Pricing insurance contracts: an incomplete market approach

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Setting the stage

- Arbitrage-free pricing: standard method for financial markets
- Assume complete market
  - Every payoff can be replicated
  - Every risk is hedgeable

- Insurance "markets" are *incomplete*
  - Insurance risks are un-hedgeable
  - But, financial risks are hedgeable

Aim of This Presentation

- Find pricing rule for insurance contracts consistent with arbitrage-free pricing
  - "Market-consistent valuation"
  - Basic workhorse: utility functions
  - Price = BE Repl. Portfolio + MVM

Outline

- Recap of Arbitrage-free pricing
- Utility Functions
- Optimal Wealth
- Principle of Equivalent Utility
- Pricing Insurance Contracts

Arbitrage-free Pricing-Formulas

- "Risk-neutral" formula:

  $$f_0 = e^{-r}\,[\pi^* f_u + (1-\pi^*)\, f_d\,]$$

  $$\pi^* = (e^r - d)/(u-d);\ 1-\pi^* = (u-e^r)/(u-d)$$

- $f_0 = e^{-r}\, E^Q[\, f_1\,]$

- Deflator formula:

  $$f_0 = [\, p\, f_u\, R_u + (1-p)\, f_d\, R_d\,]$$

  $$R_u = e^{-r}\,\pi^*/p;\ R_d = e^{-r}\,(1-\pi^*)/(1-p)$$

- $f_0 = E^P[\, f_1\, R_1\,]$

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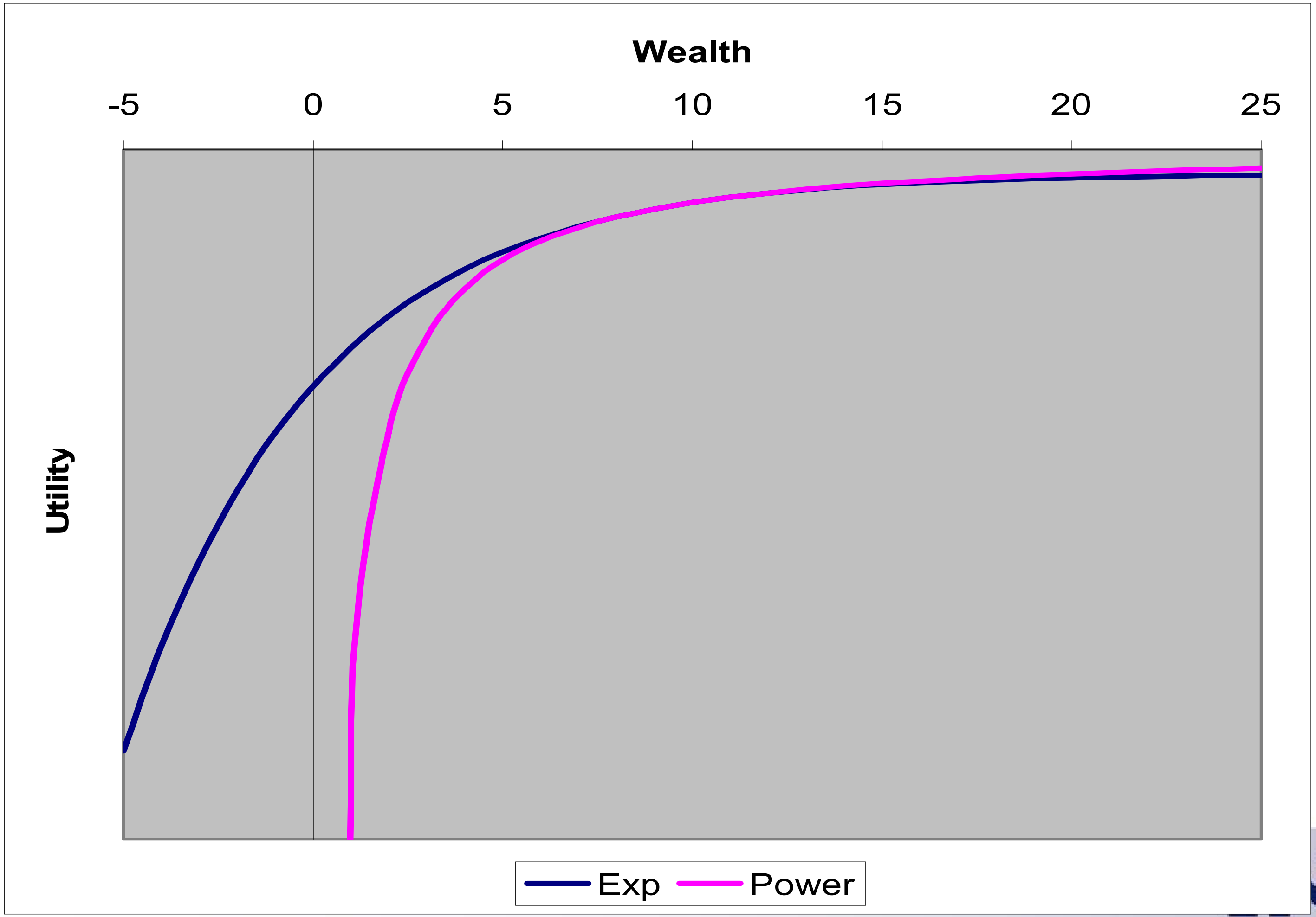
Utility Functions

