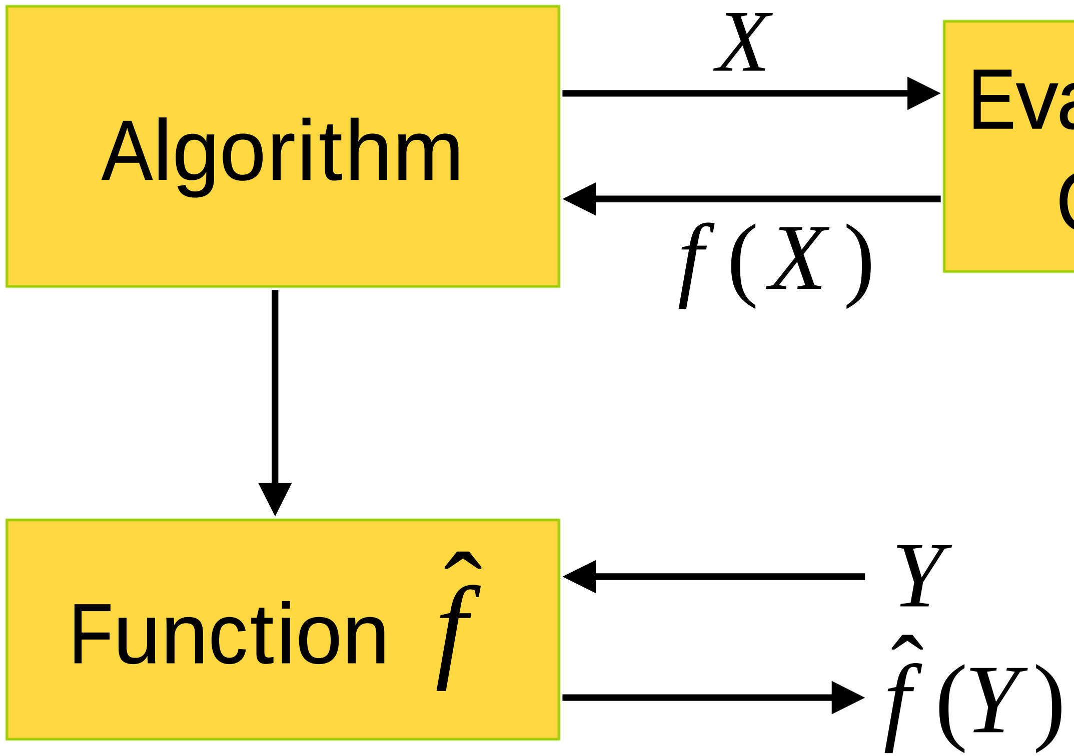
Monotone

$$\hat{f}(X) \leq \frac{1}{|X|}\sum_{v \in X} f(\{v\}) \leq \max_{v \in X} f(\{v\}) \leq f(X)$$

$$f(X) \leq \sum_{v \in X} f(\{v\}) = n\hat{f}(X)$$

Submodular

Approximating Submodular Functions



Approximating Submodular Functions

Results

- Algorithm with $\alpha(n) = \sqrt{n+1}$ for matroid rank functions.
- Algorithm with $\alpha(n) = O(\sqrt{n} \log n)$ for monotone submodular functions.
- No polynomial algorithm can achieve a factor better than $\alpha(n) = \Omega(\sqrt{n} / \log n)$ even for matroid rank functions.

Submodular Load Balancing

Svitkina & Fleischer (2008)

$f_1, \ldots, f_m$: Monotone Submodular Functions

$$\min_{\{V_1,\ldots,V_m\}} \max_j f_j(V_j)?$$

$$f_j(X) := \sum_{v \in X} p_{jv}$$ → Scheduling

2-Approximation Algorithm

Lenstra, Shmoys, Tardos (1990)

$O(\sqrt{n}\log n)$-Approximation Algorithm

Submodular Max-Min Fair Allocation

Golovin (2005)
Khot & Ponnuswami (2007)

$f_1, \dots, f_m$: Monotone Submodular Functions

$$\max_{\{V_1, \dots, V_m\}} \min_j f_j(V_j)?$$

Santa Claus

$$f_j(X) := \sum_{v \in X} p_{jv}$$ → Asadpour & Saberi (2007)

