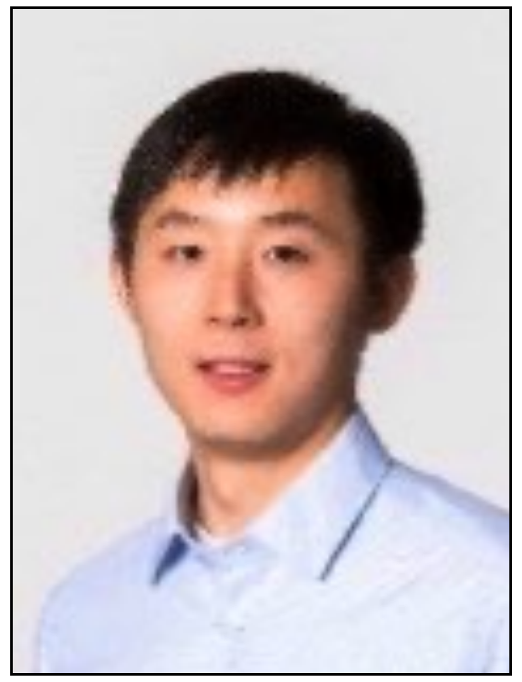
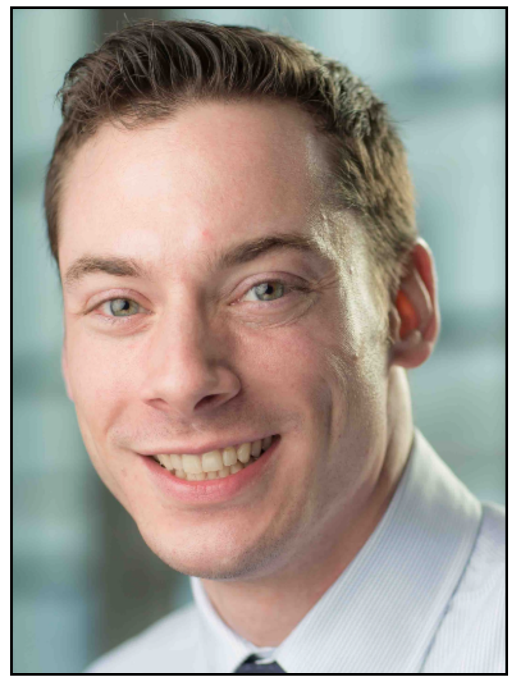
# Randomized Assortment Optimization



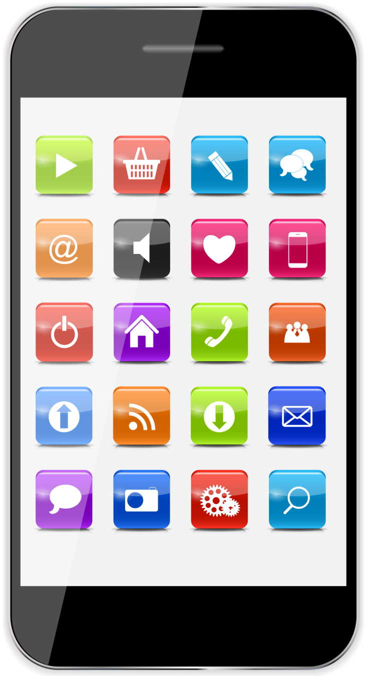


Zhengchao Wang Heikki Peura Wolfram Wiesemann

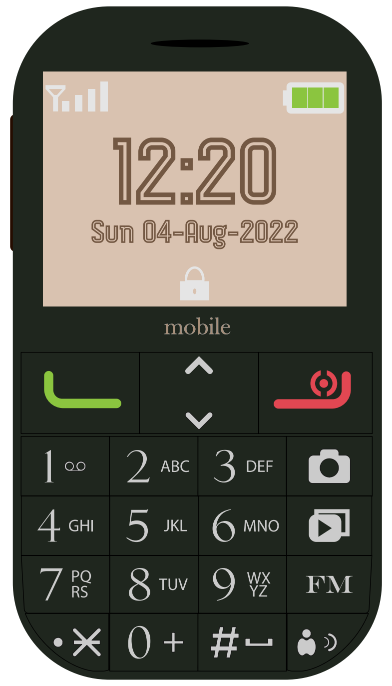
**Department of Analytics, Marketing & Operations**
**Imperial College Business School**