- Economic agent has to make decisions under uncertainty
- Choose “optimal” investment strategy for wealth W_0
- Decision today $\Rightarrow$ uncertain wealth W_T
- Make a trade-off between gains and losses

Utility Functions - Examples

- **Exponential utility:**
  - $U(W) = -e^{-aW} / a$
  - $RA = a$
- **Power utility:**
  - $U(W) = W^{(1-b)} / (1-b)$
  - $RA = b/W$



Optimal Wealth

- Maximise expected utility $E^P[U(W_T)]$
- By choosing optimal wealth W_T

$$\max_{W_T} \mathbf{E}^{\mathbf{P}}\left[U(W_T)\right]$$
$$\text{s.t. } \mathbf{E}^{\mathbf{P}}\left[W_T R_T\right] = W_0$$

- First order condition for optimum
  - $U'(W_T^*) = \lambda R_T$
- Solution: $W_T^* = (U')^{-1}(\lambda R_T)$
  - Intuition: buy "cheap" states & spread risk
  - Solve λ from budget constraint

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Optimal Wealth - Example

- Black-Scholes economy
- Exp Utility: $U(W) = -e^{-aW} / a$
- Condition for optimal wealth:
  - $U'(W_T^*) = \lambda R_T$
  - $\exp\{-aW_T^*\} = \lambda\, C\, S_T^{-\theta}$
  - $\theta=(\mu-r)/\sigma^2$ (market price of risk)
- Optimal wealth: $W_T^* = C^* + \theta/a \ln(S_T)$
  - Solve C^* from budget constraint

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Principle of Equivalent Utility

- Economic agent thinks about selling a (hedgeable) financial risk H_T
  - Wealth at time $T = W_T - H_T$

- What price π_0 should agent ask?
- Agent will be *indifferent* if expected utility is unchanged:

$$\max_{W_T^*} \mathbf{E}^{\mathbf{P}}\left[U(W_T^*)\right] = \max_{W_T^{*\pi},\pi_0} \mathbf{E}^{\mathbf{P}}\left[U(W_T^{*\pi} - H_T)\right]$$

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Principle of Equivalent Utility (2)

- Principle of Equivalent Utility is consistent with arbitrage-free pricing for *hedgeable* claims
- Principle of Equivalent Utility can also be applied to insurance claims
  - Mixture of financial & insurance risk
  - Mixture of hedgeable & un-hedgeable risk
- Literature:
  - Musiela & Zariphopoulou
  - V. Young

Pricing Insurance Contracts

- Assume insurance claim: $H_T I_T$,
- Hedgeable risk H_T
  - cash amount C
  - stock-price S_T
- Insurance risk I_T
  - number of policyholders alive at time T
  - 1 / 0 if car-accident does (not) happen
  - salary development of employee

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Pricing Insurance Contracts

- **Apply principle of Equivalent Utility**

$$\max_{W_T^*} \mathbf{E}^{\mathbf{P}}\left[U(W_T^*)\right] = \max_{W_T^{*\pi}, \pi_0} \mathbf{E}^{\mathbf{P}}\left[U(W_T^{*\pi} - H_T I_T)\right]$$

- **Find price π_0 via solving "right-hand" optimisation problem**

$$\max_{W_T^{*\pi}} \mathbf{E}^{\mathbf{P}}\left[U(W_T^{*\pi} - H_T I_T)\right]$$

$$\text{s.t. } \mathbf{E}^{\mathbf{P}}\left[W_T^{*\pi} R_T\right] = W_0 + \pi_0$$

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Pricing Insurance Contracts (2)