$O(\sqrt[4]{m}\sqrt{\log n}^3)$-Approximation Algorithm

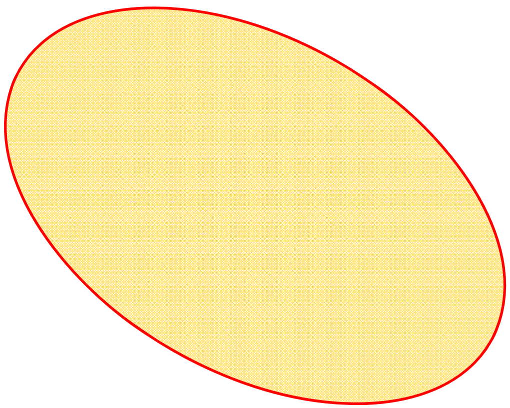
$O(\sqrt{n}\sqrt[4]{m}\log n\sqrt{\log n}^3)$-Approximation Algorithm

Ellipsoidal Approximation

$A:$ Positive Definite Symmetric Matrix

Ellipsoid $\qquad \| x \|_A = \sqrt{x'Ax}$

$$E(A) := \{x \mid x'Ax \le 1\}$$



$$\mathrm{vol}(E(A)) \propto \frac{1}{\sqrt{\det A}}$$

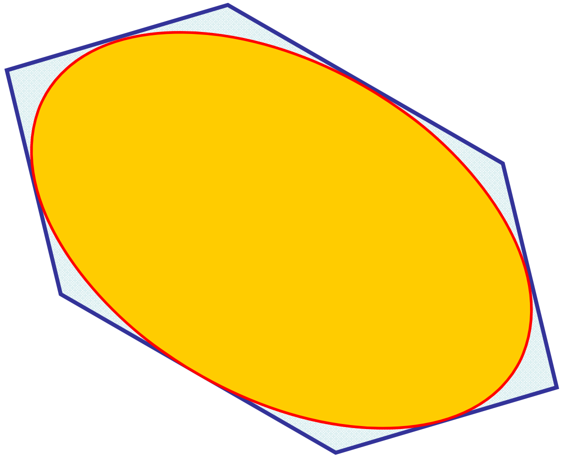
$$\max\{c'x \mid x \in E(A)\} = \sqrt{c'A^{-1}c}$$

Ellipsoidal Approximation

K : Centrally Symmetric Convex Body

$$(x \in K \Rightarrow -x \in K)$$

Maximum Volume Inscribed Ellipsoid
(The John Ellipsoid)

