Minimizing Submodular Functions

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Outline

- Submodular Functions
  - Examples
  - Discrete Convexity
- Submodular Function Minimization
  - Min-Max Theorem
  - Combinatorial Algorithms
- Applications
- Conclusion

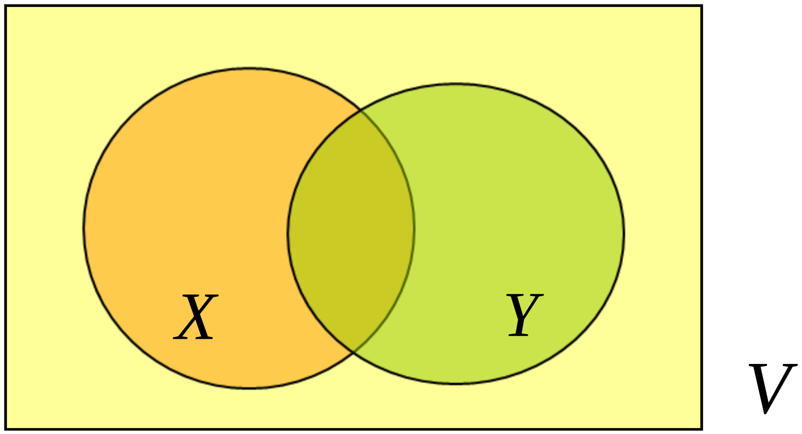
Submodular Functions

V : Finite Set

$$f : 2^V \to \mathbb{R} \qquad \forall X, Y \subseteq V$$

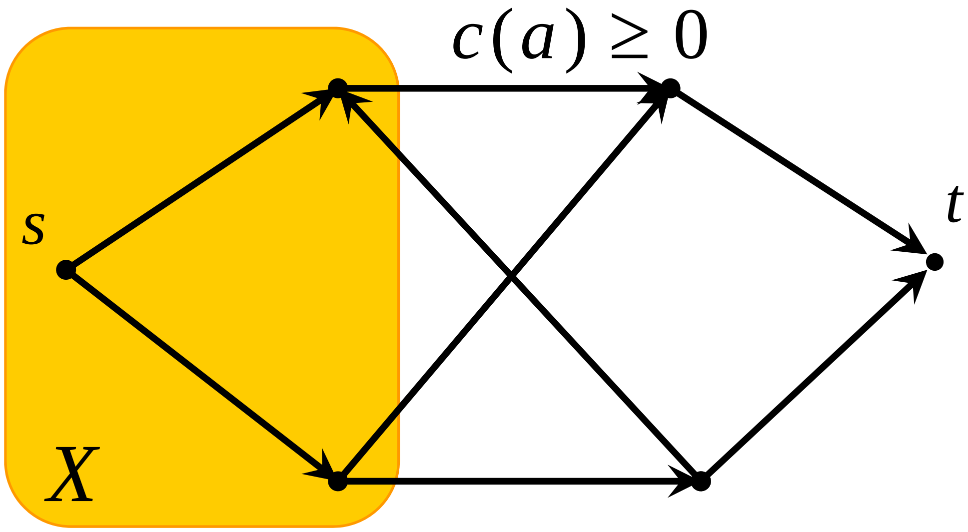
$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

- Cut Capacity Functions
- Matroid Rank Functions
- Entropy Functions



Cut Capacity Function

Cut Capacity $\kappa(X) = \sum\{c(a) \mid a : \text{leaving } X\}$



Max Flow Value＝Min Cut Capacity

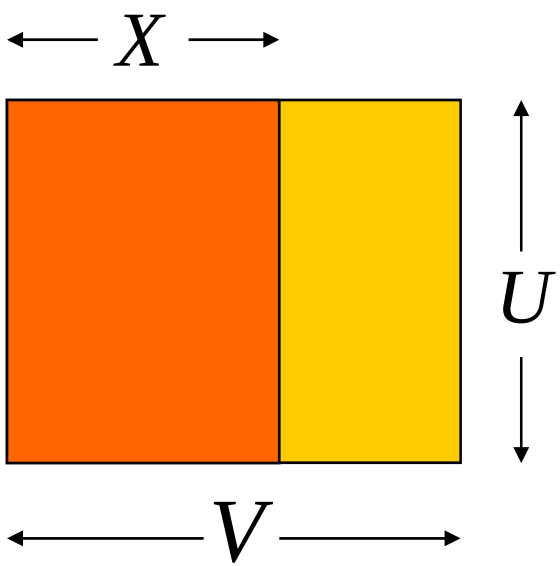
Matroid Rank Functions

Matrix Rank Function

Whitney (1935)

$$\rho(X) = \operatorname{rank} A[U, X]$$

$A =$



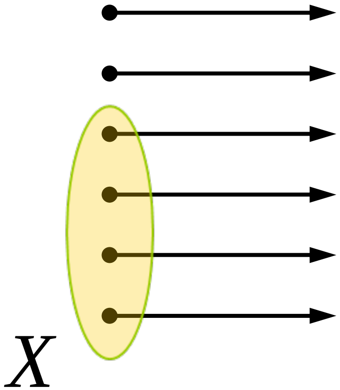
$$\forall X \subseteq V, \rho(X) \leq |X|$$

$$X \subseteq Y \Rightarrow \rho(X) \leq \rho(Y)$$

ρ : Submodular

Entropy Functions

Information Sources



$$h(\phi) = 0$$

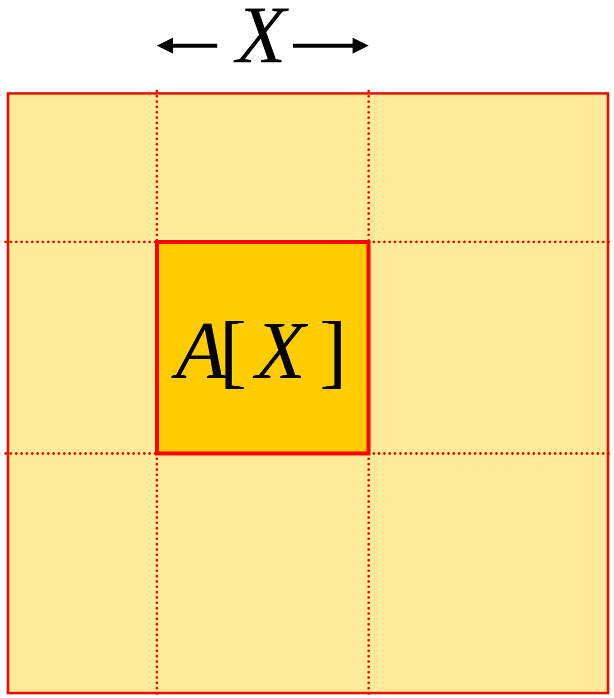
$h(X)$: Entropy of the Joint Distribution

$$h(X) + h(Y) \geq h(X \cap Y) + h(X \cup Y)$$

Conditional Mutual Information ≥ 0

Positive Definite Symmetric Matrices

$A =$



$$f(\phi) = 0$$

$$f(X) = \log\det A[X]$$

Ky Fan's Inequality

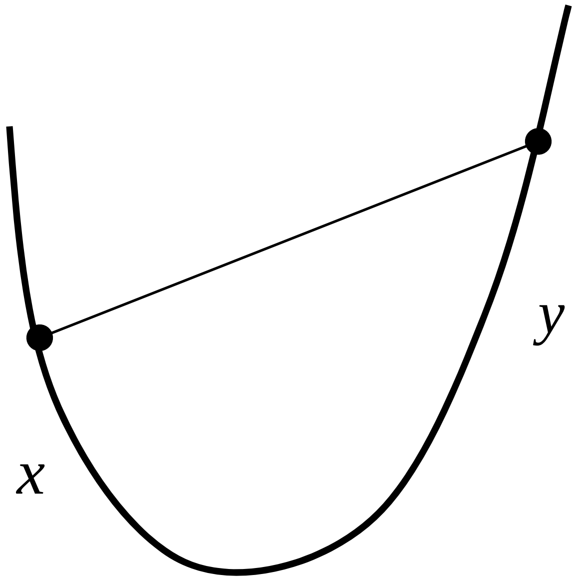
$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

