A Stochastic Control Approach to Financial Tracking Problems

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Outline

- Track a fixed growth rate, track a market index
- Stochastic linear quadratic (SLQ) control
  - indefinite cost matrices
  - solution via semidefinite programming (SDP)
- Semidefinite programming:
  - preliminaries and duality
  - solving SLQ via SDP
- Numerical examples

Tracking a Growth Rate

- m stocks: $S_i(t)$, $i = 1, ..., m$,

$$dS_i(t) = b_i S_i(t)dt + \sum_{j=1}^{m} \sigma_{ij} S_i(t) dW_j(t), \quad S_i(0) = S_{i0},$$

and a riskless asset (bond),

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = S_{00}.$$

- Given a portfolio of n ($n << m$) stocks (out of the m) and a given wealth x_0, want to do dynamic asset allocation among the n stocks and the bond, to follow as closely as possible $x_0 e^{\mu t}$, where $\mu > 0$ is given, over a "long" time horizon.

- The wealth process, $x(\cdot)$, under an admissible control $\pi(\cdot)$, satisfies: $x(0) = x_0$, and

$$dx(t) = \Big\{ rx(t) + \sum_{i=1}^{n} \big[b_i - r \big] \pi_i(t) \Big\} dt + \sum_{j=1}^{m} \sum_{i=1}^{n} \sigma_{ij} \pi_i(t) dW_j(t),$$

- Rewrite:

$$dx(t) = [rx(t) + b^T \pi(t)]dt + \pi^T \sigma_n dW(t), \qquad x(0) = x_0.$$

Our objective is:

$$\min \mathsf{E} \int_0^\infty e^{-2\rho t}[x(t) - x_0 e^{\mu t}]^2 dt,$$

where $2\rho > 0$ is a discount factor.

- Applying a transformation of variables:

$$y(t) := e^{-\rho t}[x(t) - x_0 e^{\mu t}], \qquad \bar{\pi}(t) := e^{-\rho t}\pi(t),$$

we have

$$\begin{array}{ll} \min & \mathsf{E}\int_0^\infty |y(t)|^2 dt \\ \text{s.t.} & dy(t) = \Big\{(r-\rho)y(t) + b^T\bar{\pi}(t) + (r-\mu)x_0 e^{(\mu-\rho)t}\Big\}dt \\ & \qquad +\bar{\pi}(t)^T \sigma_n dW(t), \\ & y(0) = 0. \end{array}$$

- Lemma. If $\rho > \max\{\mu, r - \frac{1}{2}b^T\Sigma_n^+ b\}$, then the above SLQ problem is stabilizable (i.e., there exists a feedback control under which the state process $\lim_{t\to\infty} \mathsf{E}[x(t)^T x(t)] \to 0$.

- To absorb the nonhomogeneous term in the drift part, let $y_0(t) := x_0 e^{(\mu-\rho)t}$. Then,

$$\begin{aligned} &\min \quad \mathsf{E}\int_0^\infty |y(t)|^2 dt \\ &\text{s.t.} \quad dy_0(t) = (\mu-\rho)y_0(t)dt \\ &\qquad dy(t) = \Big\{(r-\mu)y_0(t) + (r-\rho)y(t) + b^T\bar{\pi}(t)\Big\}dt + \bar{\pi}(t)^T\sigma_n dW(t) \\ &\qquad \begin{bmatrix} y_0(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}. \end{aligned}$$

- A canonical formulation of the SLQ problem is as follows:

$$\begin{aligned} \text{(SLQ)} \qquad & \min \mathsf{E}\int_0^\infty [y(t)^T Q y(t) + u(t)^T R u(t)]dt \\ \text{s.t.} \qquad & dy(t) = [Ay(t) + Bu(t) + f(t)]dt \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{k}[C_j y(t) + D_j u(t) + g_j(t)]dW_j(t),\\
&y(0) = y_0.
\end{aligned}
$$

- To relate to the above SLQ problem, we have,

$$
Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n};
$$

$$
A = \begin{bmatrix} \mu-\rho & 0 \\ r-\mu & r-\rho \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ b_1 - r & b_2 - r & \cdots & b_n - r \end{bmatrix}_{2\times n};
$$

$$
C_j = 0, \qquad D_j = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \sigma_{1j} & \sigma_{2j} & \cdots & \sigma_{nj} \end{bmatrix}_{2\times n}, \quad f(t) \equiv 0,\ g_j(t) \equiv 0;
$$

for $j = 1, ..., n$. Note, here Q and R are both singular. Hence, we are in the realm of the SLQ with *indefinite* cost matrices.