# The Assortment Optimization Problem

£3 profit



£10 profit

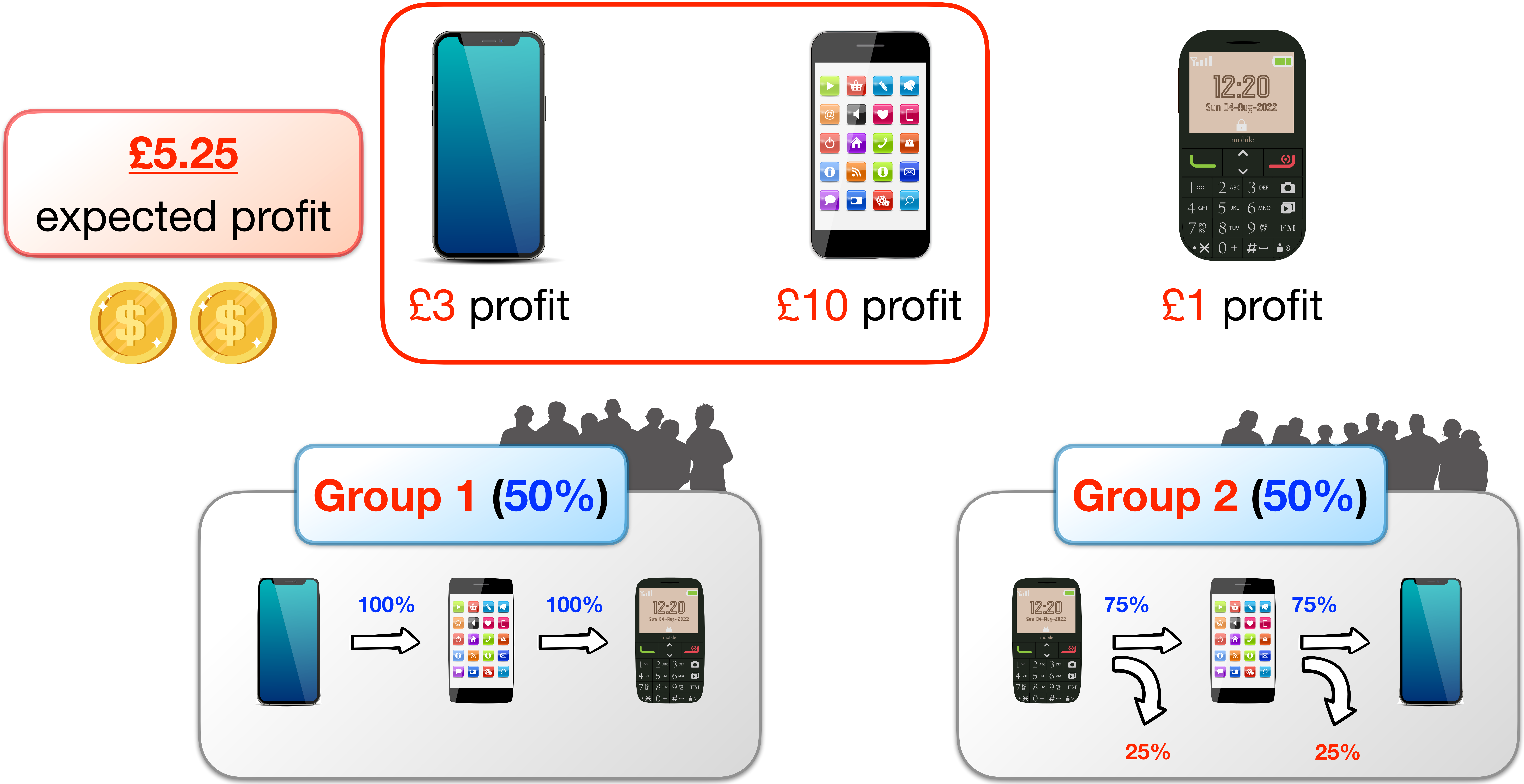



£1 profit

# The Assortment Optimization Problem


£3 profit | £10 profit | £1 profit


# The Assortment Optimization Problem



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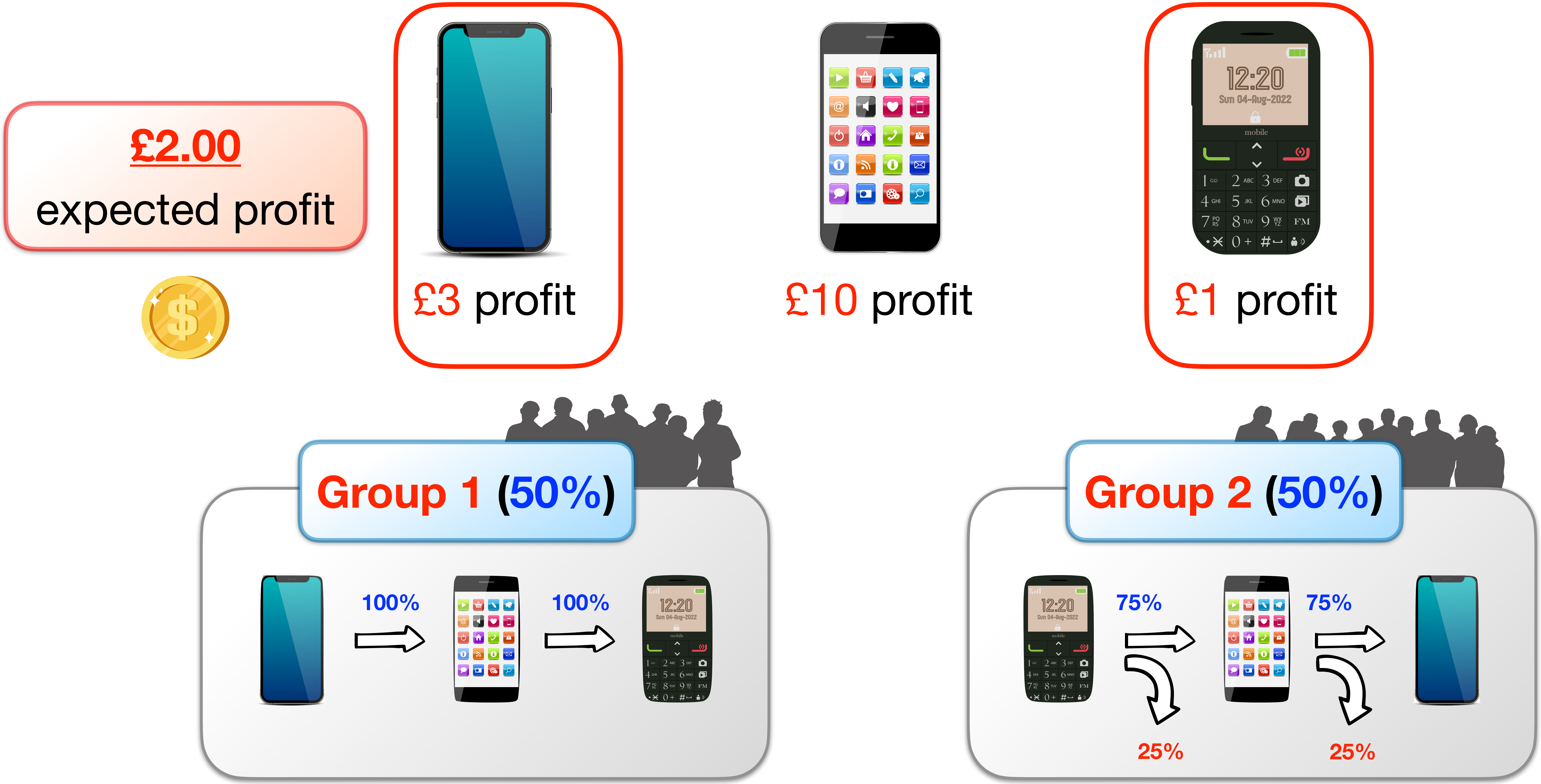
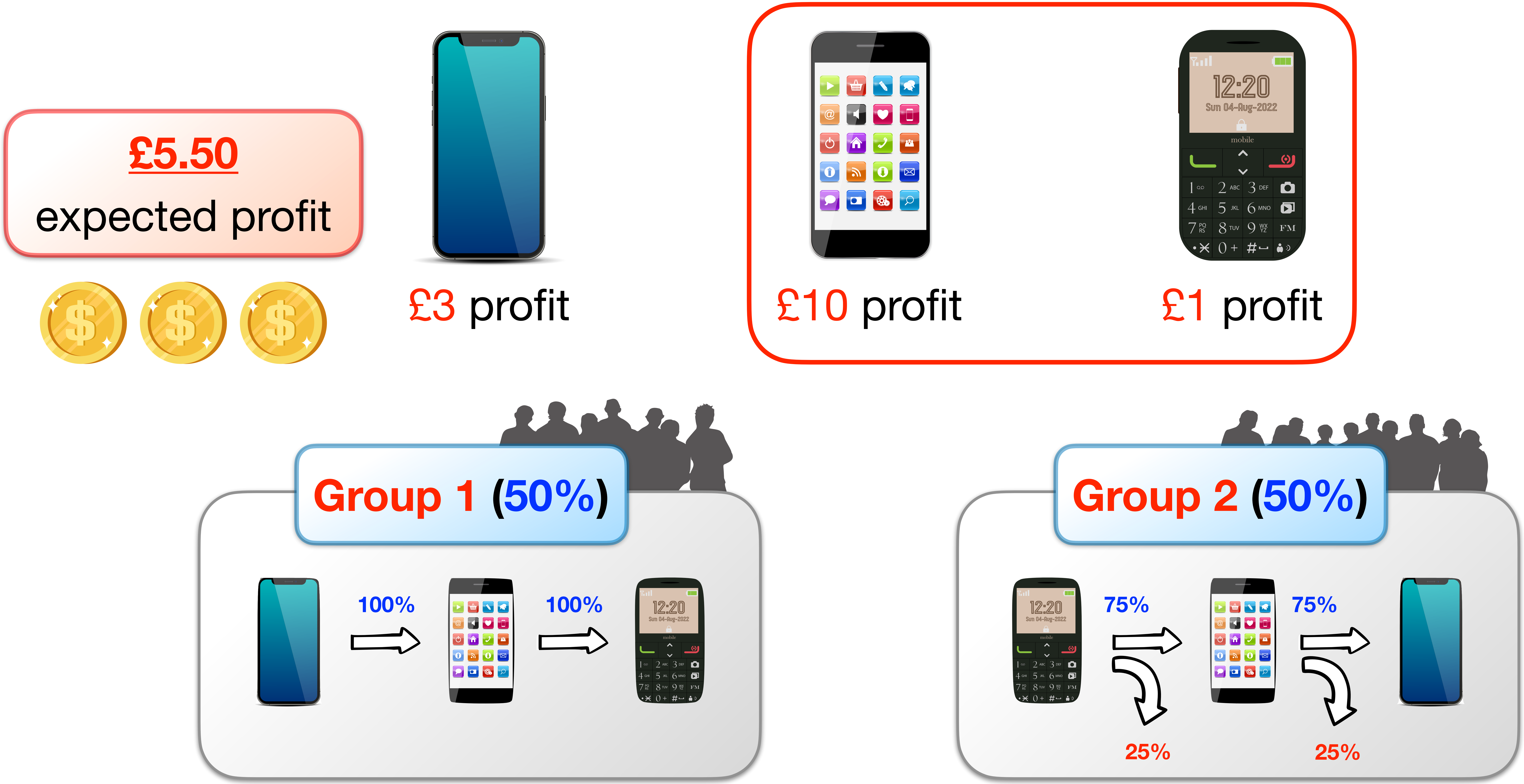


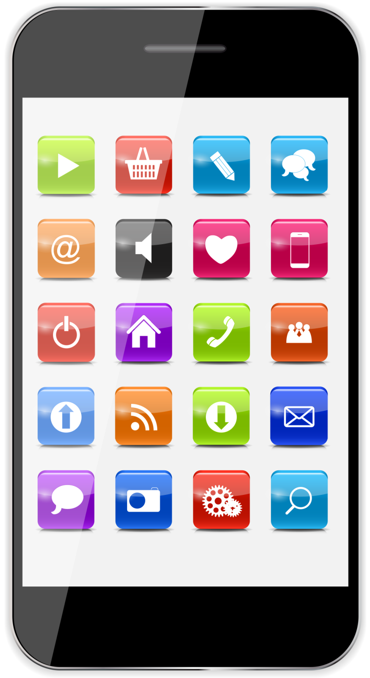
# The Assortment Optimization Problem



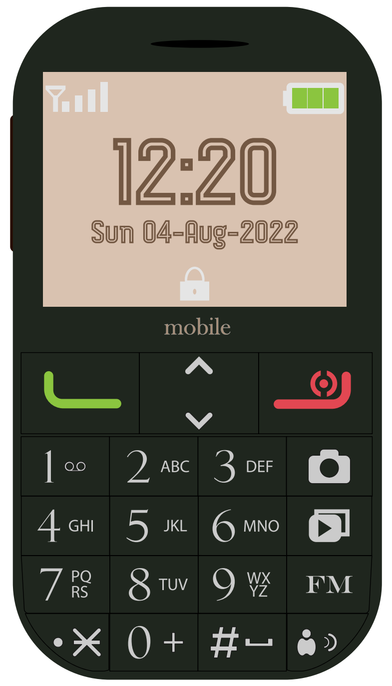
- **Multinomial logit model:**
  - Choice model: Luce ('59), Plackett ('75)
  - Optimization: Talluri & van Ryzin ('04), Rusmevichientong et al. ('10) and Davis et al. ('13) for cardinality constraints
- **Markov chain model:**
  - Choice model: Zhang & Cooper ('05), Blanchet et al. ('16), Simsek & Topaloglu ('18) for estimation
  - Optimization: Blanchet et al. ('16), Feldman & Topaloglu ('17), Désir et al. ('20) for cardinality constraints
- **Preference ranking model:**
  - Choice model: Farias et al. ('13), van Ryzin & Volcano ('15, '17)
  - Optimization: Honhon et al. ('12), Aouad et al. ('18, '21), Paul et al. ('18), Bertsimas & Mišić ('19)

# The Assortment Optimization Problem

£3 profit



£10 profit




£1 profit

**Group 1 (50%)**

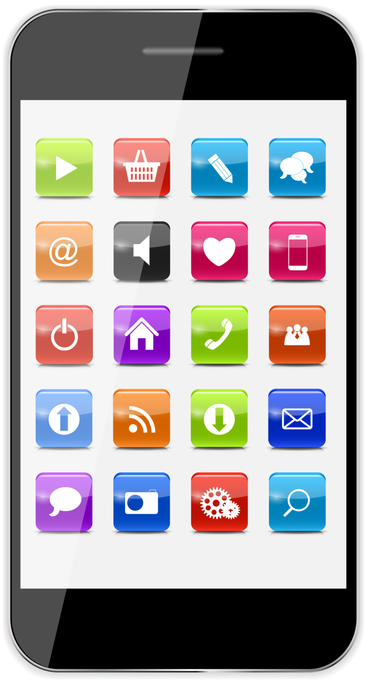
**Group 2 (50%)**

**£5.50**

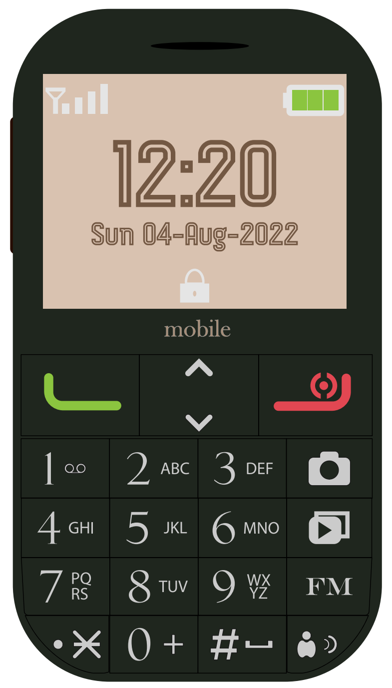
exp. profit

# The Assortment Optimization Problem

£3 profit



£10 profit




£1 profit

Group 1 (50%)

Group 2 (50%)

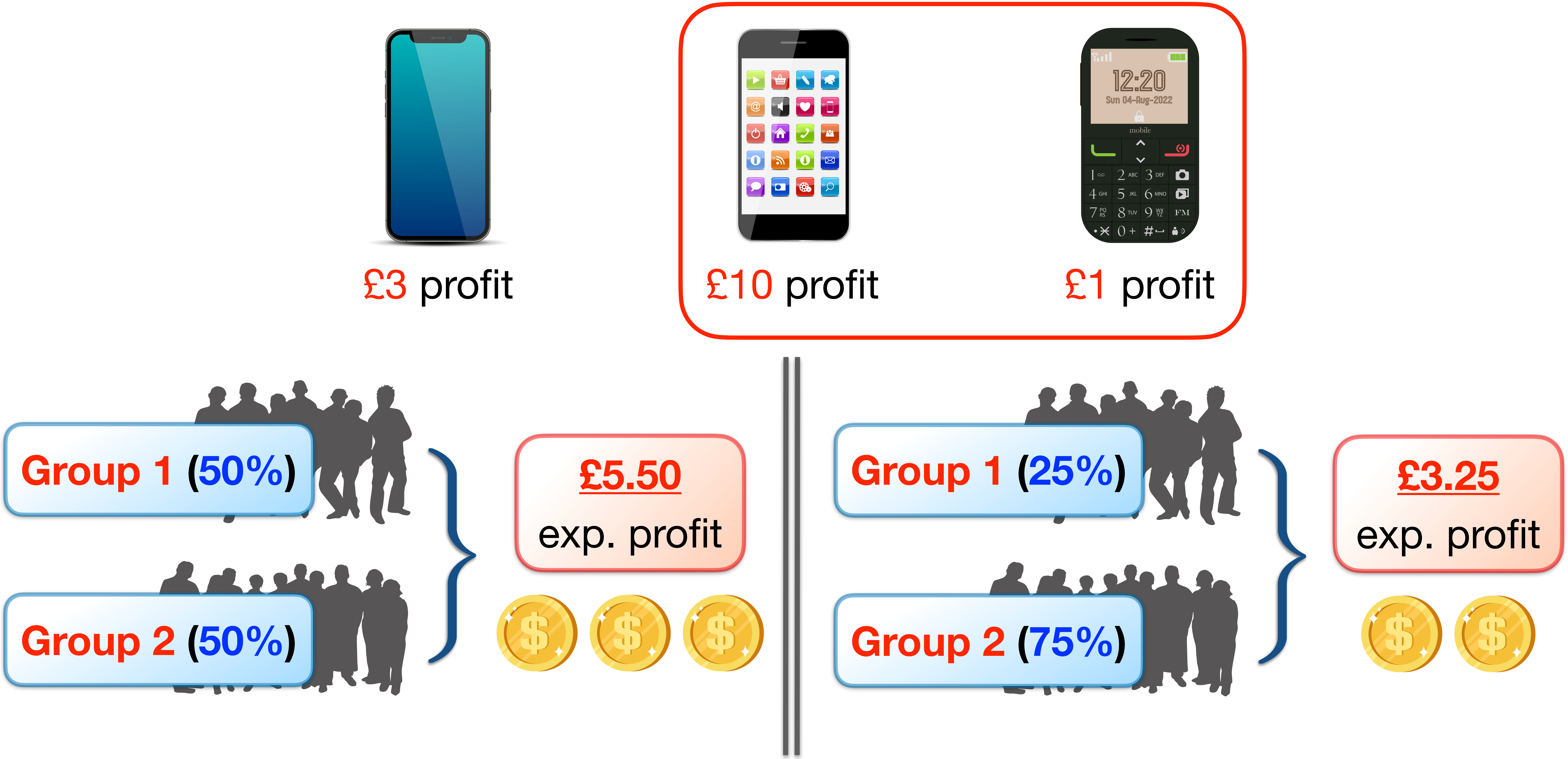
**£5.50**

exp. profit

Group 1 (25%)