Extension of the Hadamard Inequality

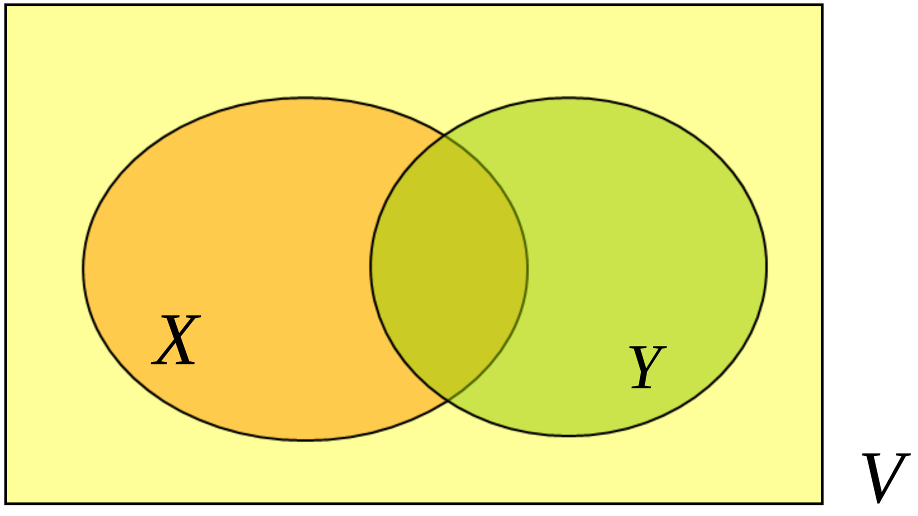
$$\det A \leq \prod_{i \in V} A_{ii}$$

Discrete Convexity

Convex Function

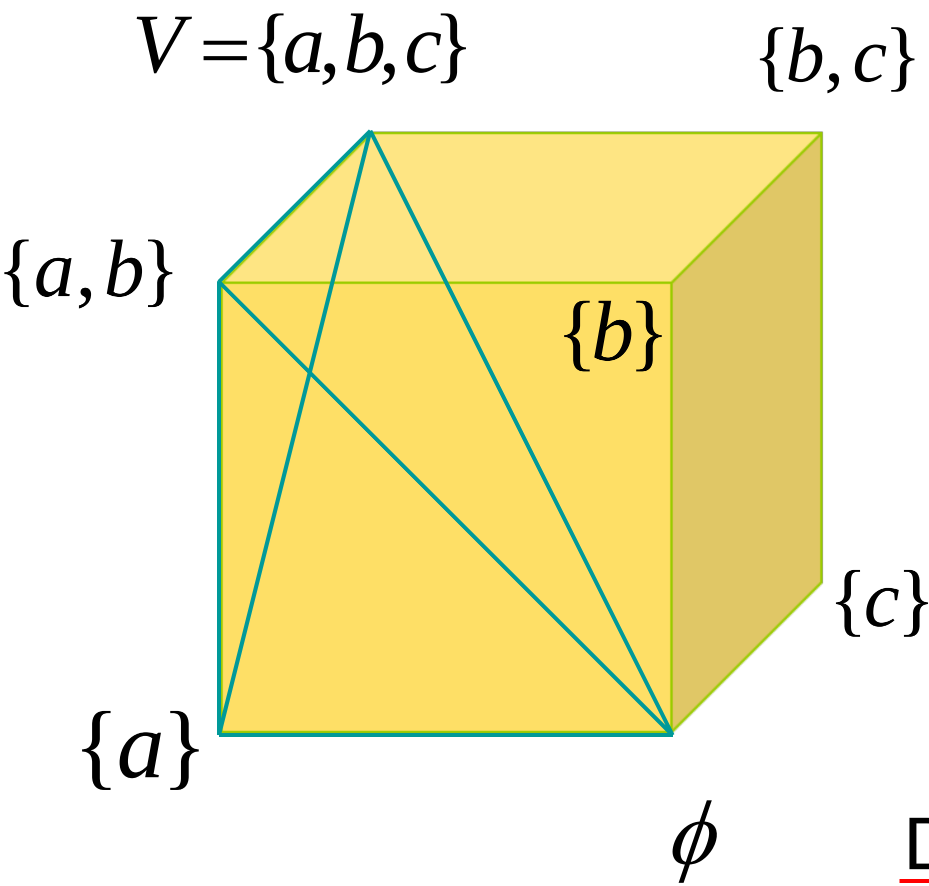


$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$



Discrete Convexity

Lovász (1983)



$\hat{f}$: Linear Interpolation

$\hat{f}$: Convex

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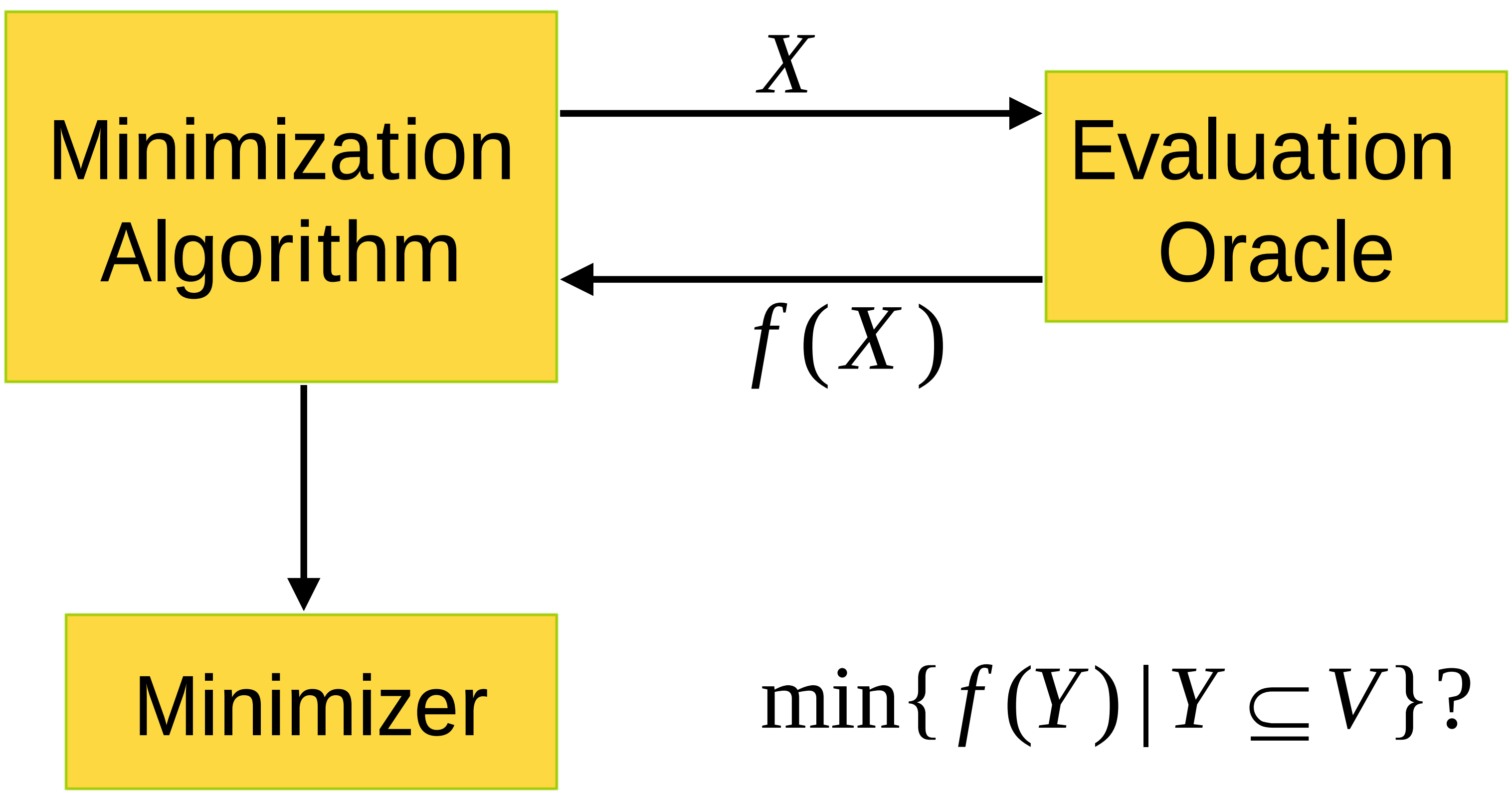
f : Submodular