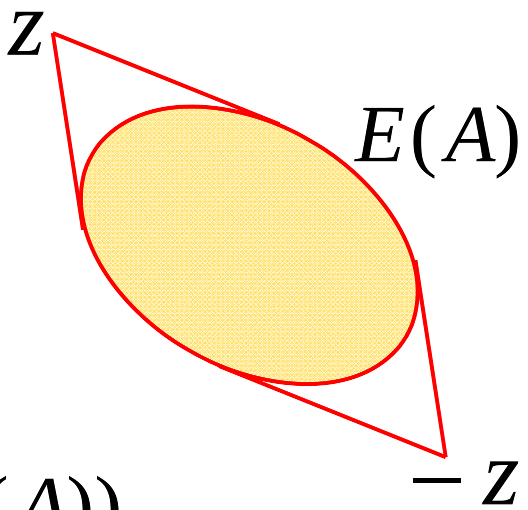


Ellipsoidal Approximation

K : Centrally Symmetric Convex Body

$E(A)$: John Ellipsoid of K

$$\Rightarrow \; E(A) \subseteq K \subseteq \sqrt{n} E(A)$$

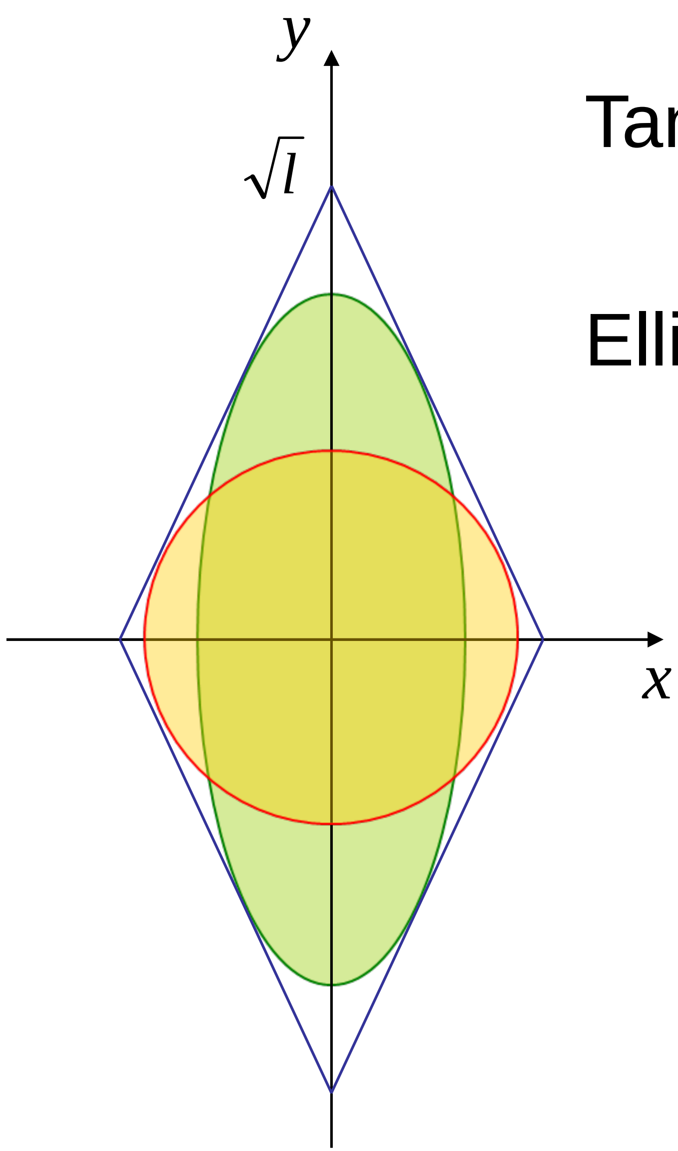


$\because$) Suppose $\exists z \in K, \; \| z \|_A > \sqrt{n}.$

Consider the John Ellipsoid $E(L(A,z))$ of $\mathrm{conv}\{E(A) \cup \{z, -z\}\}.$