Group 2 (75%)

# The Assortment Optimization Problem

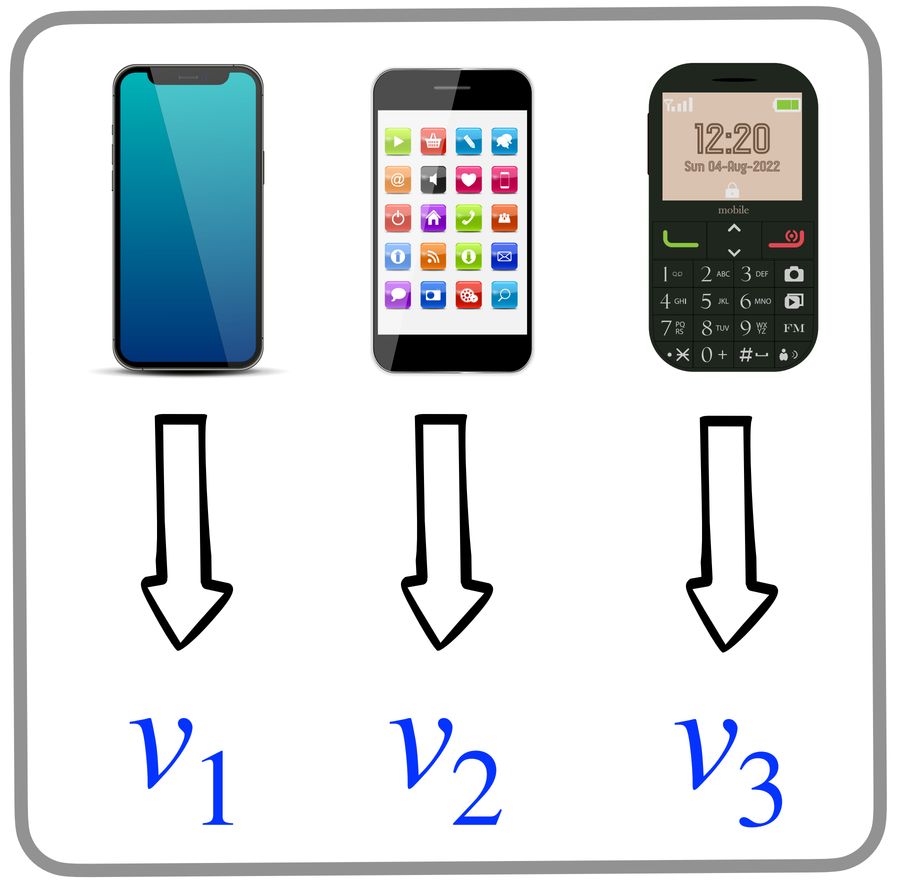


The bias-variance tradeoff in choice models:

## The bias-variance tradeoff in choice models:

**MNL model**

- large bias
- small variance



# The bias-variance tradeoff in choice models:

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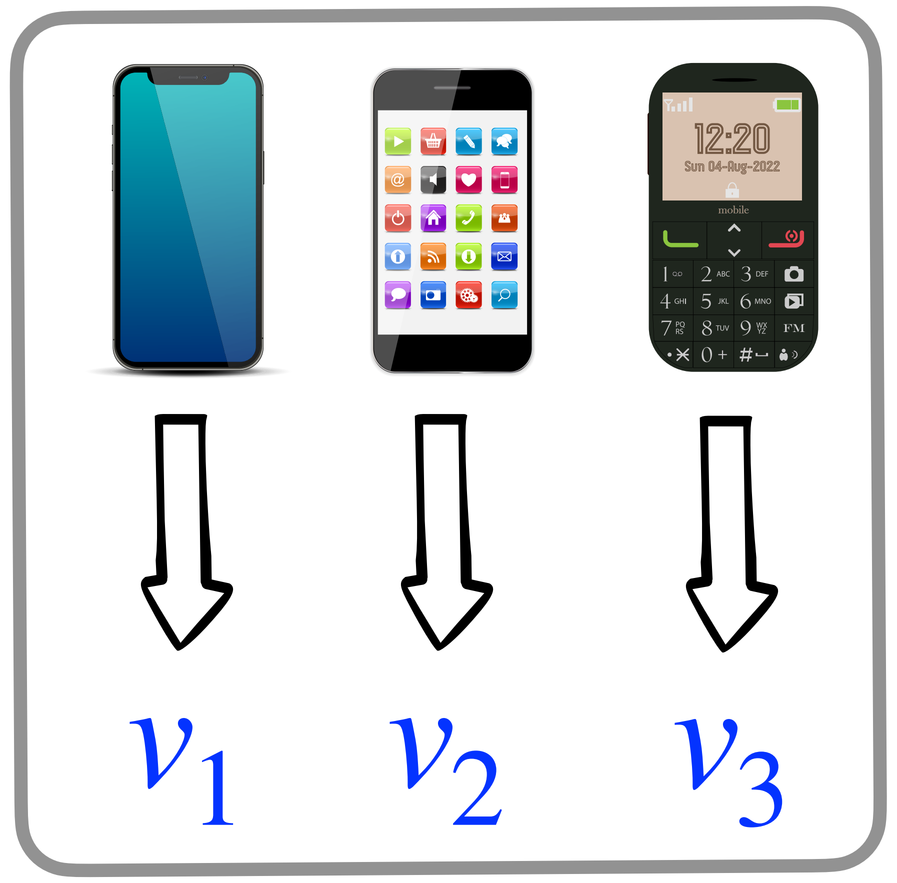
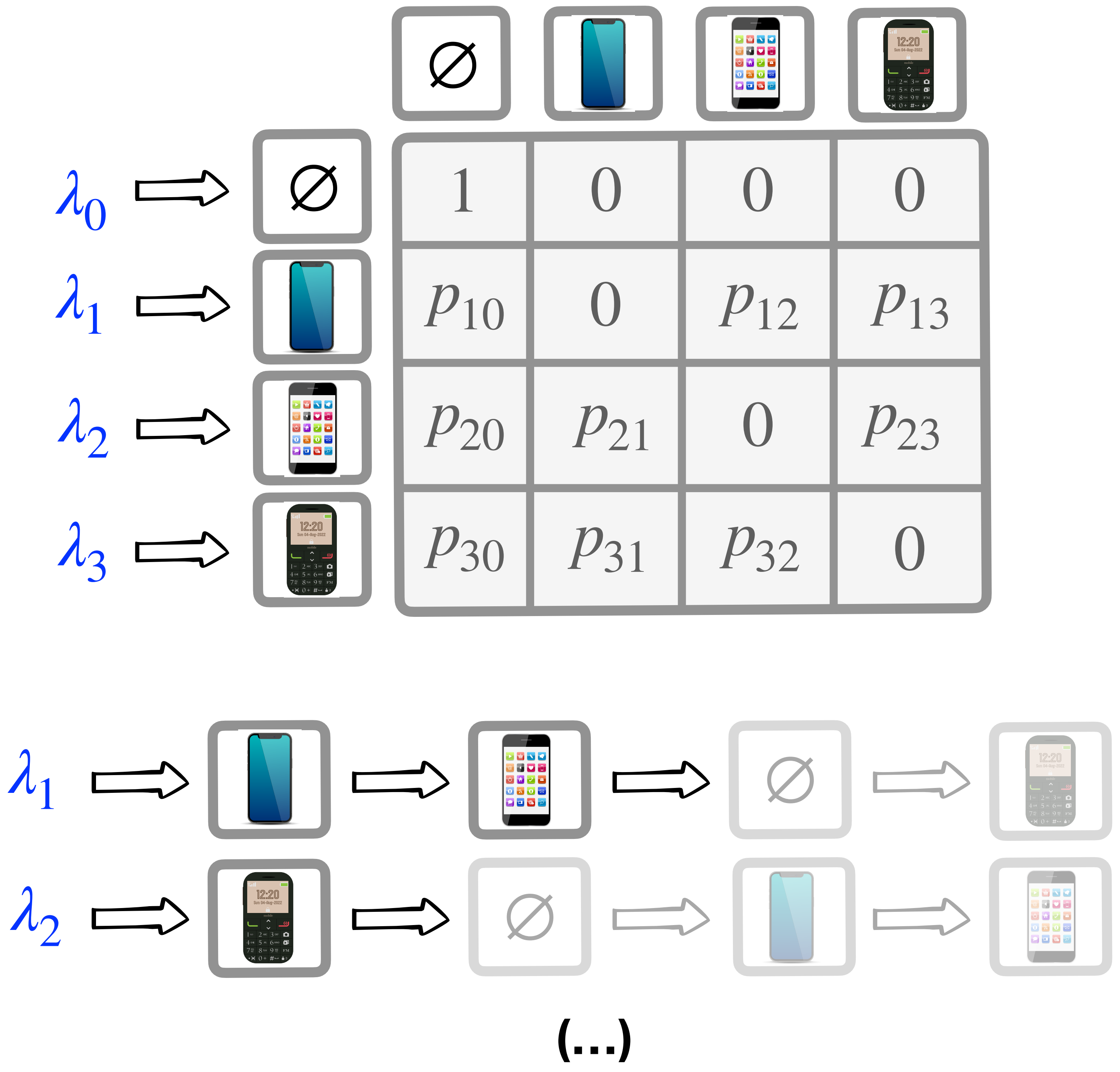