Discrete Convex Analysis

Murota (2003)

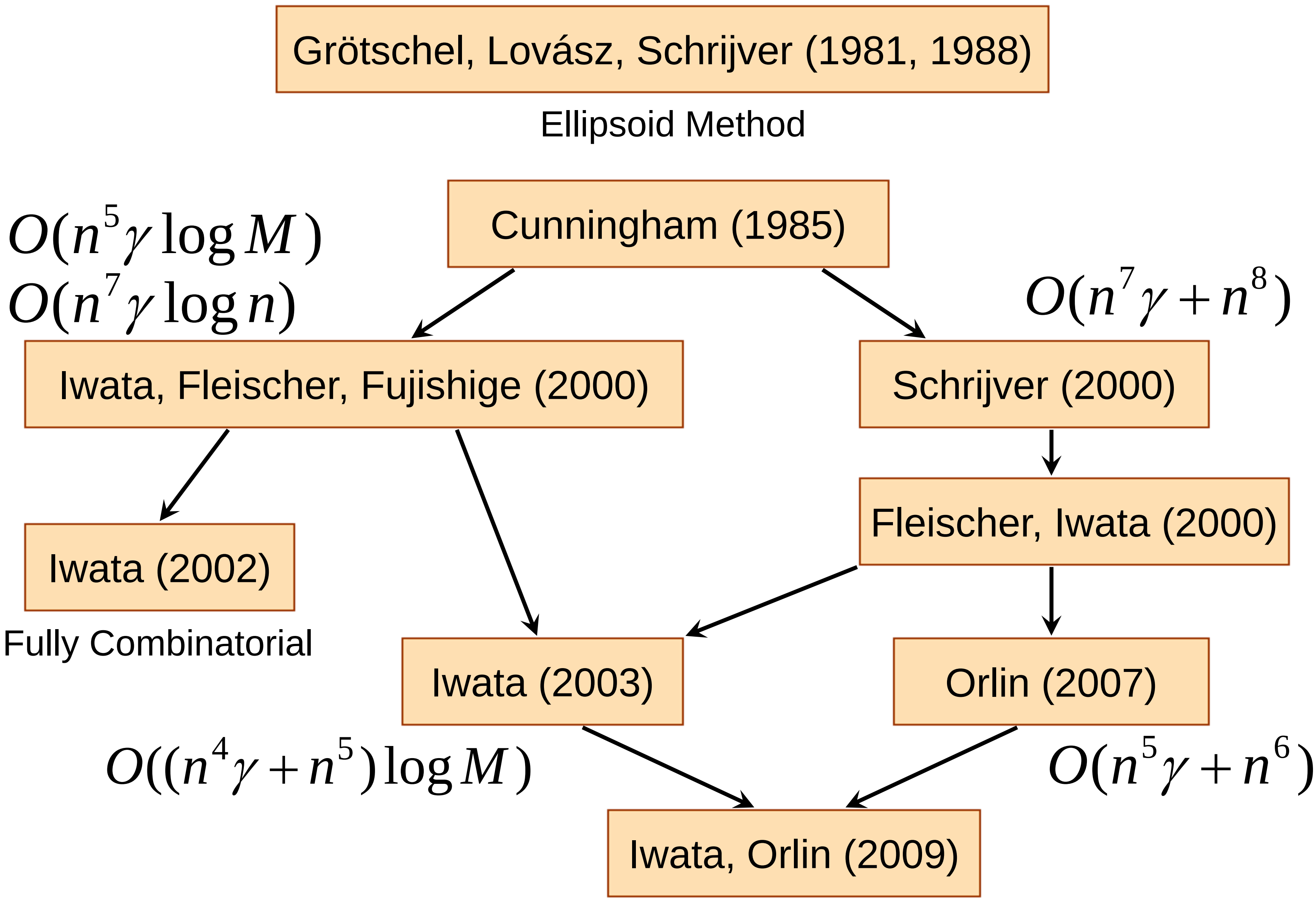
Submodular Function Minimization

Assumption: $f(\phi) = 0$



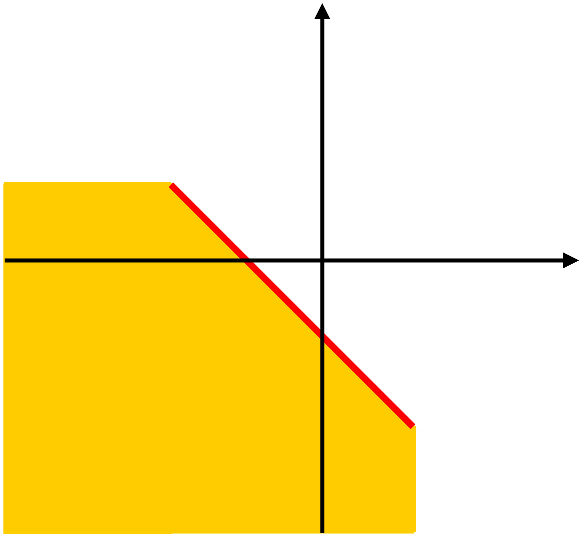
$$\min\{ f(Y) \mid Y \subseteq V \} ?$$

Submodular Function Minimization



Base Polyhedra

$$\mathbf{R}^V = \{x \mid V \to \mathbf{R}\}$$



$$x(Y) = \sum_{v \in Y} x(v)$$

Submodular Polyhedron

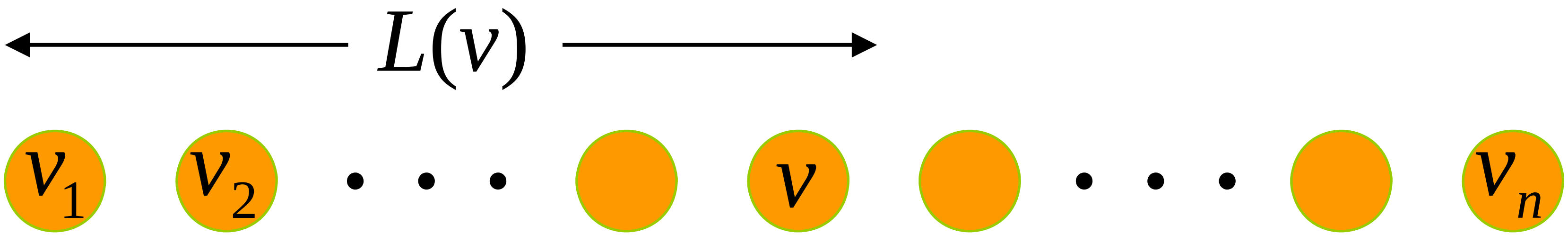
$$P(f) = \{x \mid x \in \mathbf{R}^V, \forall Y \subseteq V, x(Y) \leq f(Y)\}$$