Tracking a Market Index

- Let the market index be represented as follows:

$$I(t) = \sum_{j=1}^{m} \alpha_j S_j(t), \qquad I(0) = I_0.$$

Our objective is,

$$\min \mathsf{E} \int_0^\infty e^{-2\rho t}[x(t) - I(t)]^2 dt,$$

subject to the wealth equation. Assume w.l.g. $x(0) = I(0)$.

- If $\rho > \max \left\{ r - \frac{1}{2} b^T \Sigma_n^+ b; \quad b_i + \frac{1}{2} \sum_{j=1}^m \sigma_{ij}^2,\ i = 1, 2, \cdots, m \right\}$, then the SLQ problem is stabilizable.

- Going through similar transforms as before, we have

$$
\begin{aligned}
\min \quad & \mathsf{E} \int_0^\infty [\bar{x}(t) - \bar{I}(t)]^2 dt \\
\text{s.t.} \quad & d\bar{x}(t) = \left\{ (r-\rho)\bar{x}(t) + \sum_{i=1}^{n} \left[b_i - r \right] \bar{\pi}_i(t) \right\} dt + \sum_{j=1}^{m} \sum_{i=1}^{n} \sigma_{ij} \bar{\pi}_i(t) dW_j(t) \\
& d\bar{I}(t) = -\rho \bar{I}(t) dt + \sum_{i=1}^{m} \alpha_i b_i \bar{S}_i(t) dt + \sum_{i=1}^{m} \sum_{j=1}^{m} \alpha_i \sigma_{ij} \bar{S}_i(t) dW_j(t) \\
& d\bar{S}_i(t) = (b_i - \rho)\bar{S}_i(t) dt + \sum_{j=1}^{m} \sigma_{ij} \bar{S}_i(t) dW_j(t), \ i = 1, \dots, m, \\
& (\bar{x}(0), \bar{I}(0), \bar{S}_i(0)) = (x_0, x_0, S_{i0}).
\end{aligned}
$$

- To relate to the canonical form, here the state (vector) is

$$
y(t) := [\bar{x}(t), \bar{I}(t), \bar{S}_1(t), \cdots, \bar{S}_m(t)]^T
$$

and the control (vector) is

$$u(t) := [\bar{\pi}_1(t), \cdots, \bar{\pi}_n(t)]^T.$$

The coefficient matrices are:

$$Q := \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+2)\times(m+2)}$$

$$R := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{n\times n}$$

$$A := \begin{bmatrix} r-\rho & 0 & 0 & \cdots & 0 \\ 0 & -\rho & \alpha_1 b_1 & \cdots & \alpha_m b_m \\ 0 & 0 & b_1-\rho & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b_m-\rho \end{bmatrix}_{(m+2)\times(m+2)}$$

$$B := \begin{bmatrix} b_1-r & b_2-r & \cdots & b_n-r \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+2)\times n}$$

$$C_j := \begin{bmatrix} 0 & 0 & 0 & q \cdots & 0 \\ 0 & 0 & \alpha_1\sigma_{1j} & \cdots & \alpha_m\sigma_{mj} \\ 0 & 0 & \sigma_{1j} & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_{mj} \end{bmatrix}_{(m+2)\times(m+2)}$$

$$D_j := \begin{bmatrix} \sigma_{1j} & \sigma_{2j} & \cdots & \sigma_{nj} \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(m+2)\times n}$$

$$f(t) \equiv 0; \qquad g_j(t) \equiv 0.$$

where $j = 1, ..., n$. Once again, in this case Q and R are both singular.