## MC/PR models

- small bias
- large variance



## The bias-variance tradeoff in choice models:

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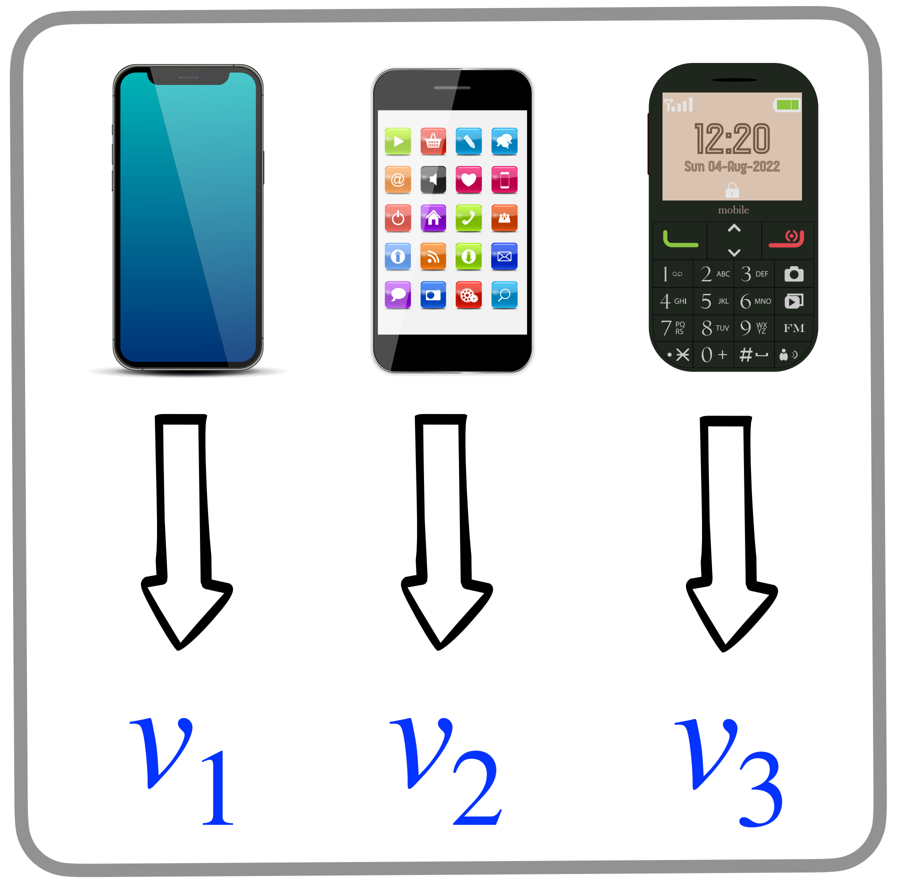


**MC/PR models**

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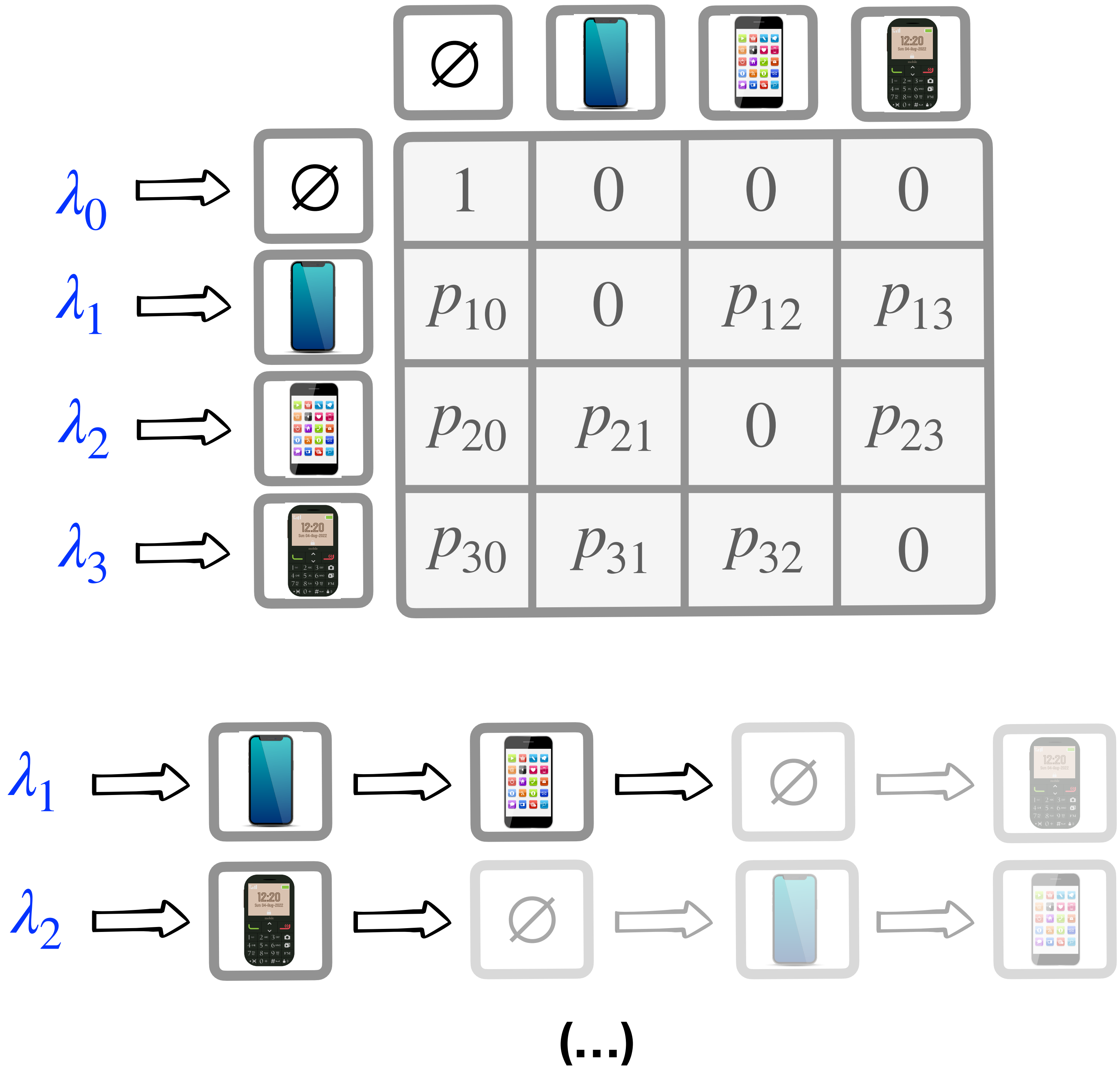
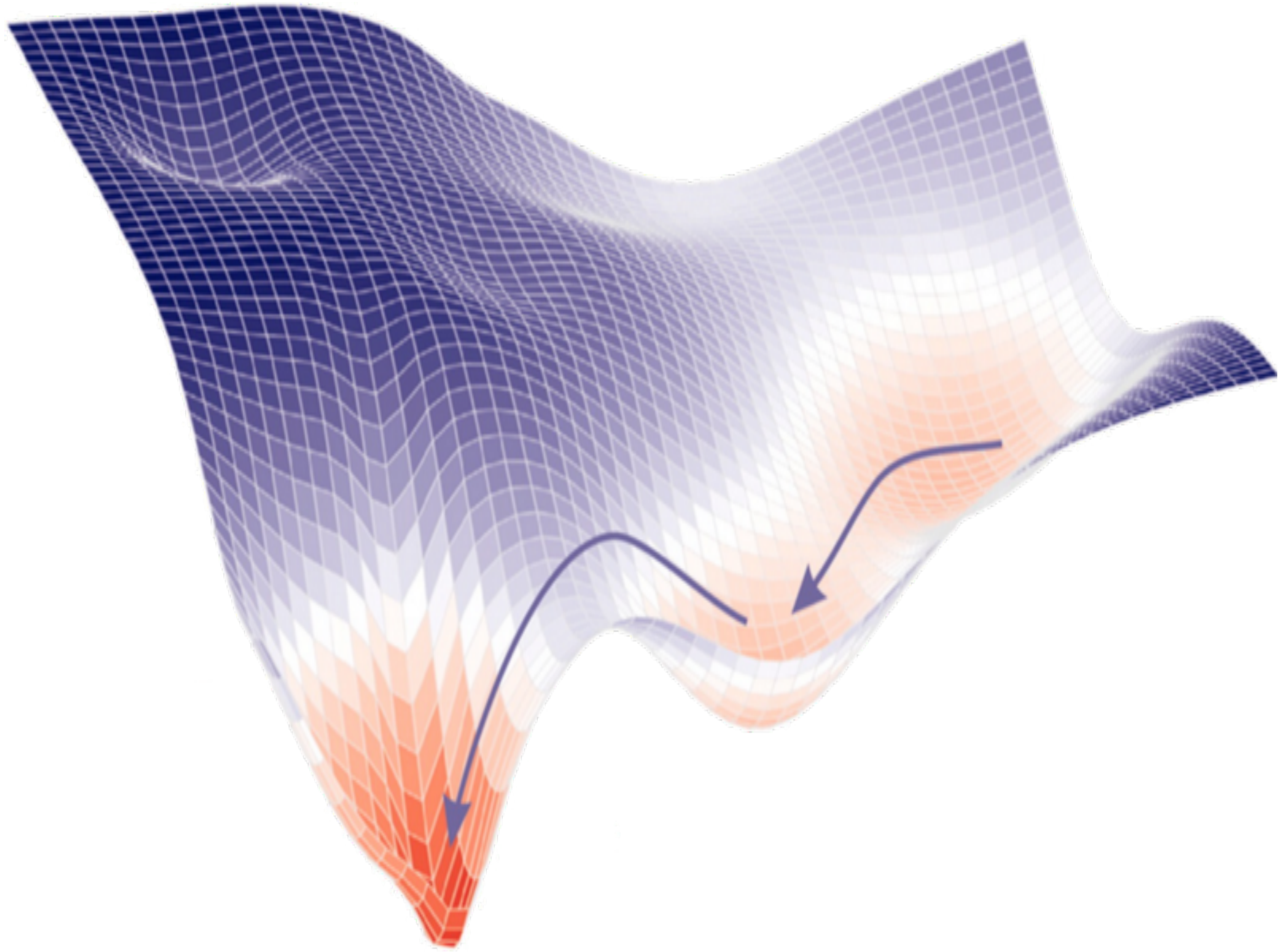



***underfitting***

***overfitting***

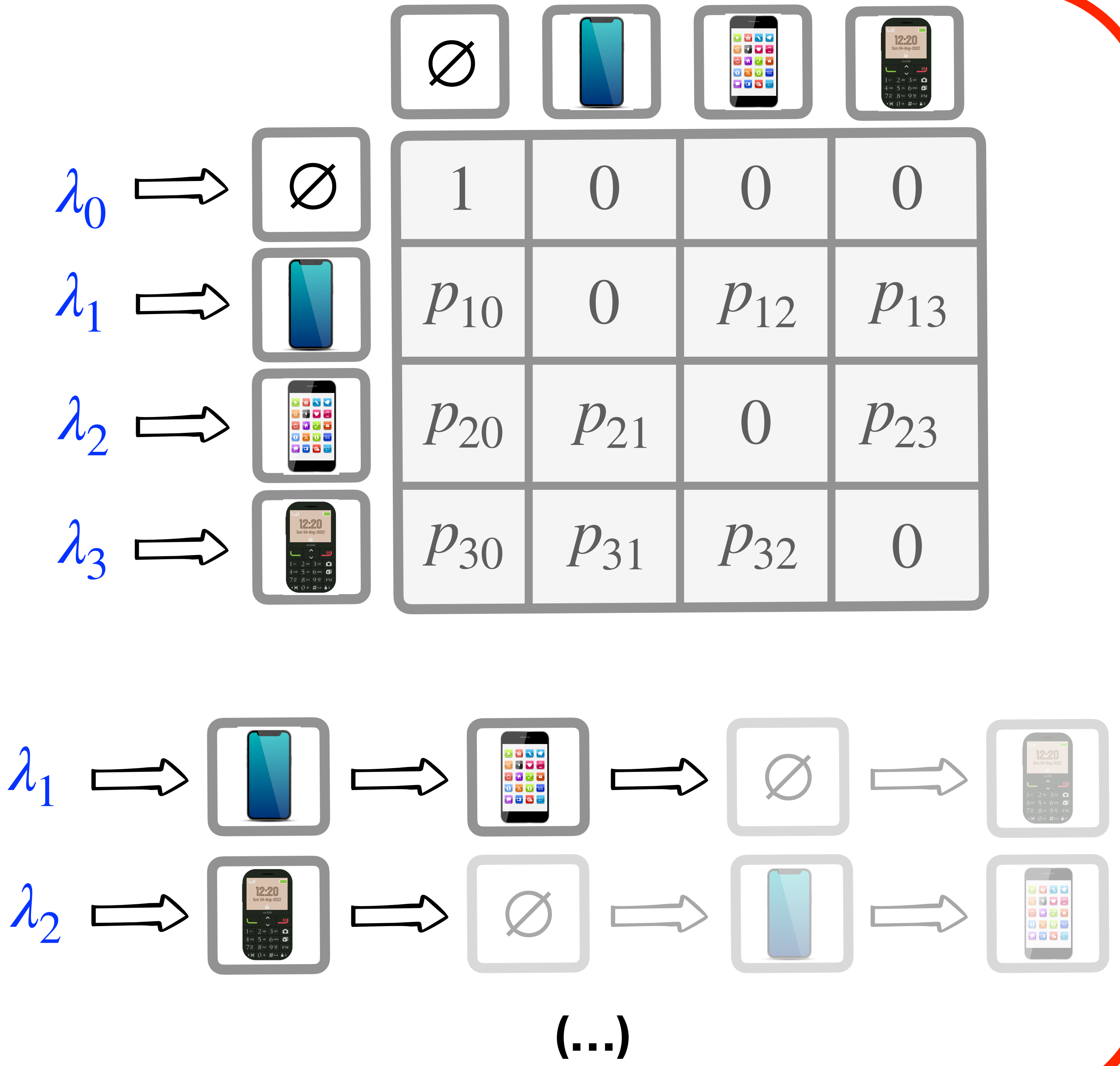
Bishop (2006), Pattern Recognition and Machine Learning (Springer).
Hastie et al. (2009), The Elements of Statistical Learning: Data Mining, Inference, and Prediction (Springer).