- Consider specific example: life insurance contract
  - Portfolio of N policyholders
  - Survival probability p until time T
  - Pay each survivor the cash amount C
- Payoff at time $T = Cn$
  - n is (multinomial) random variable
  - $E[n] = Np$
  - $Var[n] = Np(1-p)$

Pricing Insurance Contracts (3)

- Solve optimisation problem:

$$\max_{W_T^{*\pi}} \mathbf{E}^{\mathrm{P}}\left[U(W_T^{*\pi} - Cn)\right]$$

$$\text{s.t. } \mathbf{E}^{\mathrm{P}}\left[W_T^{*\pi} R_T\right] = W_0 + \pi_0$$

- Mixture of random variables
  - *W,R* are “financial” risks
  - *n* is “insurance” risk
- “Adjusted” F.O. condition

Pricing Insurance Contracts (4)

- **Necessary condition for optimal wealth:**

$$\mathbf{E}^{\mathbf{PU}}\left[U'(W_T^{*\pi} - Cn)\right] = \lambda R_T$$

  - **E^{PU}[ ] does not affect "financial" W_T**
  - **E^{PU}[ ] only affects "insurance" risk *n***
- **Exponential utility:**
  - $U'(W\text{-}Cn) = e^{-a(W-Cn)} = e^{-aW} e^{aCn}$

$$\mathbf{E}^{\mathbf{PU}}[e^{-a\left(W_T^{*\pi} - Cn\right)}] = e^{-aW_T^{*\pi}}\mathbf{E}^{\mathbf{PU}}[e^{aCn}] = \lambda R_T$$

Pricing Insurance Contracts (5)

- For large N: $n \rightarrow n(\, Np \,,\, Np(1-p) \,)$
  - MGF of $z \sim n(m,V)$: $E[e^{tz}] = \exp\{\, t\,m + \tfrac{1}{2}t^2 V \,\}$
- Hence:
  - $E^{PU}[\, e^{aCn} \,] = \exp\{\, aC\,Np + \tfrac{1}{2}(aC)^2\, Np(1-p) \,\}$
- Equation for optimal wealth:

$$e^{-aW_T^{*\pi}}\, e^{aCNp+\frac{1}{2}a^2C^2Np(1-p)} = \lambda R_T$$

Pricing Insurance Contracts (6)

- Solution for optimal wealth:

$$W_T^{*\pi} = -\frac{1}{a}\ln(\lambda R_T) + \left(CNp + \frac{1}{2}aC^2Np(1-p)\right)$$

$\underbrace{-\frac{1}{a}\ln(\lambda R_T)}_{\text{Surplus } W^*}$ + $\big(\underbrace{CNp}_{\text{BE}} + \underbrace{\tfrac{1}{2}aC^2Np(1-p)}_{\text{MVM}}\big)$

- Price of insurance contract:
  - $\pi_0 = e^{-rT}(CNp + \frac{1}{2}aC^2Np(1-p))$
  - "Variance Principle" from actuarial lit.

Observations on MVM

- MVM $\propto$ ½ Risk-av * Var(Unhedg. Risk)
  - Note: MVM for "binomial" risk!
  - "Diversified" variance of *all* unhedgeable risks

- Price for *N*+1 contracts:
  - $e^{-rT}(C(N+1)p + \frac{1}{2}aC^2(N+1)p(1-p))$
- Price for extra contract:
  - $e^{-rT}(Cp + \frac{1}{2}aC^2p(1-p))$

Observations on MVM (2)

- Sell 1 contract that pays C if policyholder N dies
  - Death benefit to N's widow
  - Certain payment C + (N-1) uncertain
  - Price: $e^{-rT}(C + C(N-1)p + \frac{1}{2}aC^2(N-1)p(1-p))$
- Price for extra contract:
  - $e^{-rT}(C(1-p) - \frac{1}{2}aC^2p(1-p))$
  - BE + negative MVM!
  - Give bonus for *diversification benefit*

Incremental Pricing of Insurance

- Existing liability portfolio *L*
  - Rep = $BE_L + \frac{1}{2} RA\, \sigma_L^2$
  - σ_L^2 is variance of unhedgeable risk of *L*
- Additional claim *C*
  - Correlation ρ between unhedg. risks *L*,*C*
  - Rep = $(BE_L + BE_C) + \frac{1}{2} RA\, (\sigma_L^2 + 2\rho\sigma_L\sigma_C + \sigma_C^2)$
- Rep for contract *C*:
  - $BE_C + \frac{1}{2} RA\, (2\rho\sigma_L\sigma_C + \sigma_C^2)$
  - MVM depends on *existing* portfolio *L*

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