Base Polyhedron

$$B(f) = \{x \mid x \in P(f), x(V) = f(V)\}$$

Greedy Algorithm

Edmonds (1970)
Shapley (1971)



$$y(v) = f(L(v)) - f(L(v) - \{v\}) \qquad (v \in V)$$

y : Extreme Base

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} y(v_1) \\ y(v_2) \\ \vdots \\ y(v_n) \end{bmatrix} = \begin{bmatrix} f(L(v_1)) \\ f(L(v_2)) \\ \vdots \\ f(L(v_n)) \end{bmatrix}$$

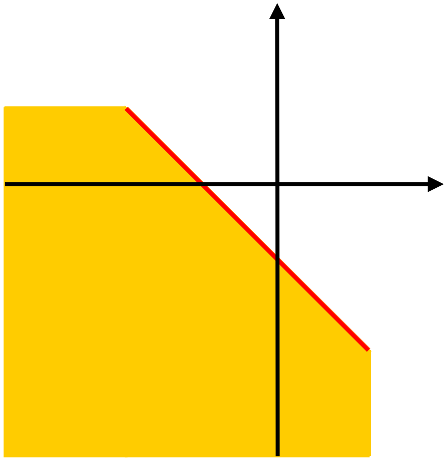
Min-Max Theorem

Theorem Edmonds (1970)

$$\min_{Y \subseteq V} f(Y) = \max\{x^-(V) \mid x \in B(f)\}$$

$$x^-(v) := \min\{0, x(v)\}$$

$$x^-(V) \le x(Y) \le f(Y)$$



Combinatorial Approach

Extreme Base $y_L \in B(f)$

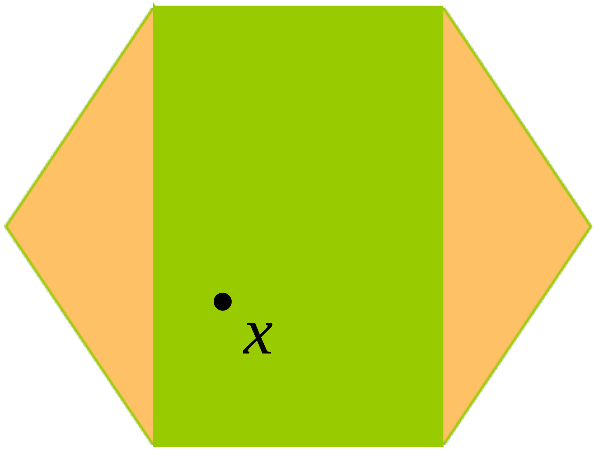
Convex Combination

$$x = \sum_{L \in \Lambda} \lambda_L y_L$$

Cunningham (1985)

$$O(n^6 M\gamma \log nM)$$

$$M = \max_{X \subseteq V} |f(X)|$$



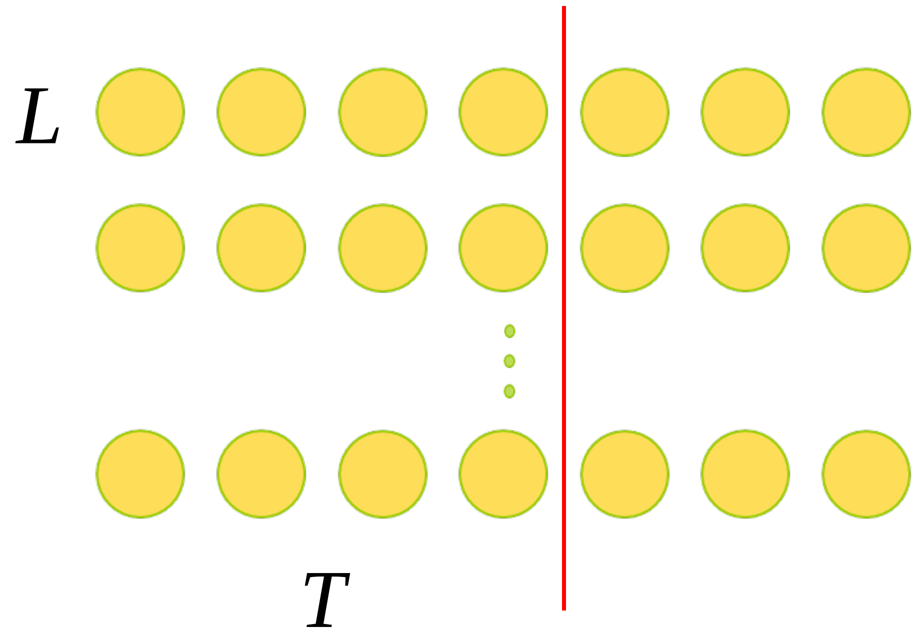
Combinatorial Approach

$$x = \sum_{L \in \Lambda} \lambda_L y_L$$

y_L : Extreme Base

$$x(v) \leq 0, \quad \forall v \in T$$

$$x(v) \geq 0, \quad \forall v \notin T$$



$$y_L(T) = f(T), \quad \forall L \in \Lambda. \quad \therefore x(T) = f(T)$$

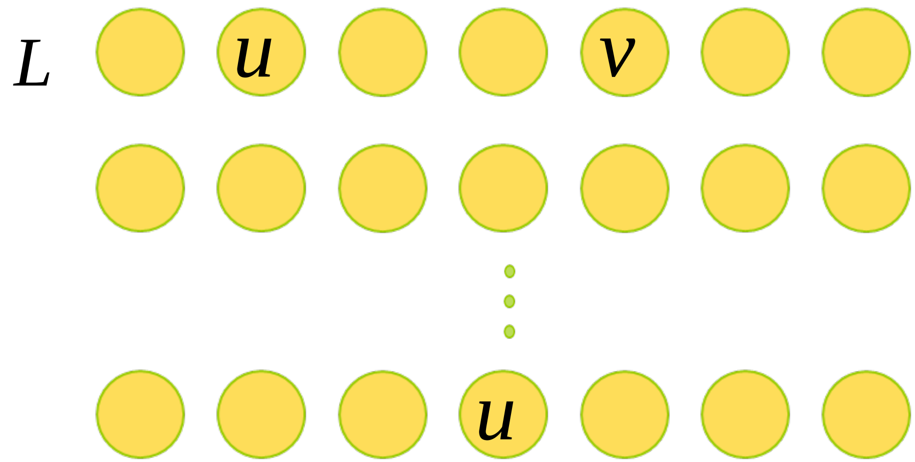
$$x^-(V) = x(T) = f(T)$$

T : Minimizer

Distance Labeling

$$x = \sum_{L \in \Lambda} \lambda_L y_L$$

y_L : Extreme Base



Labeling

$$d_L : V \to \mathbf{Z} \qquad (L \in \Lambda)$$

$$x(u) \leq 0 \;\Rightarrow\; d_L(u) = 0, \quad \forall L \in \Lambda.$$

$$u \prec_L v \;\Rightarrow\; d_L(u) \leq d_L(v).$$

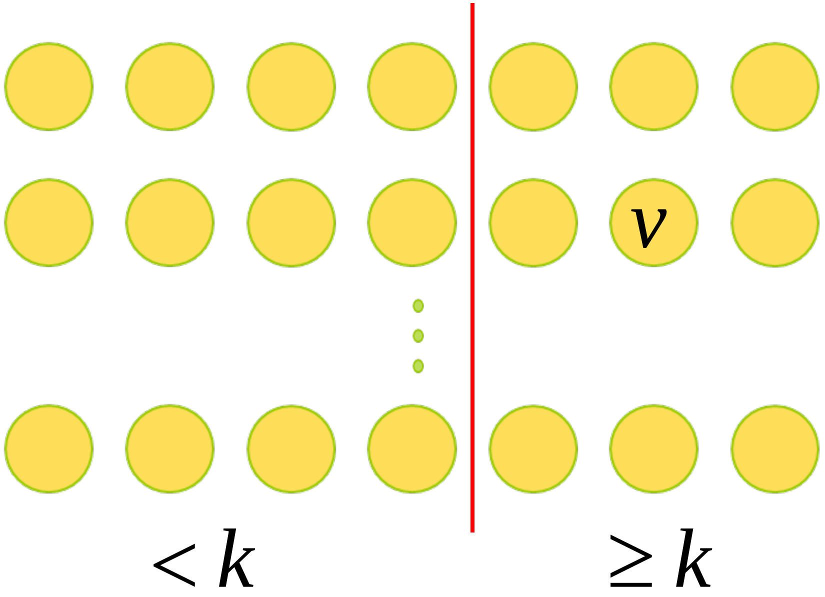
$$| d_L(u) - d_K(u) | \leq 1, \;\; \forall L, K \in \Lambda, \forall u \in V.$$