## Combining estimation with optimization amplifies errors:



**MC/PR models**

- 👍 small bias
- 👎 large variance




| | ∅ | 📱 | 📱 | 📱 |
|---|---|---|---|---|
| ∅ | 1 | 0 | 0 | 0 |
| 📱 | $p_{10}$ | 0 | $p_{12}$ | $p_{13}$ |
| 📱 | $p_{20}$ | $p_{21}$ | 0 | $p_{23}$ |
| 📱 | $p_{30}$ | $p_{31}$ | $p_{32}$ | 0 |

*"Post-decision disappointment"*
*"error-maximization effect of optimization"*

Michaud (1989), Financial Analysts Journal 45(1):31–42. Smith & Winkler (2006), Management Science 52(3):311–322.
DeMiguel & Nogales (2009), Operations Research 57(3):560–577.

**The robust optimization paradigm to combat estimation errors:**

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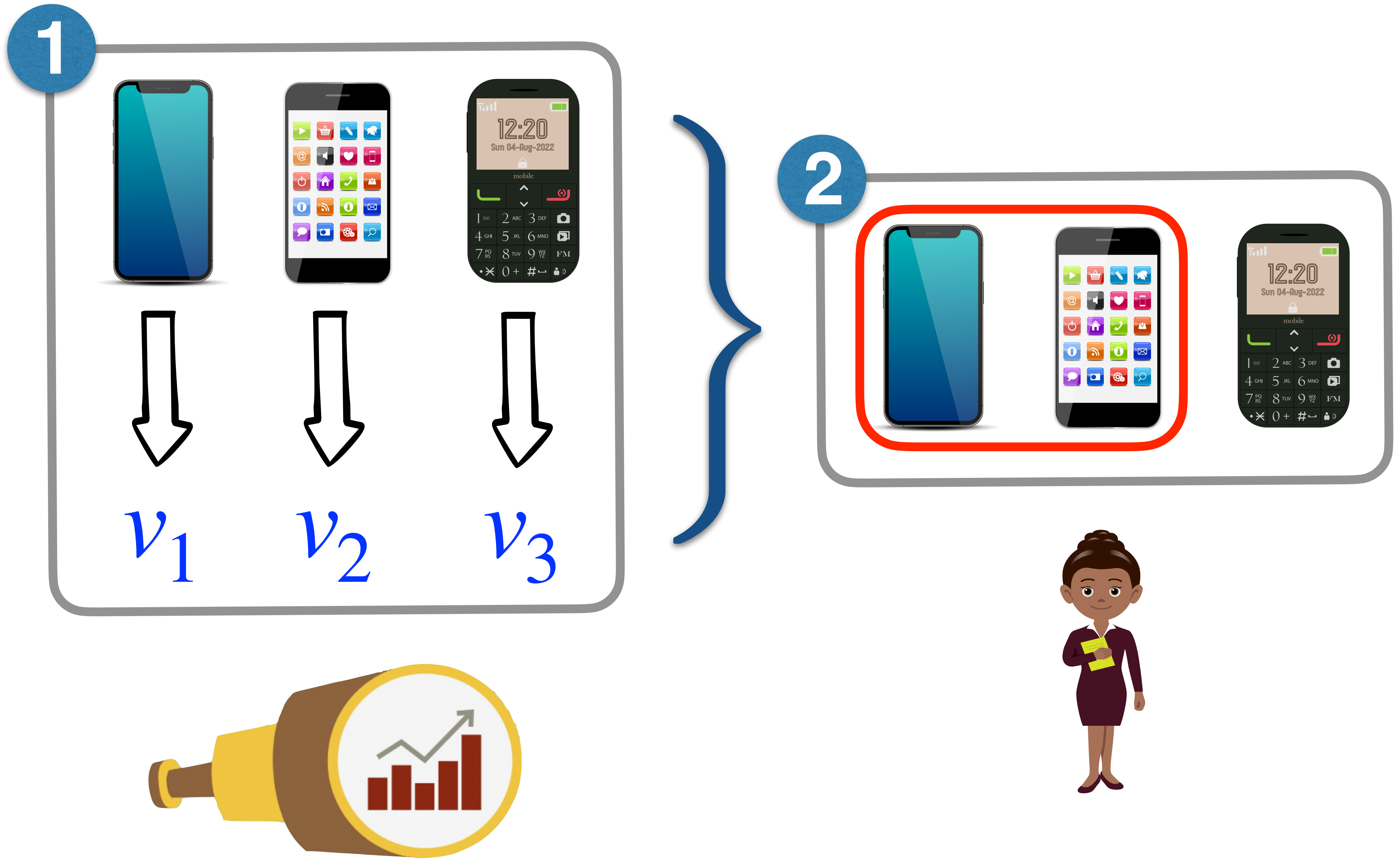
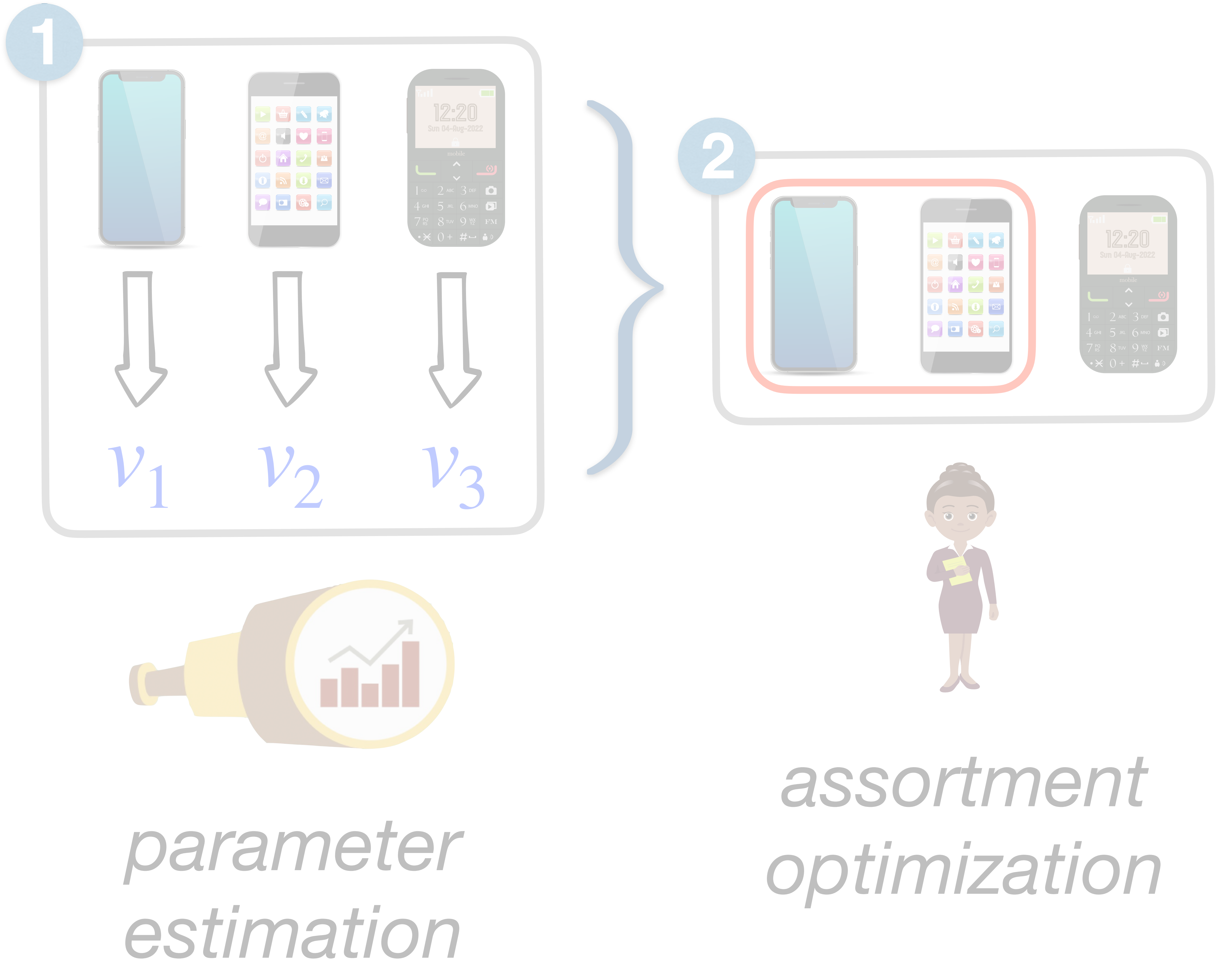
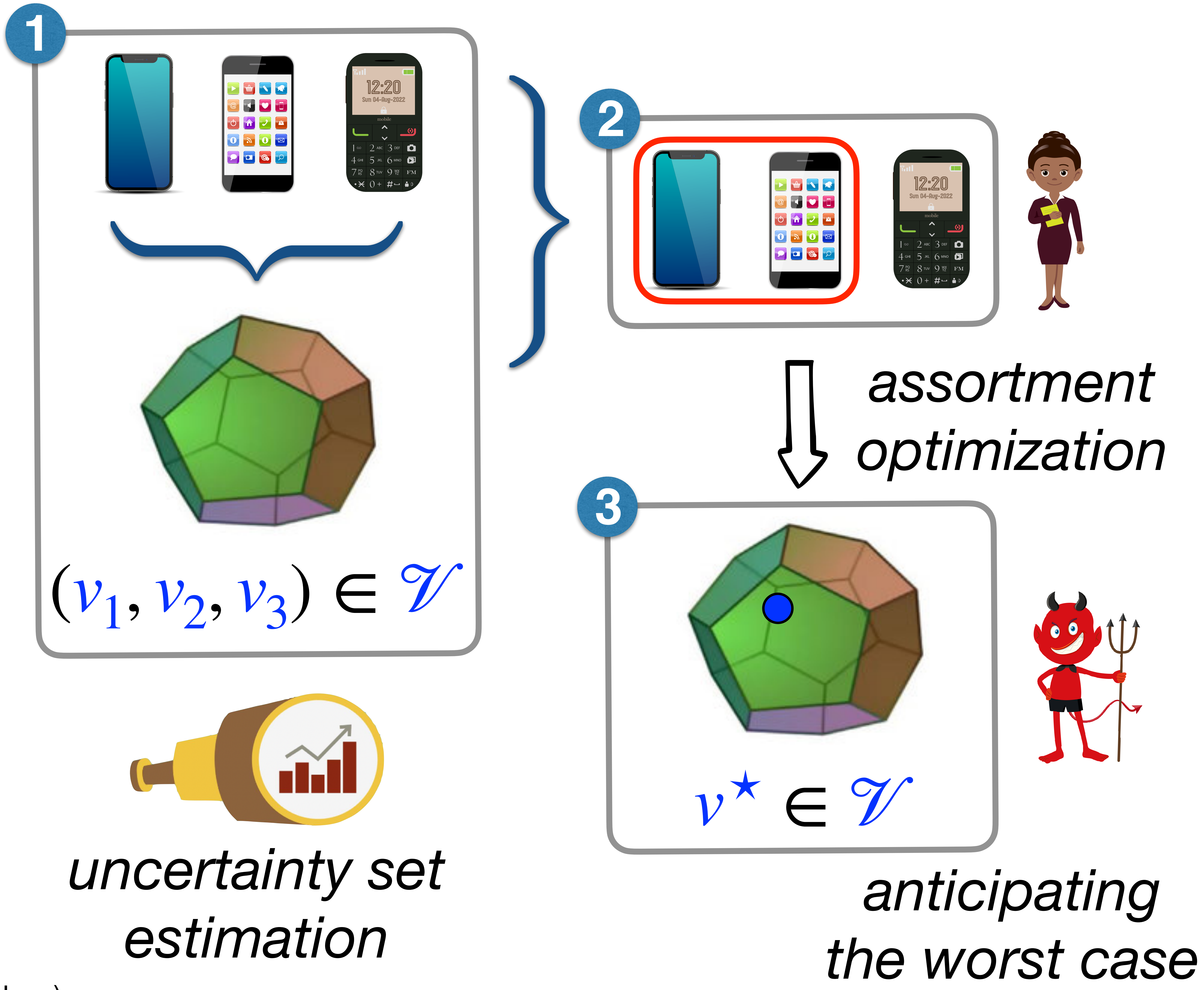