Classical LQ Theory

- Obtain P from the Riccati equation:

$$A^T P + PA + Q + C^T PC - (PB + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T PC) = 0$$

- Then,

$$u^*(t) = -(R + D^T P^* D)^{-1}(B^T P^* + D^T P^* C)x^*(t)$$

is the optimal control.

- Requires: $R + D^T P^* D \succ 0$; guaranteed only when

$$Q \succeq 0, \qquad R \succ 0$$

Motivation and SDP Preliminaries

- Recall, we need to solve the Riccati equation:

$$A^T P + PA + Q + C^T PC - (PB + C^T PD)(R + D^T PD)^{-1}(B^T P + D^T PC) = 0$$

- To illustrate the basic idea, consider the quadratic equation:

$$ax^2 + 2bx + c = 0.$$

One way to solve the quadratic equation is to rewrite it as an optimization problem:

$$\begin{array}{ll} \text{maximize} & x \\ \text{subject to} & -ax^2 - 2bx - c \geq 0 \end{array}$$

(i.e., subject to $(x - x_1)(x - x_2) \leq 0$); and

$$\begin{array}{ll} \text{minimize} & x \\ \text{subject to} & -ax^2 - 2bx - c \geq 0 \end{array}$$

- The above problems are equivalent to:

$$\begin{array}{ll} \text{optimize} & x \\ \text{subject to} & \begin{bmatrix} -2bx - c & x \\ x & a^{-1} \end{bmatrix} \succeq 0 \end{array}$$

- The same idea applies to the matrix equation:

$$X^T A X + (X^T B^T + BX) + C = 0$$

where A is positive definite and C symmetric. We can solve an

SDP problem:

$$\begin{array}{ll} \text{optimize} & \operatorname{tr} X \\ \text{subject to} & \begin{bmatrix} -XB^T - BX - C & X \\ X & A^{-1} \end{bmatrix} \succeq 0 \end{array}$$

- As an example, consider

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},\ C = \begin{bmatrix} -2 & 1 \\ 1 & -3 \end{bmatrix}.$$

Use an SDP solver, let “optimize” be “maximize” and “minimize” respectively. Two solutions appear:

$$X = \begin{bmatrix} 0.6052 & -0.2249 \\ -0.2249 & 0.8166 \end{bmatrix}$$

for the maximization problem and

$$X = \begin{bmatrix} -5.2808 & 3.1844 \\ 3.1844 & -2.9788 \end{bmatrix}$$

for the minimization, and they indeed solve the quadratic matrix equation!

The SDP Approach

- Allowing indefinite Q and R, we introduce

$$F(P) = A^T P + PA + Q + C^T PC - (PB + C^T PD)(R + D^T PD)^+ (B^T P + D^T PC)$$

where M^+ denotes the pseudo-inverse; we want to solve the *generalized* Riccati equation: $F(P) = 0$.

- Consider the following SDP:

$$\begin{array}{lll} \text{(P)} & \max & \langle I, P \rangle \\ & \text{s.t.} & \mathcal{L}(P) \succeq 0 \\ & & P \in \mathcal{S}^{n \times n}, \end{array}$$

where

$$\mathcal{L}(P) := \begin{bmatrix} R + D^T PD, & B^T P + D^T PC \\ PB + C^T PD, & Q + C^T PC + A^T P + PA \end{bmatrix}.$$

- The dual of (P) is

$$
\begin{array}{lll}
\text{(D)} & \min & \langle R, Z_B\rangle + \langle Q, Z_N\rangle \\
 & \text{s.t.} & I + Z_U^T B^T + BZ_U + Z_N A^T + AZ_N \\
 & & +DZ_U C^T + CZ_U^T D^T + CZ_N C^T + DZ_B D^T = 0 \\
 & & \begin{bmatrix} Z_B, & Z_U \\ Z_U^T, & Z_N \end{bmatrix} \succeq 0.
\end{array}
$$

- Let P^* and Z^* be the primal and dual solutions. The optimal control to the SLQ problem is:

$$u^*(t, P^*) = -(R + D^T P^* D)^+ (B^T P^* + D^T P^* C) x^*(t)$$

and $u^*(t, Z^*) = Z_U^* (Z_N^*)^{-1} x^*(t)$

SDP Duality

- *Weak duality* always holds, i.e. any feasible solution to the dual (minimization) problem provides an upper bound to the primal (maximization) problem.