Distance Labeling

$$d_{\min}(u) := \min\{d_L(u) \mid L \in \Lambda\}$$

Gap of Level k

$$\exists v \in V,\ d_{\min}(v) = k.$$
$$\forall v \in V,\ d_{\min}(v) \neq k-1.$$



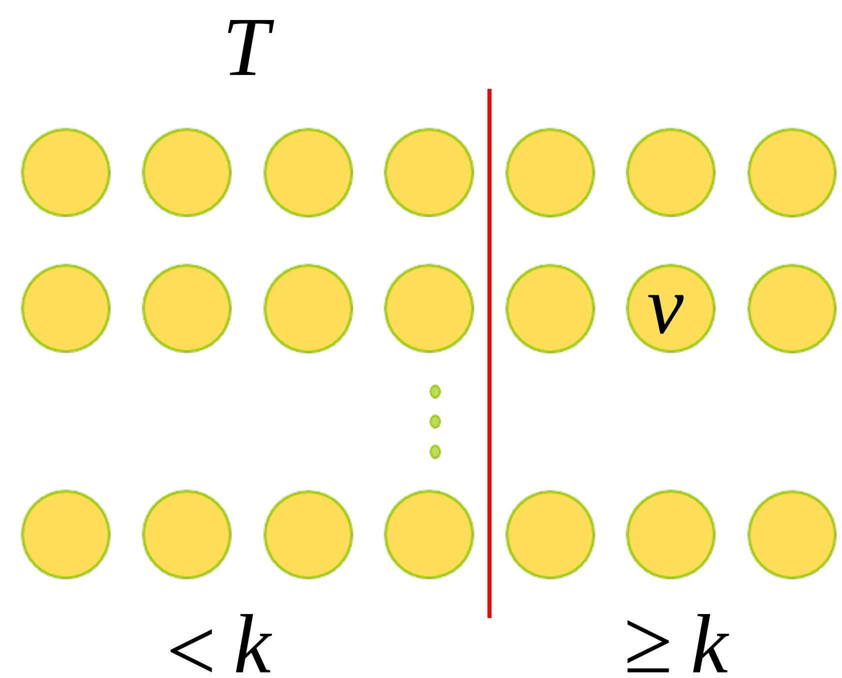
$$d_{\min}(v) \geq k \Rightarrow \underline{v \notin Y},\ \forall Y : \text{Minimizer of } f.$$

Distance Labeling

$$X \setminus T \neq \phi \Rightarrow$$

$$f(T) = x(T) < x(X \cup T) \leq f(X \cup T).$$

$$\therefore f(X) > f(X \cap T).$$

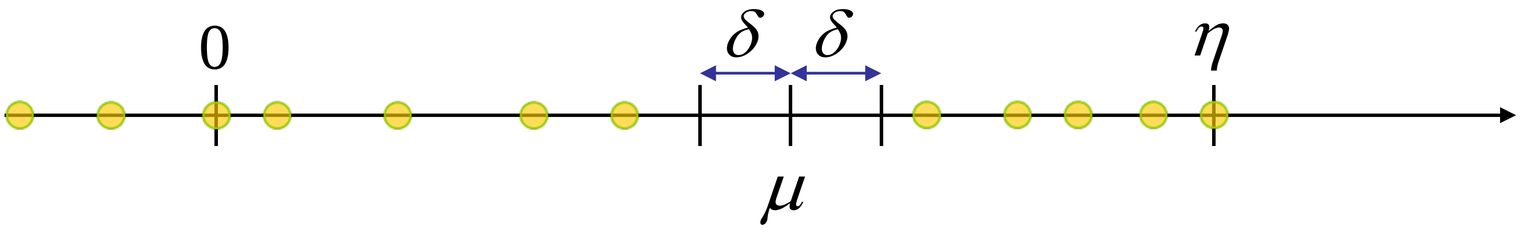


$$d_{\min}(v) \geq k \Rightarrow \underline{v \notin Y}, \ \forall Y : \text{Minimizer of } f.$$

Iteration

$$\eta := \max\{x(v) \mid v \in V\} \qquad \delta := \frac{\eta}{4n}$$

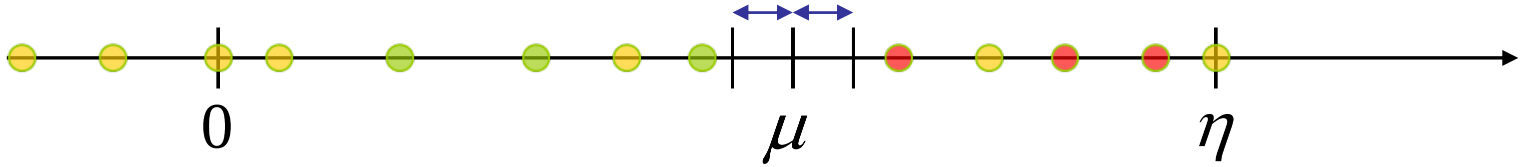
Find μ such that $|x(v) - \mu| > \delta, \forall v \in V.$



$$l := \min\{d_L(u) \mid u \in V, L \in \Lambda, x(u) > \mu\}$$

Select $u \in V$ and $L \in \Lambda$ such that $l = d_L(u)$.

New Permutation

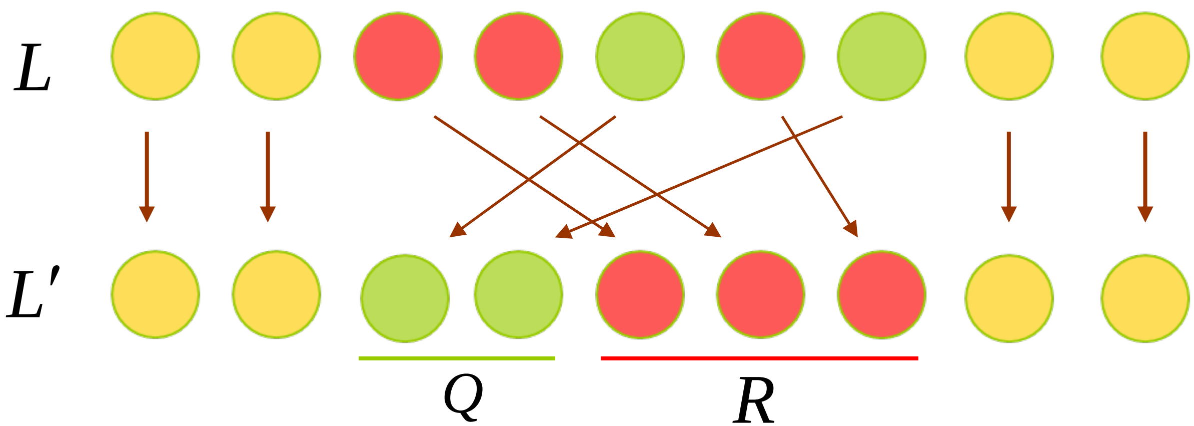


$$\text{New\_Permutation}(L, \mu, l)$$

$$d_{L'}(v) := \begin{cases} d_L(v) & (v \notin R) \\ d_L(v) + 1 & (v \in R) \end{cases}$$