Show that $\mathrm{vol}(E(L(A,z))) > \mathrm{vol}(E(A)).$

Ellipsoidal Approximation

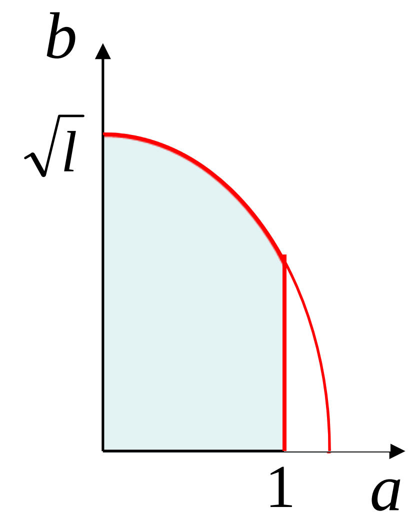


Tangential Line $$\sqrt{1-\frac{1}{l}}\,x+\frac{1}{\sqrt{l}}\,y=1$$

Ellipse $$\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$$

$$(1-\frac{1}{l})a^2+\frac{b^2}{l}\le 1$$

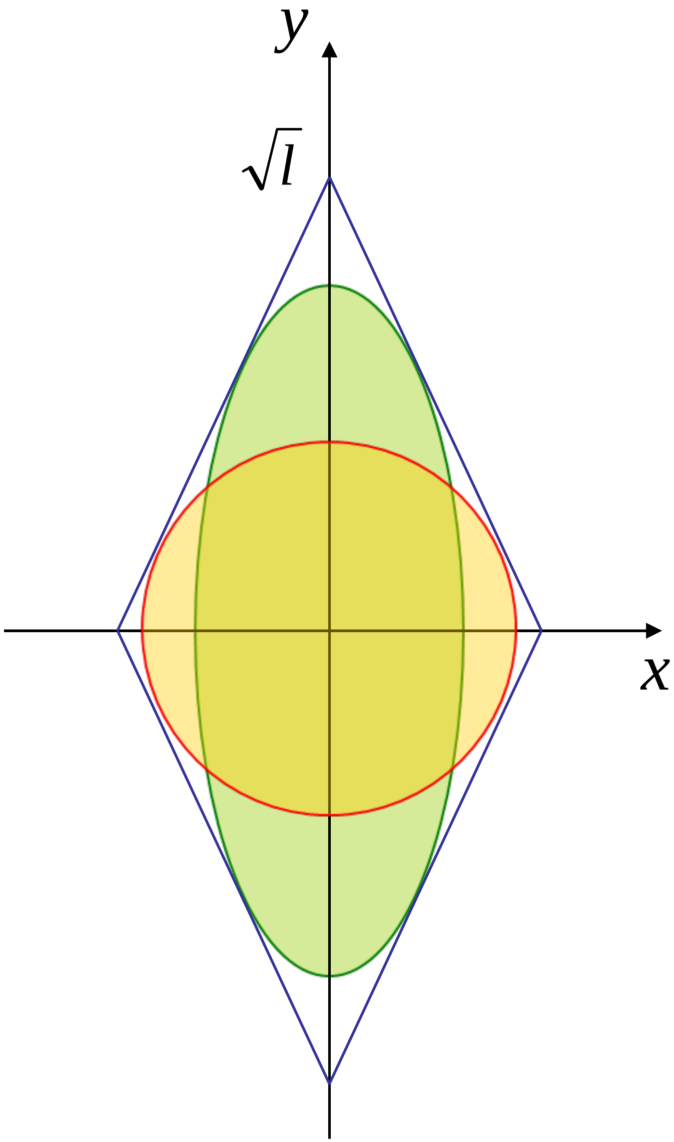
$$0\le a\le 1,\quad b\ge 0$$



$$a^{n-1}b\to \max$$

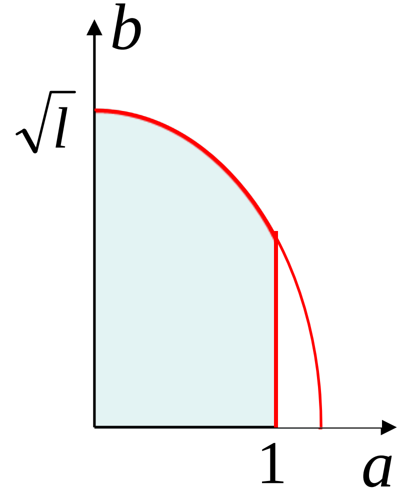
$$(n-1)\log a+\log b\to \max$$

Ellipsoidal Approximation



$$(1-\frac{1}{l})a^2+\frac{b^2}{l}\le 1$$

$$0\le a\le 1,\quad b\ge 0$$



$$(n-1)\log a+\log b\to \max$$

$$a=\sqrt{\frac{n-1}{n}\frac{l}{l-1}}\qquad b=\sqrt{\frac{l-1}{n-1}\frac{l}{n}}$$

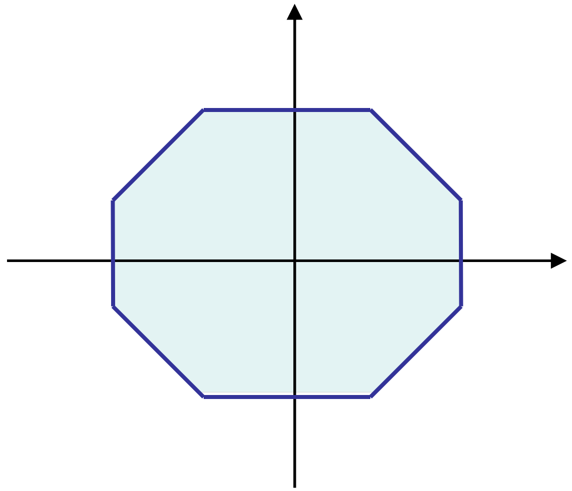
$$a^{n-1}b=\sqrt{\left(\frac{l}{n}\right)^n\left(\frac{n-1}{l-1}\right)^{n-1}}\ge 1\qquad (l\ge n)$$