*parameter estimation*

*assortment optimization*

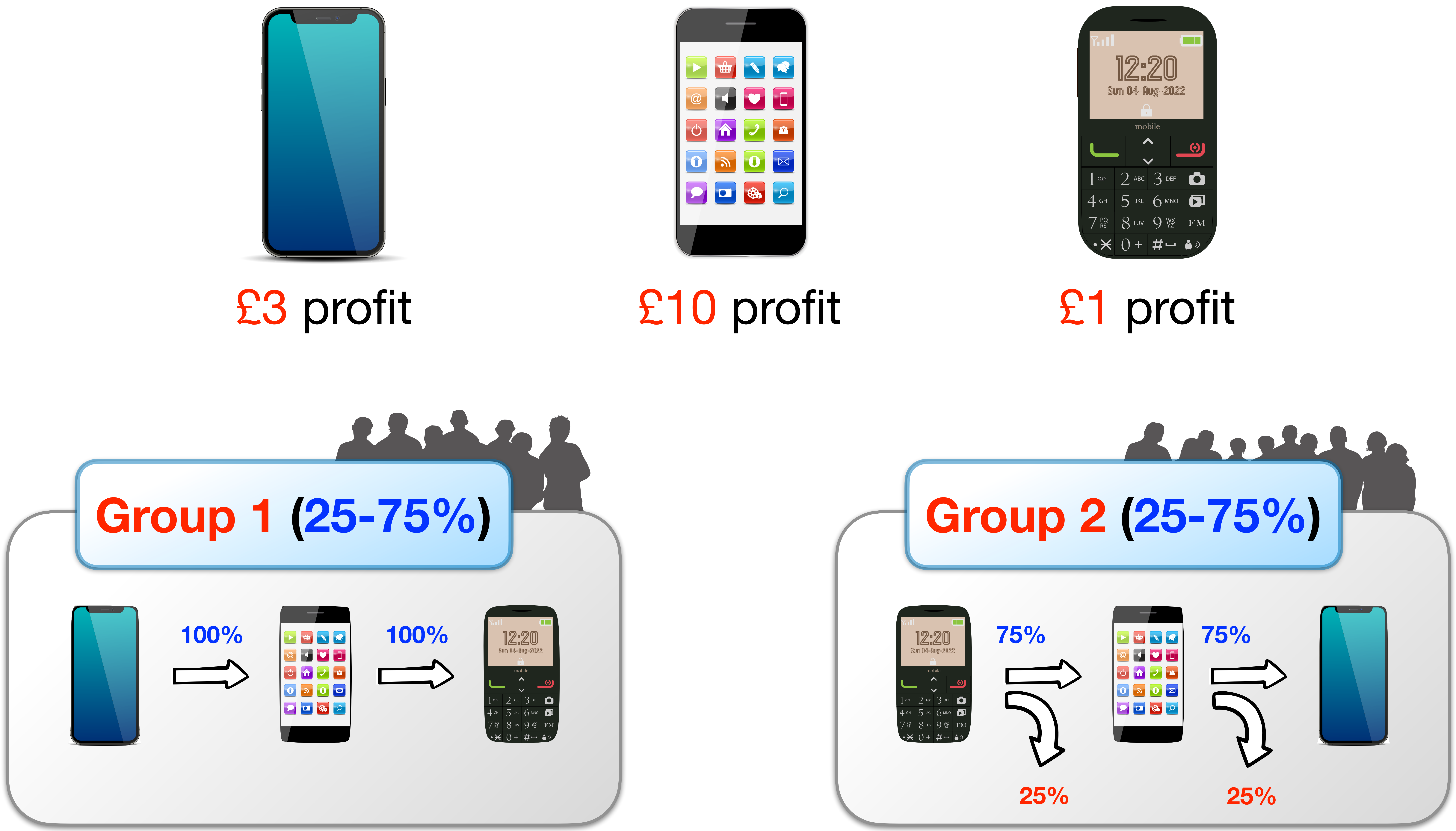
## The robust optimization paradigm to combat estimation errors:




Ben-Tal et al. (2009), Robust Optimization (Princeton University Press).
Bertsimas & den Hertog (2022), Robust and Adaptive Optimization (Dynamic Ideas).