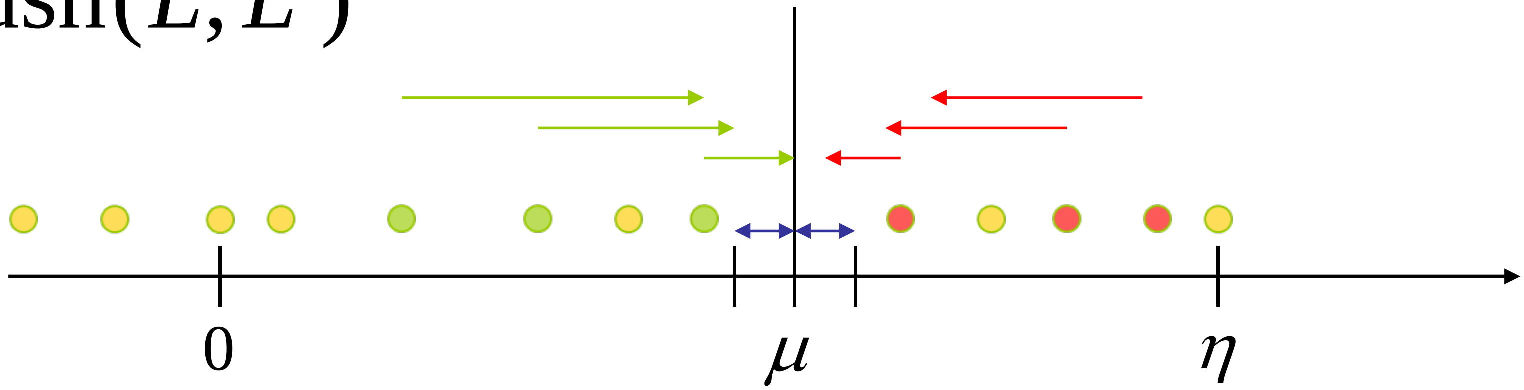
$$S = \{v \mid v \in V, d_L(v) = l\}$$

$$R = \{v \mid v \in S, x(v) > \mu\}, \quad Q = S \setminus R$$



Push Operation

$\text{Push}(L, L')$



$$\beta := \min\left\{ \frac{x(v) - \mu}{y_L(v) - y_{L'}(v)} \,\middle|\, v \in S, y_L(v) \neq y_{L'}(v) \right\}$$

$$\alpha := \min\{\lambda_L, \beta\}$$

$$\lambda_{L'} := \alpha$$

$$\lambda_L := \lambda_L - \alpha$$

$\alpha = \lambda_L$ Saturating

$\alpha = \beta$ Nonsaturating

Potential Function

$$\Phi(x) = \sum_{v \in V} x^+(v)^2$$

$$x^+(v) = \max\{x(v), 0\}$$

Nonsaturating Push Moves x to x'

$$\Rightarrow \quad \Phi(x) - \Phi(x') \geq \Phi(x) / 16n^3$$

Initially, $\Phi(x) \leq nM^2$.

After $O(n^3 \log nM)$ Nonsaturating Pushes,

$\Phi(x) < 1/n^2$.

$\eta < 1/n.$

$$\eta := \max\{x(v) \mid v \in V\}$$

Algorithm Termination

$$f : 2^V \to \mathbf{Z}$$

$$\eta := \max\{x(v) \mid v \in V\}$$

$\eta < \frac{1}{n} \;\Rightarrow\; V :$ Maximal Minimizer

$$\because) \;\; f(X) \geq \underline{x^-(V) > x(V) - 1} = f(V) - 1.$$

$$|\Lambda| = O(n^3 \log nM)$$

Running Time Bound

$$\Gamma(\Lambda) = \sum_{L\in\Lambda}\sum_{v\in V}[n - d_L(v)]$$

A Saturating Push Decreases $\Gamma(\Lambda)$.

A Nonsaturating Push Increases $|\Lambda|$ by One and $\Gamma(\Lambda)$ by at Most n^2.

Total Increase of $\Gamma(\Lambda)$	$O(n^5 \log nM)$
# Saturating Pushes	$O(n^5 \log nM)$
Running Time	$O(n^6 \gamma \log nM)$

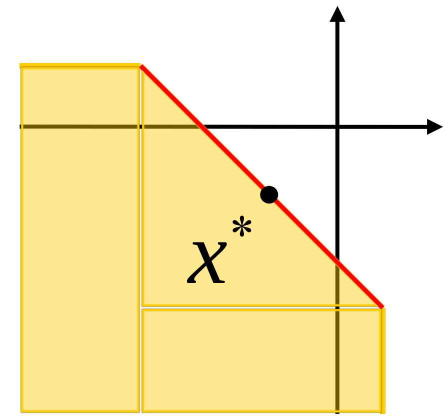
Improvements

- A Simple Algorithm $O(n^6 \gamma \log nM)$
- A Faster Weakly Polynomial Algorithm
  $O((n^4 \gamma + n^5) \log nM)$
- A Strongly Polynomial Algorithm
  $O((n^5 \gamma + n^6) \log n)$
- A Fully Combinatorial Algorithm
  $O((n^7 \gamma + n^8) \log n)$

The Minimum-Norm Base

Minimize $\|x\|^2$

subject to $x \in B(f)$



Theorem   Fujishige (1984)

x^* : opt.sol.

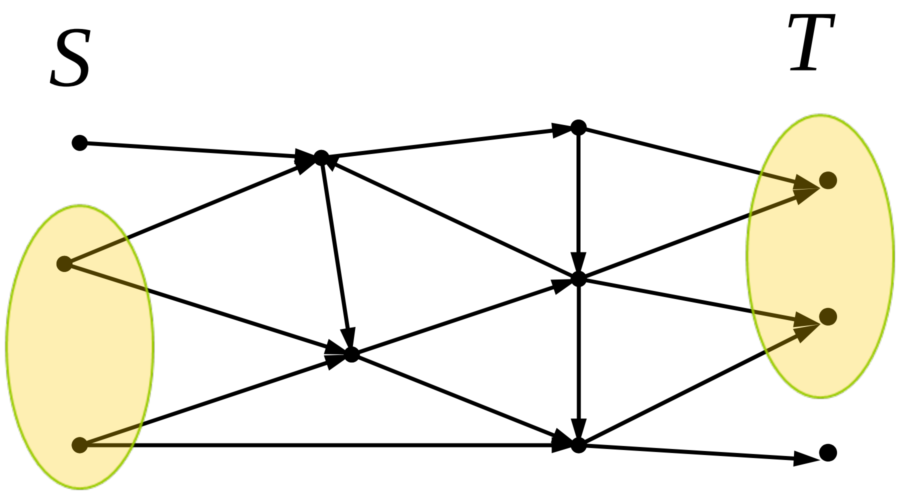
$X_- := \{v \mid x^*(v) < 0\}$ ⟶ Minimal Minimizer

$X_0 := \{v \mid x^*(v) \leq 0\}$ ⟶ Maximal Minimizer

Remark   Nagano (2007)

The minimum-norm base minimizes $\sum_{v \in V} g(x(v))$ in $B(f)$ for any convex function g.