Symmetrized Polymatroids

f : Monotone Submodular Function $f(\phi)=0$

$$P(f)=\{x \mid x \in \mathbf{R}_+^V, \sum_{v\in S} x(v) \le f(S), \forall S \subseteq V\}$$

$$Q(f)=\{x \mid x \in \mathbf{R}^V, \sum_{v\in S} |x(v)| \le f(S), \forall S \subseteq V\}$$



John Ellipsoid of $Q(f)$

Axis-Aligned $E(D)$

D : Diagonal

Symmetrized Polymatroids

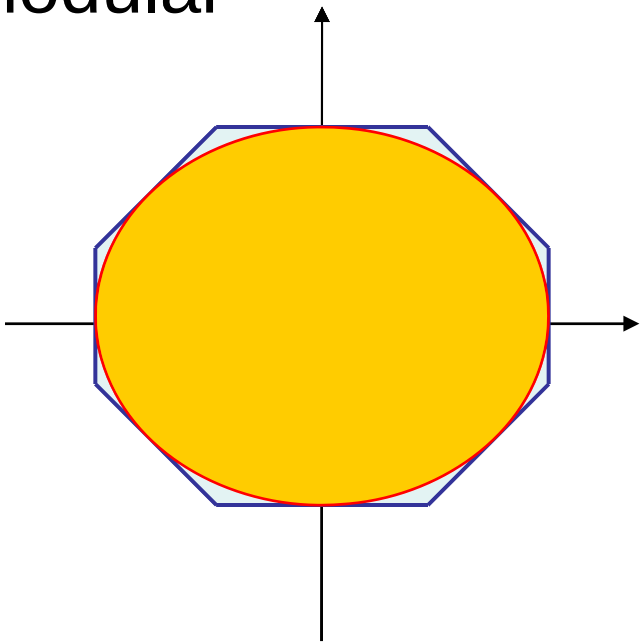
$E(D)$: John Ellipsoid of $Q(f)$ $\quad p_u = \dfrac{1}{D_{uu}} \quad (u \in V)$

$$\hat{f}(S) := \sqrt{\sum_{u \in S} p_u} = \max\{\chi_S y \mid y \in E(D)\}$$

$\hat{f}$: Monotone Submodular

$$f(S) = \max\{\chi_S y \mid y \in Q(f)\}$$

$$E(D) \subseteq Q(f) \subseteq \sqrt{n} E(D)$$



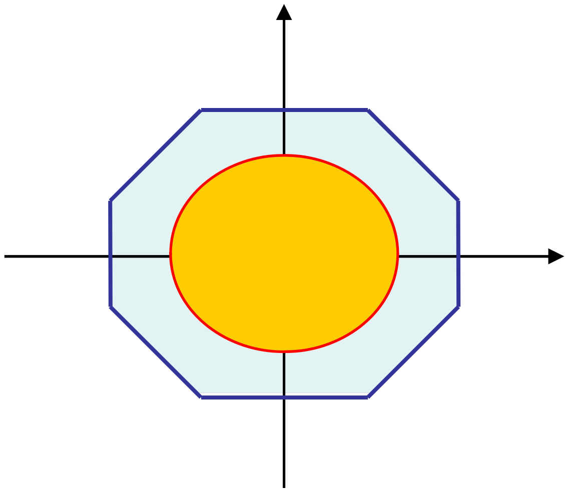
$$\hat{f}(S) \le f(S) \le \sqrt{n}\, \hat{f}(S)$$

Approximate Decision Problem

Axis-Aligned Ellipsoid $E(D) \subseteq Q(f)$

- Find an $z \in Q(f)$ such that $\| z \|_D > \sqrt{n+1}$, or
- Certify that $\| x \|_D \leq \sqrt{n+1} / \beta, \quad \forall x \in Q(f)$.

$(0 < \beta \leq 1)$

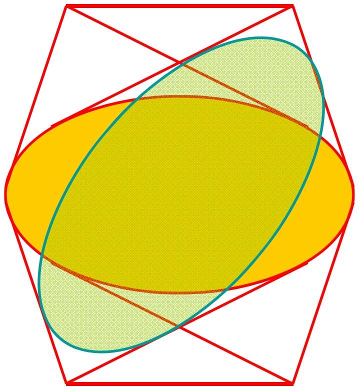


$$Q(f) \subseteq \frac{\sqrt{n+1}}{\beta} E(D)$$

Axis-Aligned Ellipsoidal Approximation

$$\| z \|_D > \sqrt{n+1}$$

Compute the John Ellipsoid $E(A)$ of $\text{conv}\{E(D) \cup \{z, -z\}\}$.