# The Robust Assortment Optimization Problem



# The Robust Assortment Optimization Problem

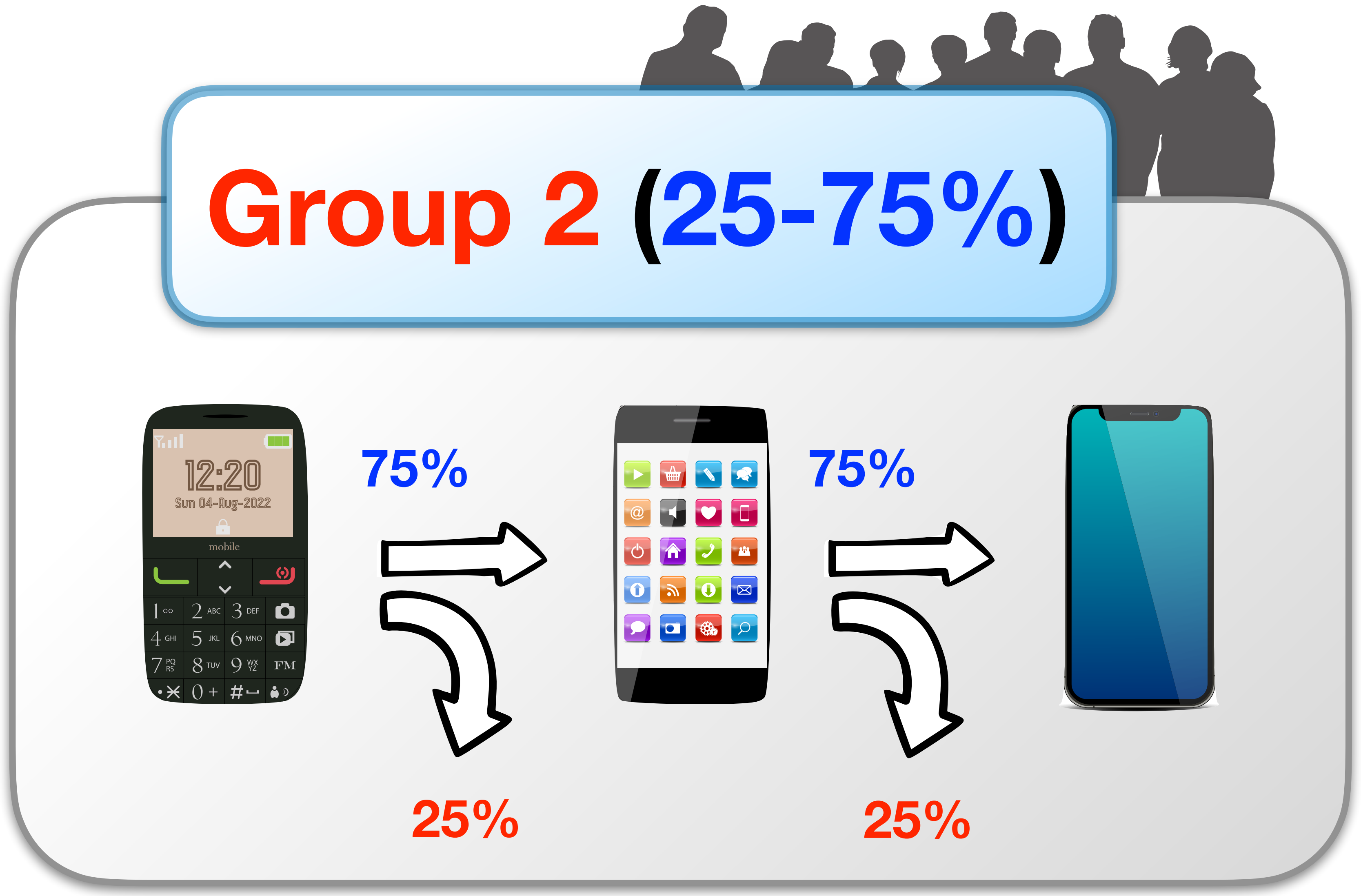
**£3.25** *worst-case* expected profit

£3 profit

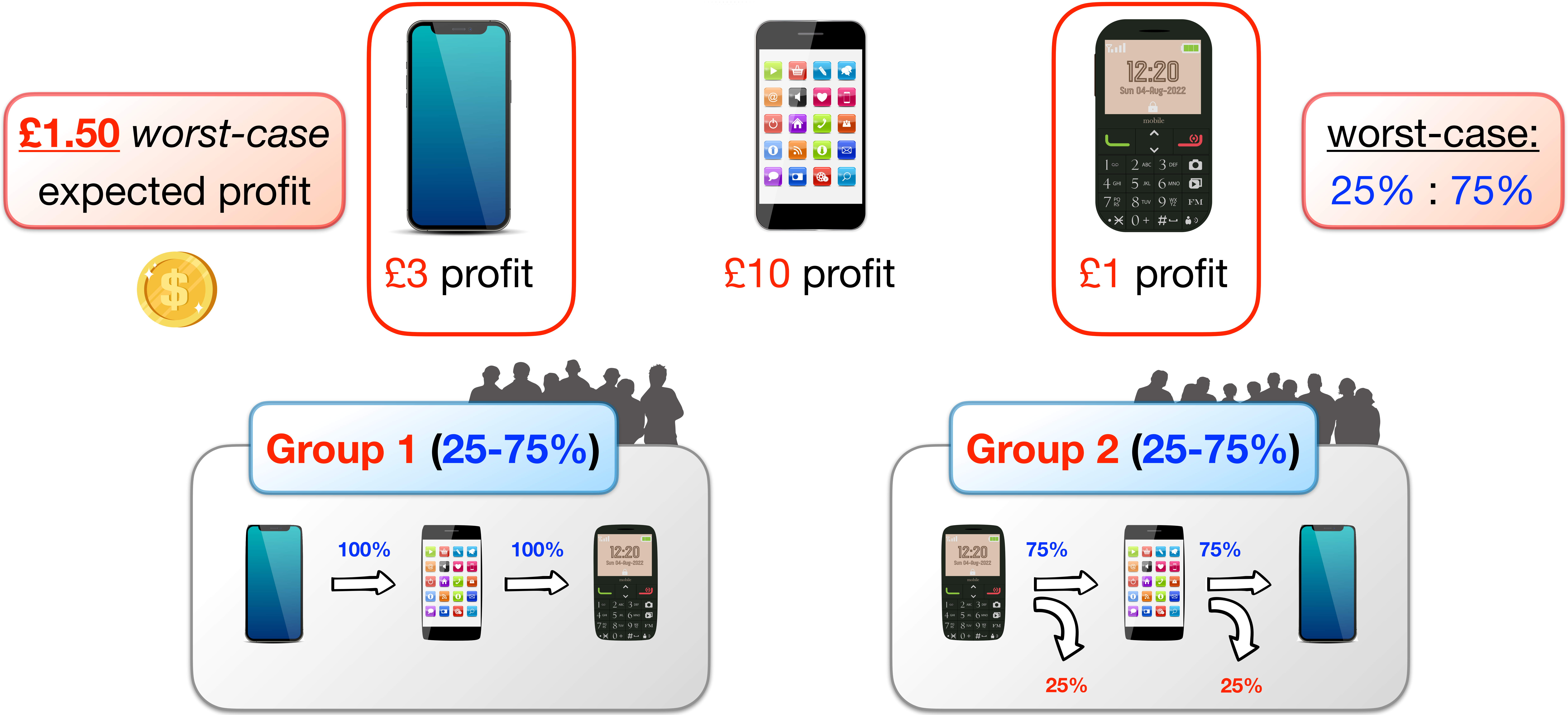
£10 profit

£1 profit

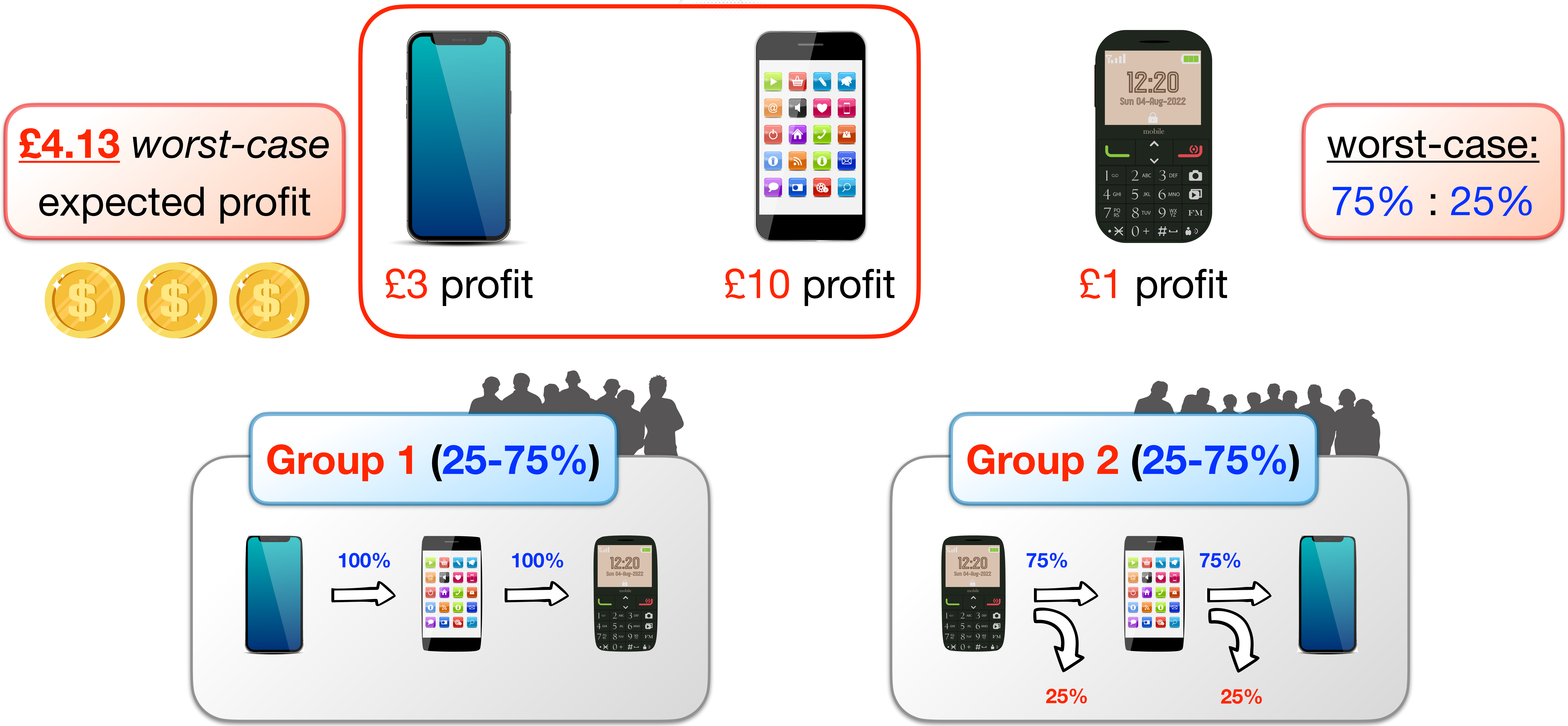
worst-case:
25% : 75%

# The Robust Assortment Optimization Problem



# The Robust Assortment Optimization Problem

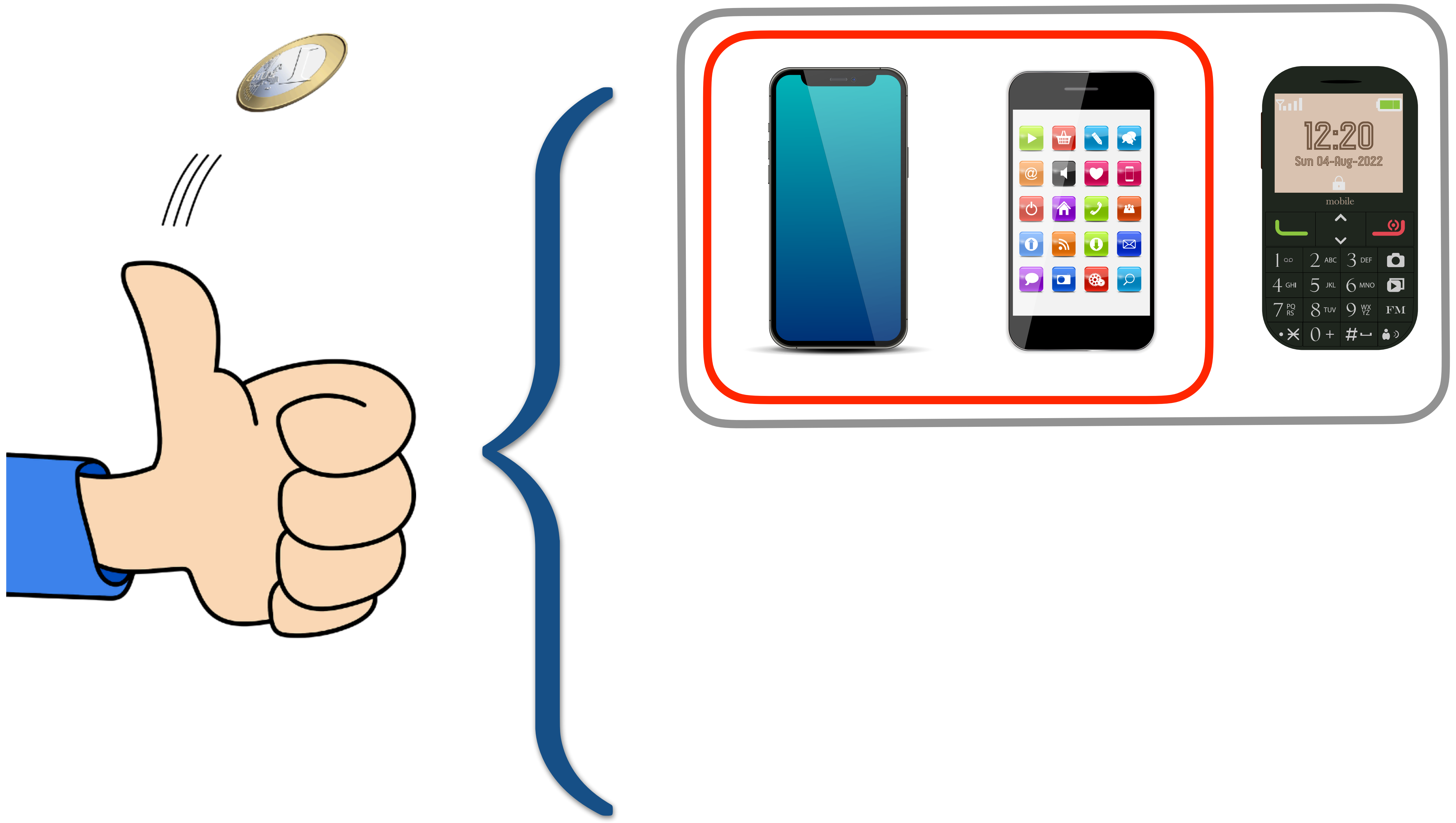


# The Robust Assortment Optimization Problem

- **Multinomial logit model:**
  - Rusmevichientong & Topaloglu ('12) solve robust assortment optimization problem under uncertain product valuations; *revenue-ordered assortments remain optimal*
- **Markov chain model:**
  - Désir et al. ('21) use *robust MDP-type algorithms* to solve robust assortment optimization problem under uncertain arrival rates and transition probabilities
- **Preference ranking model:**
  - Farias et al. ('13) estimate worst-case revenues for fixed assortment under uncertain preference distributions
  - Bertsimas & Mišić ('17) solve robust assortment optimization problem under uncertain preference distributions