Evacuation Problem（Dynamic Flow）



Hoppe, Tardos (2000)

$c(a)$: Capacity

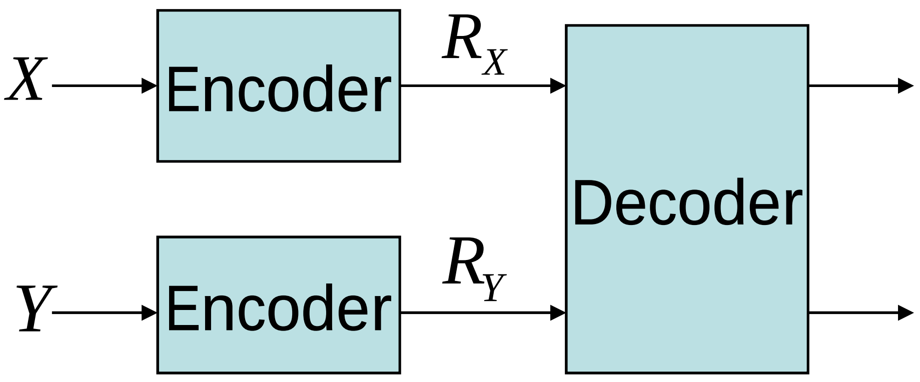
$\tau(a)$: Transit Time

$b(v)$: Supply/Demand

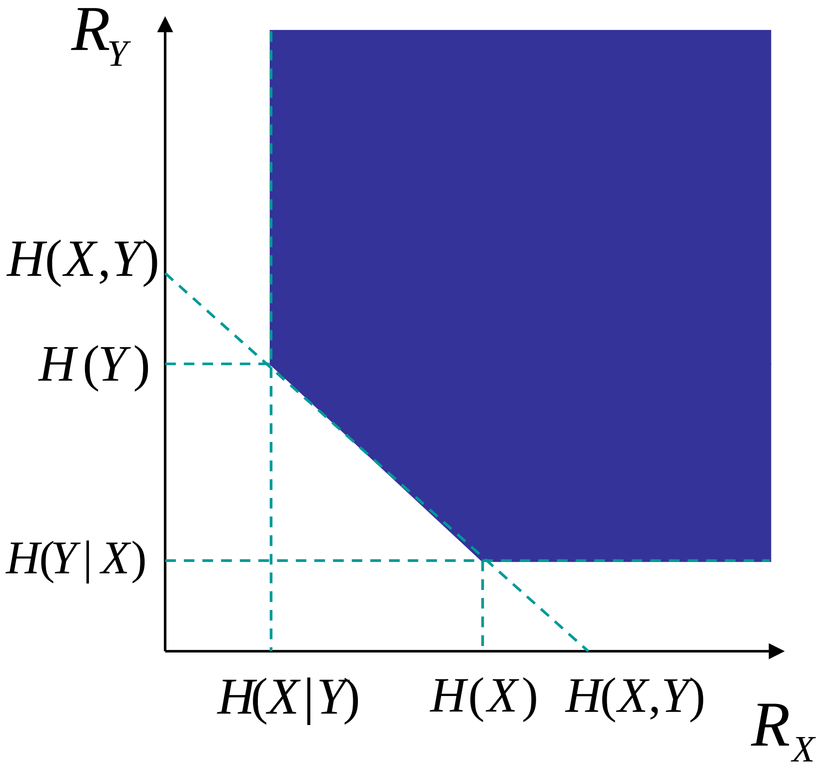
$o(X)$: Maximum Amount of Flow from $X \cap S$ to $T \setminus X$.

Feasible $\Longleftrightarrow$ $b(X) \le o(X), \forall X \subseteq S \cup T$

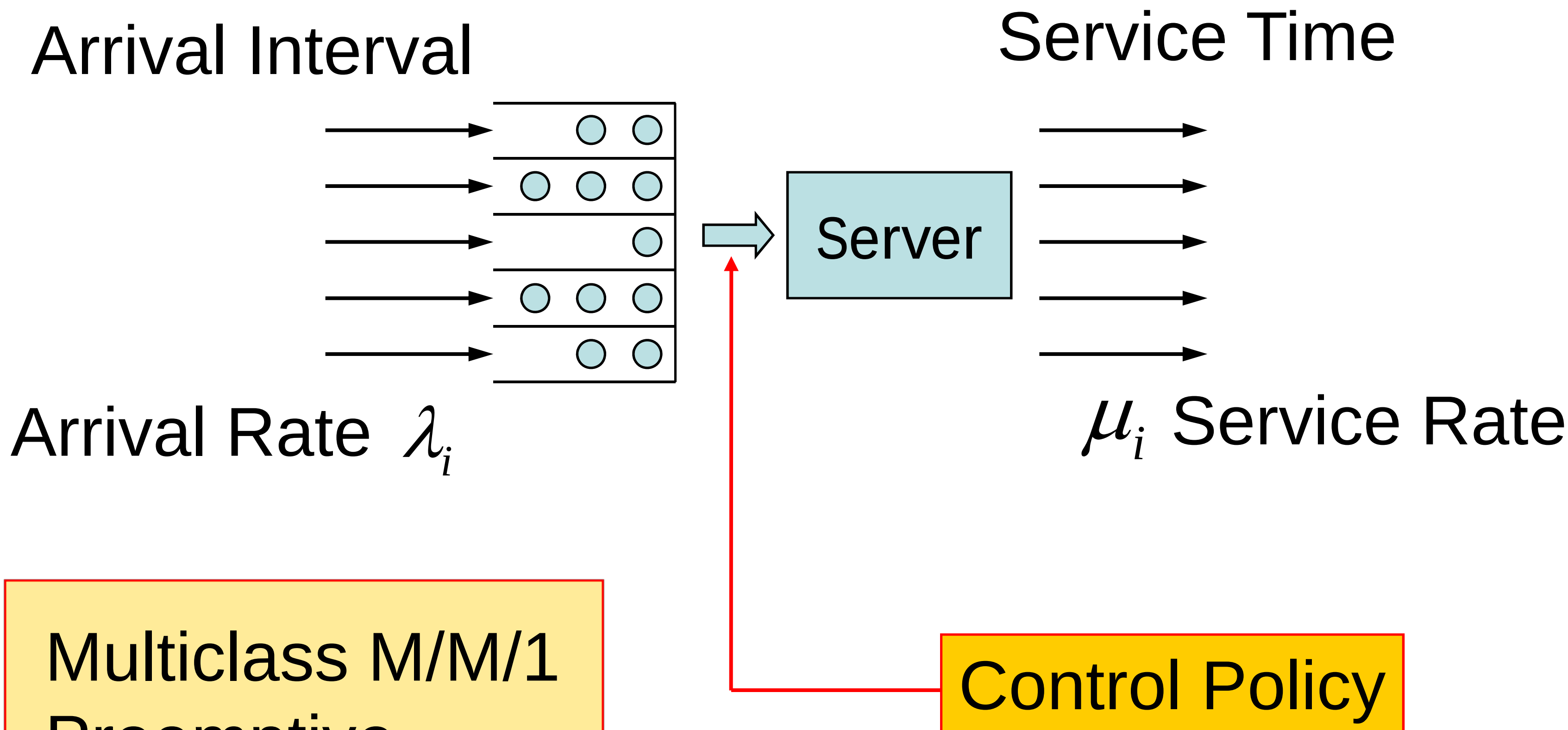
Multiterminal Source Coding



Slepian, Wolf (1973)