$E(A)$ is NOT axis-aligned.

$$T(z) = \{\text{Diag}(\sigma)z \mid \sigma \in \{1, -1,\}^n\}$$

$$\text{conv}\{E(D) \cup T(z)\} \subseteq Q(f)$$

Axis-Aligned Ellipsoidal Approximation

$$B := (\mathrm{Diag}(\mathrm{diag}(A^{-1})))^{-1}$$

$E(B)$: Axis-Aligned Ellipsoid

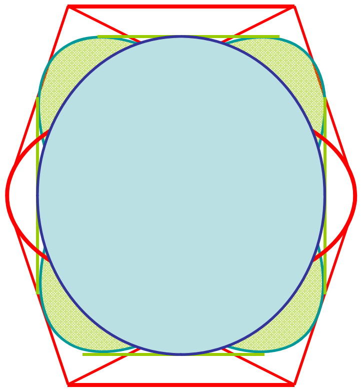
$$E(B) \subseteq \mathrm{conv}\{E(D) \cup T(z)\}.$$

Hadamard's inequality

$$\mathrm{vol}(E(B)) \geq \mathrm{vol}(E(A))$$

$$\geq \left(1+\frac{1}{n}\right)^{n}\left(1-\frac{1}{n}\right)^{n-1}\mathrm{vol}(E(D))$$

$$1+\frac{1}{4n^2}-O(1/n^3)$$



Axis-Aligned Ellipsoidal Approximation

Initial Ellipsoid $D_0 := \mathrm{Diag}\left(\frac{n}{f(u)^2}\right)$

$$E(D_0) \subseteq Q(f) \subseteq nE(D_0)$$

The volume increases by a factor of $1 + \frac{1}{4n^2} - O(1/n^3)$.

The algorithm terminates within $O(n^3 \log n)$ iterations.

$$E(D) \subseteq Q(f) \subseteq \frac{\sqrt{n+1}}{\beta} E(D)$$

Approximate Decision Problem

Maximize $\sum_{u \in V} d_u x_u{}^2$

subject to $x \in Q(f)$ ⟶ $x \in P(f)$

$$d_u = D_{uu} \quad (u \in V)$$

- For matroid rank functions, $\beta = 1$.
- For general monotone submodular functions, $\beta = 1/O(\log n)$.

Approximate Decision Problem

Maximize $\sum_{u \in V} x(u)^2$

subject to $x \in P(f)$

$$T_0 := \phi$$

$$j_k := \arg\max f(T_{k-1} \cup \{j\})$$

$$T_k := T_{k-1} \cup \{j_k\}$$

$$\hat{x}(j_k) := f(T_k) - f(T_{k-1})$$

$\hat{x} \in P(f)$ Extreme Point

$$\sum_{u \in V} \hat{x}(u)^2 \geq \left(1 - \frac{1}{\mathrm{e}}\right)^2 \max\left\{ \sum_{u \in V} x(u)^2 \,\middle|\, x \in P(f) \right\}$$

$(1-1/\mathrm{e})^2$ Approximate Solution

Approximate Decision Problem

$$f(T_k) \geq (1 - 1/\mathrm{e}) \max_{S:|S|=k} f(S).$$

Nemhauser, Wolsey, Fisher (1978)

$$h(k) := f(T_k)$$ Concave

$$g(S) := \frac{\mathrm{e}}{\mathrm{e}-1} h(|S|)$$ Submodular

$$f(S) \leq g(S), \quad \forall S \subseteq V.$$

$$\max_{x \in P(f)} \sum_{u \in V} x(u)^2 \leq \max_{x \in P(g)} \sum_{u \in V} x(u)^2 = \left(\frac{\mathrm{e}}{\mathrm{e}-1} \right)^2 \sum_{u \in V} \hat{x}(u)^2$$

Open Problem

- A Constant Factor Approximation Algorithm for Separable Convex Quadratic Function Maximization over Polymatroids.

Maximize $\sum_{u \in V} d_u x_u{}^2$

subject to $x \in P(f)$