- *Strong duality* — that the optimal values of the primal and dual problems coincide — holds if there exists a pair of *complementary optimal solutions*, i.e., $\mathcal{L}(P^*)Z^* = 0$. If, furthermore, $\mathcal{L}(P^*) + Z^* \succ 0$, then the pair is called *strictly complementary*.

- Strong duality holds, if (P) is feasible and (D) satisfies the Slater condition ($Z^* \succ 0$).

Theoretical Results

- Consider the two controls:

$$u^*(t, P^*) = -(R + D^T P^* D)^+ (B^T P^* + D^T P^* C) x^*(t)$$

and $u^*(t, Z^*) = Z_U^* (Z_N^*)^{-1} x^*(t)$

- Questions:
  – Are the controls $u^*(\cdot, P^*)$ and $u^*(\cdot, Z^*)$ stabilizing, i.e., ensuring $\lim_{t\to\infty} \mathsf{E}[x^*(t)^T x^*(t)] = 0$?
  – Does the primal SDP solution satisfy the generalized Riccati equation: $F(P^*) = 0$?
  – Does (SLQ) have an attainable optimal control (i.e., a control that achieves a finite optimum of the cost functional)?

Theoretical Results

- Introduce the following statements:

  (a) (LQ) is attainable at any $x_0 \in \Re^n$.
  (b) (P) has an optimal solution P^* satisfying:
  (i) the generalized stochastic Riccati equation $F(P^*) = 0$;
  (ii) $u^*(t, P^*)$ is stabilizing.
  (c) (P) and (D) have complementary optimal solutions P^* and Z^*, with $Z_N^* \succ 0$.

- The following implications hold:

  (a) $\Rightarrow$ **(b(i))**.
  (b) $\Rightarrow$ **(a)**, with Control $u^*(\cdot, P^*)$ being optimal.
  (b) $\Rightarrow$ **(c)**.
  (c) $\Rightarrow$ **(a)**, with Control $u^*(\cdot, Z^*)$ being optimal.

- While **(b)** $\Rightarrow$ **(c)**, we only have **(c)** $\Rightarrow$ **(b (i))** via **(a)**. That is, **(c)** does not imply that the control $u^*(\cdot, P^*)$ is stabilizing, suggesting that it can be quite different from the other control $u^*(\cdot, Z^*)$.

- When $u^*(\cdot, P^*)$ is stabilizing, the three statements **(a)**, **(b)** and **(c)** are equivalent.

- Even when $u^*(\cdot, P^*)$ is stabilizing, and hence both controls are optimal, the two controls can still be different.

- If $R + D^T P^* D \succ 0$, then [under **(c)**] the two controls coincide; and **(b)** and **(c)** become equivalent.

- When $Q \succ 0$ and $R \succ 0$, all three statements hold and are equivalent.

References

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- David D. Yao, Shuzhong Zhang and Xun Yu Zhou, Stochastic LQ control via Semidefinite Programming, *SIAM Journal on Control and Optimization*, **40** (2001) 801-823.

- David D. Yao, Shuzhong Zhang and Xun Yu Zhou, Stochastic LQ control via Primal-Dual Semidefinite Programming, *SIAM Review*, **46** (2004) 85-111.

Numerical Studies

- Choice of discount factor: insensitive (as long as stability is maintained)

- Frequency of portfolio adjustment: daily, weekly, monthly, ...

- Quality/robustness of the solution:
  - different portfolios: large/medium/small cap's
  - different market conditions: up, down, volatile ...
  - amount of borrowing