Multiclass Queueing Systems



Multiclass M/M/1
Preemptive

Performance Region

s_j : Expected Staying Time of a Job in *j*

S : Achievable

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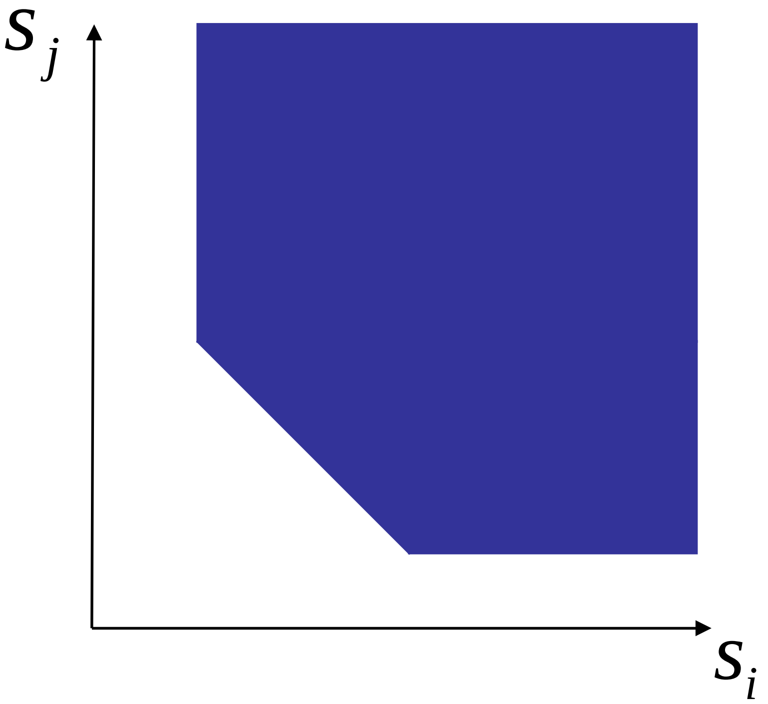
$$\sum_{i\in X}\rho_i S_i \geq \frac{\sum_{i\in X}\rho_i/\mu_i}{1-\sum_{i\in X}\rho_i}, \forall X \subseteq V$$

Coffman, Mitrani (1980)

$$\rho_i := \lambda_i/\mu_i, \quad \sum_{i\in V}\rho_i < 1$$



A Class of Submodular Functions

$x, y, z \in \mathrm{R}_+^V$

Itoko & I. (2005)

$h:$ Nonnegative, Nondecreasing, Convex

$$f(X) = z(X) - y(X)h(x(X)) \quad (X \subseteq V)$$

→ Submodular

$$\sum_{i \in X} \rho_i S_i \geq \frac{\sum_{i \in X} \rho_i / \mu_i}{1 - \sum_{i \in X} \rho_i}, \forall X \subseteq V$$

$$z_i := \rho_i S_i \qquad y_i := \frac{\rho_i}{\mu_i}$$

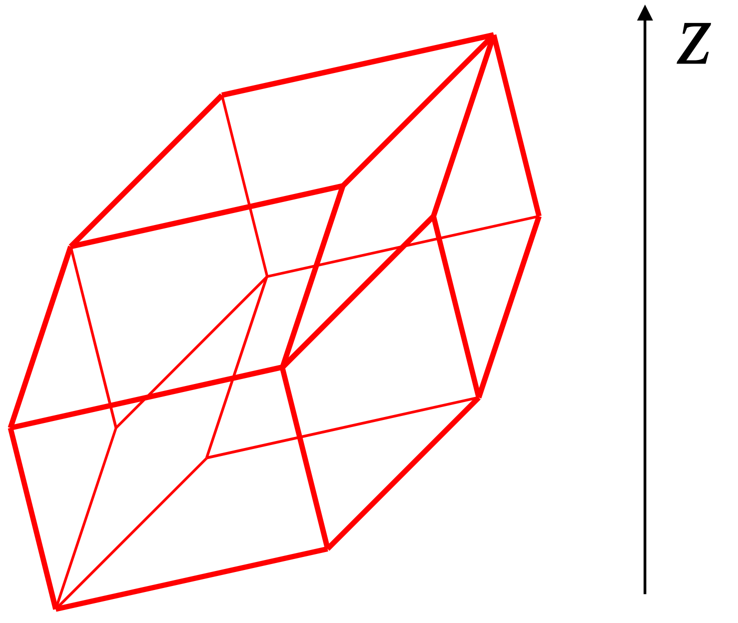
$$x_i := \rho_i \qquad h(x) := \frac{1}{1 - x}$$

Zonotope in 3D

$$w(X) = (x(X), y(X), z(X))$$

$$Z = \mathrm{conv}\{w(X) \mid X \subseteq V\}$$

Zonotope

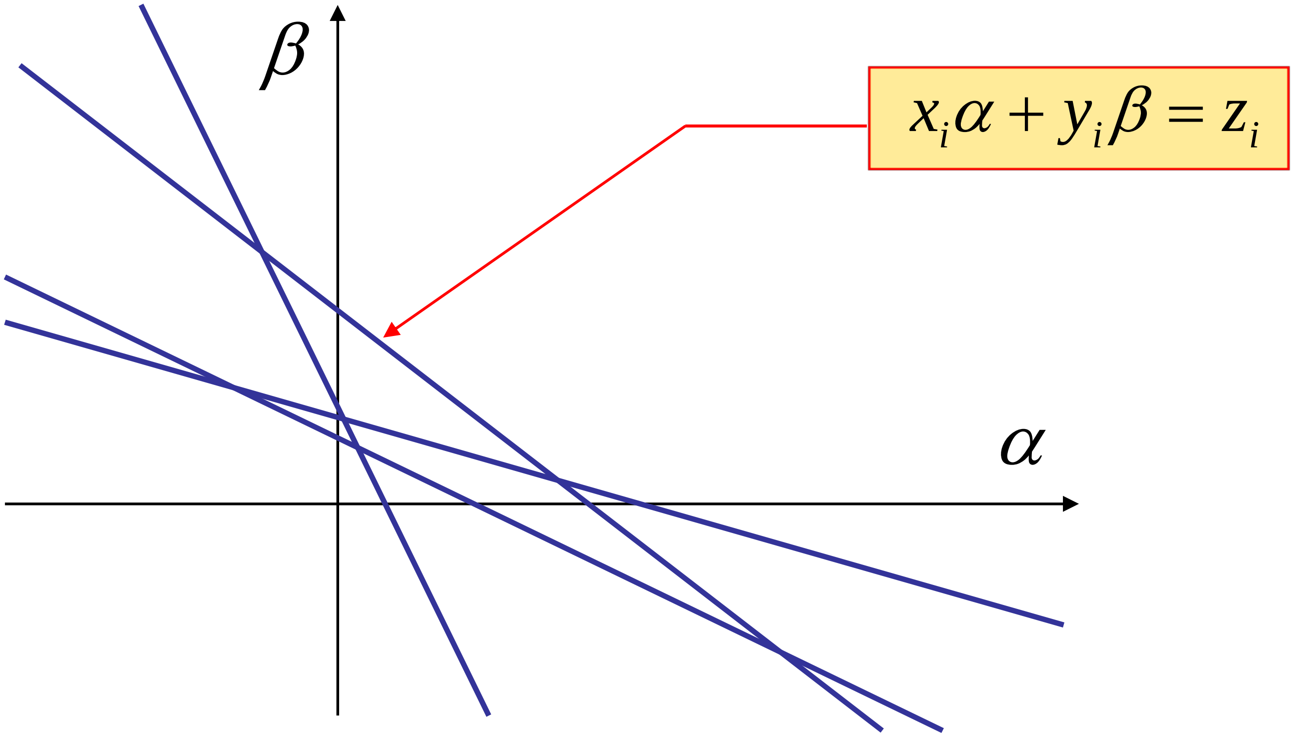


$$\tilde{f}(x, y, z) = z - yh(x)$$

$$\min\{f(X) \mid X \subseteq V\}$$
$$= \min\{\tilde{f}(x, y, z) \mid (x, y, z) : \text{Lower Extreme Point of } Z\}$$

Remark: $\tilde{f}(x, y, z)$ is NOT concave!

Line Arrangement



Enumerating All the Cells

Topological Sweeping Method
Edelsbrunner, Guibas (1989)

$\rightarrow O(n^2)$

Summary

- Submodular Functions Arise Everywhere.
- Discrete Analogue of Convexity.
- General SFM Algorithms Available.
- Exploit Special Structures of Problems.