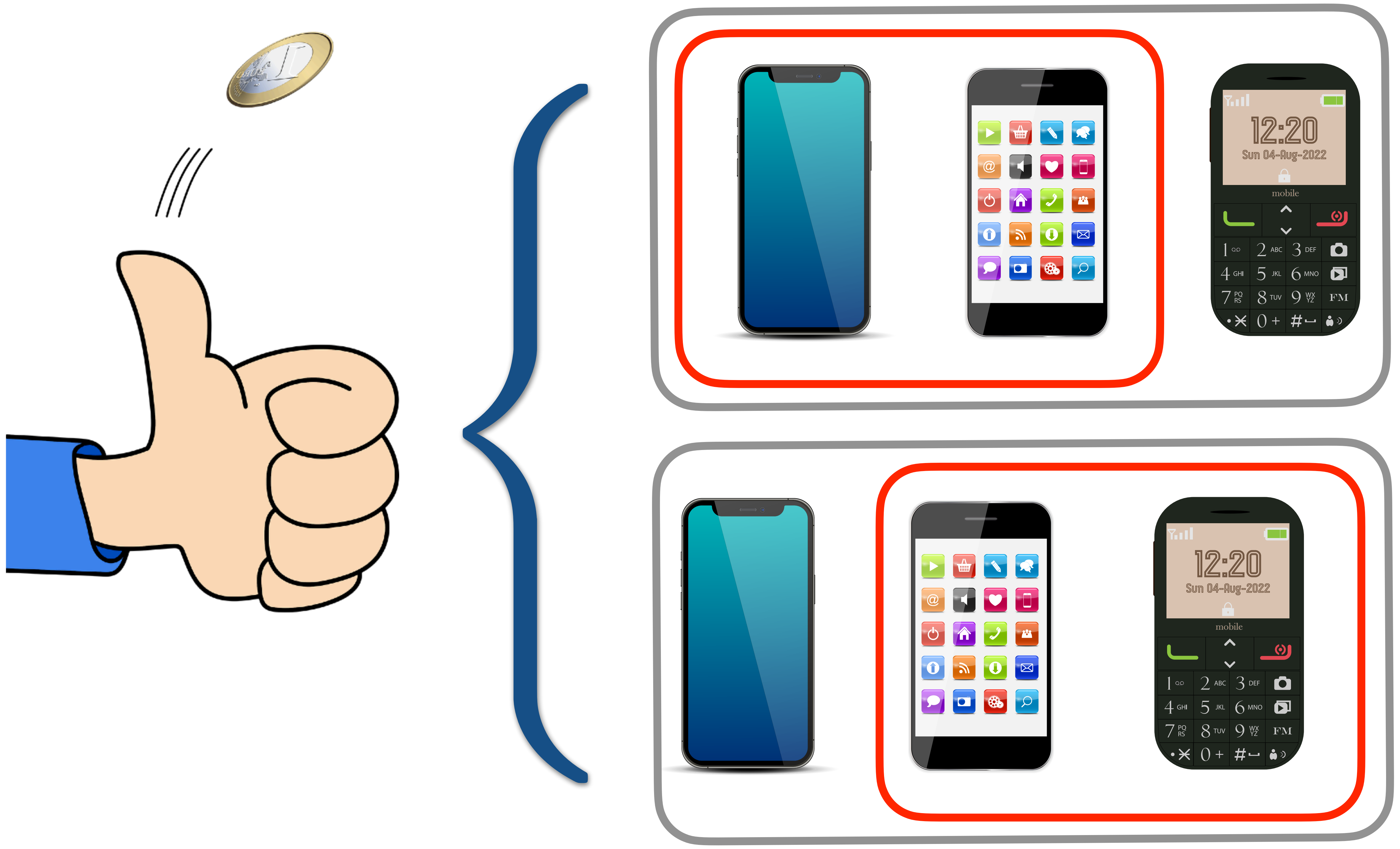
# The Randomized Robust Assortment Optimization Problem

# The Randomized Robust Assortment Optimization Problem




w. p. $\frac{1}{2}$

# The Randomized Robust Assortment Optimization Problem




w. p. $\frac{1}{2}$

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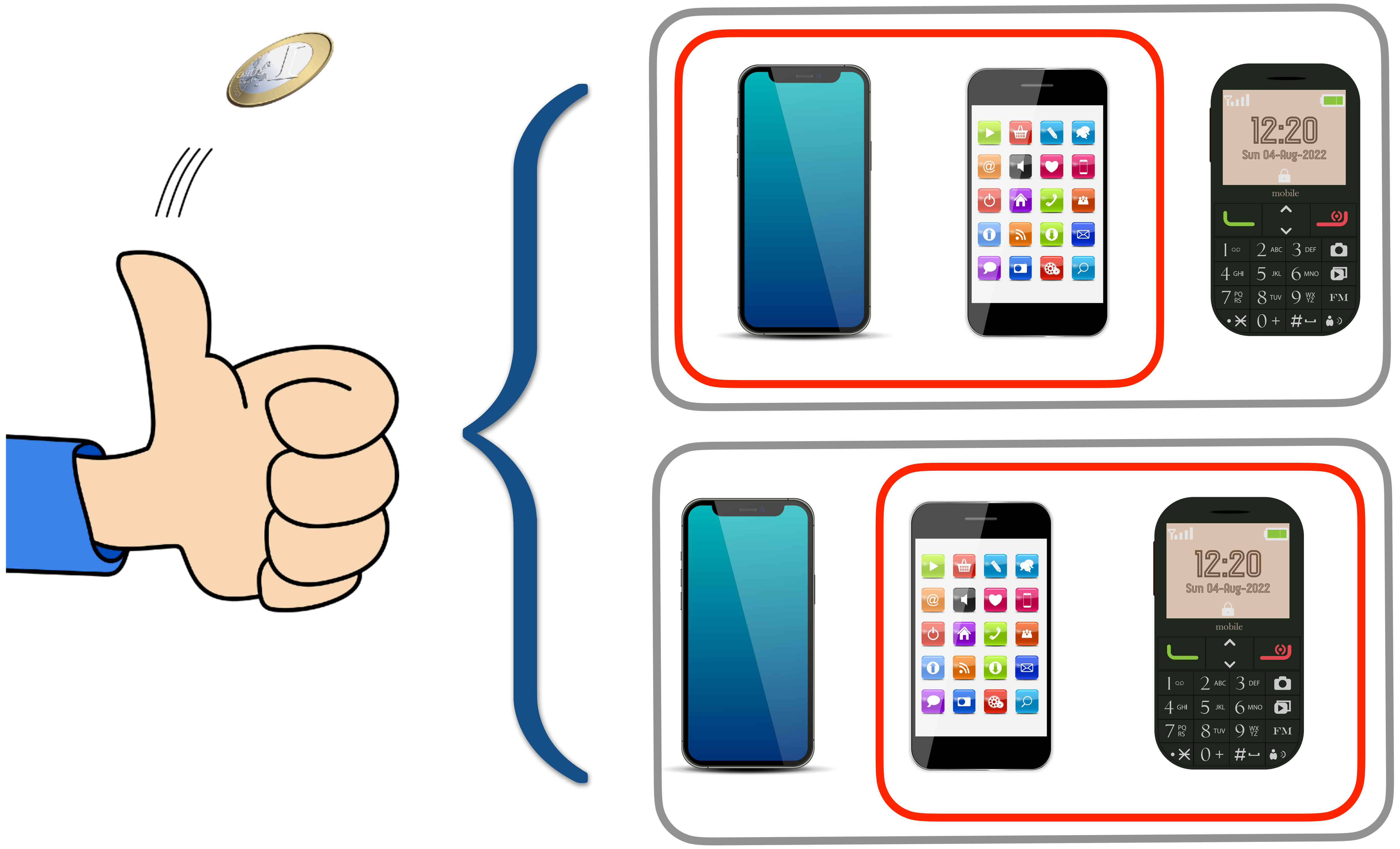
# The Randomized Robust Assortment Optimization Problem



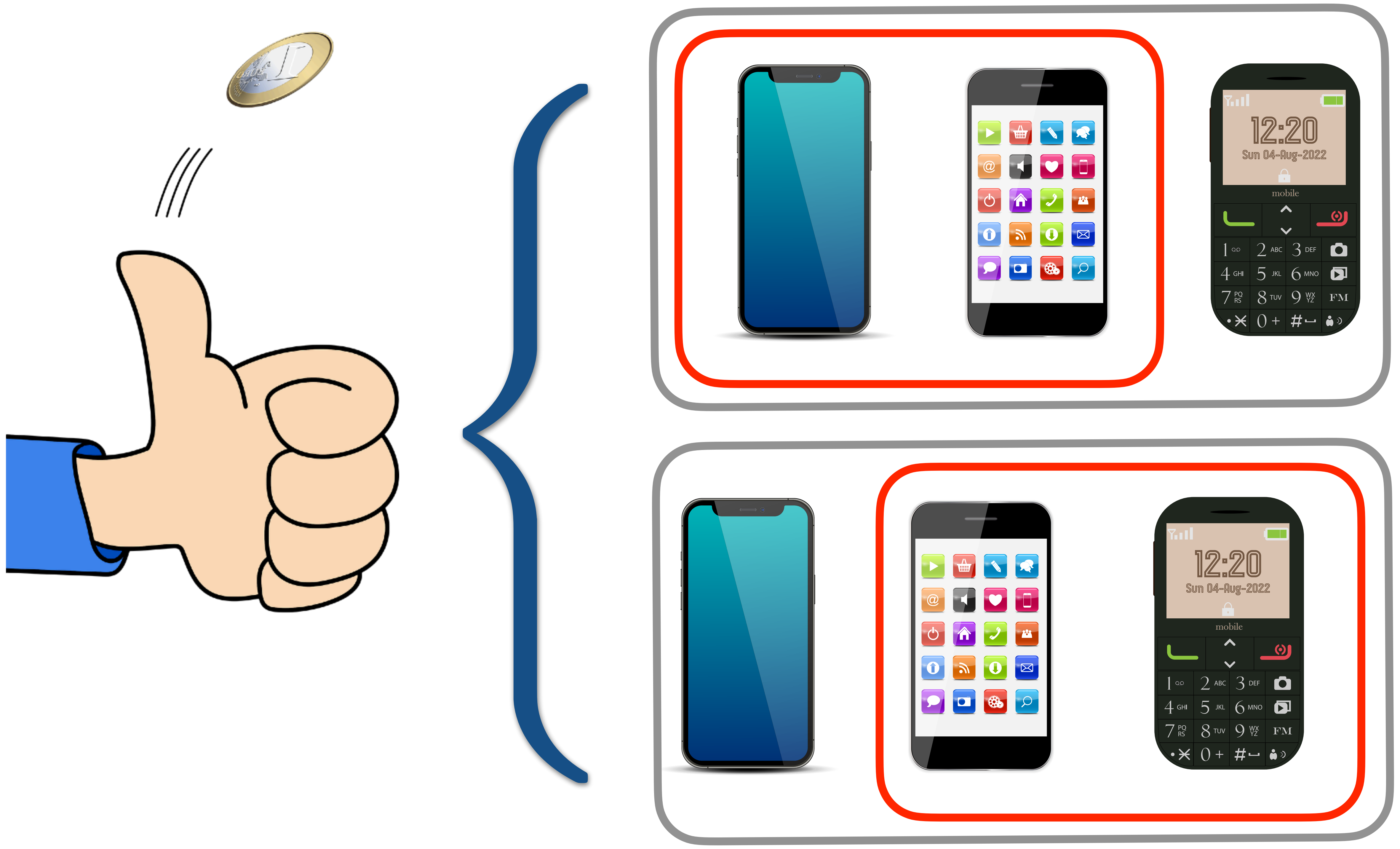
w. p. $\frac{1}{2}$

w. p. $\frac{1}{2}$

**£4.81** *worst-case* expected profit

(£5.38 nominal)

# The Randomized Robust Assortment Optimization Problem



w. p. $\frac{1}{3}$

w. p. $\frac{2}{3}$

**£5.33** *worst-case* expected profit

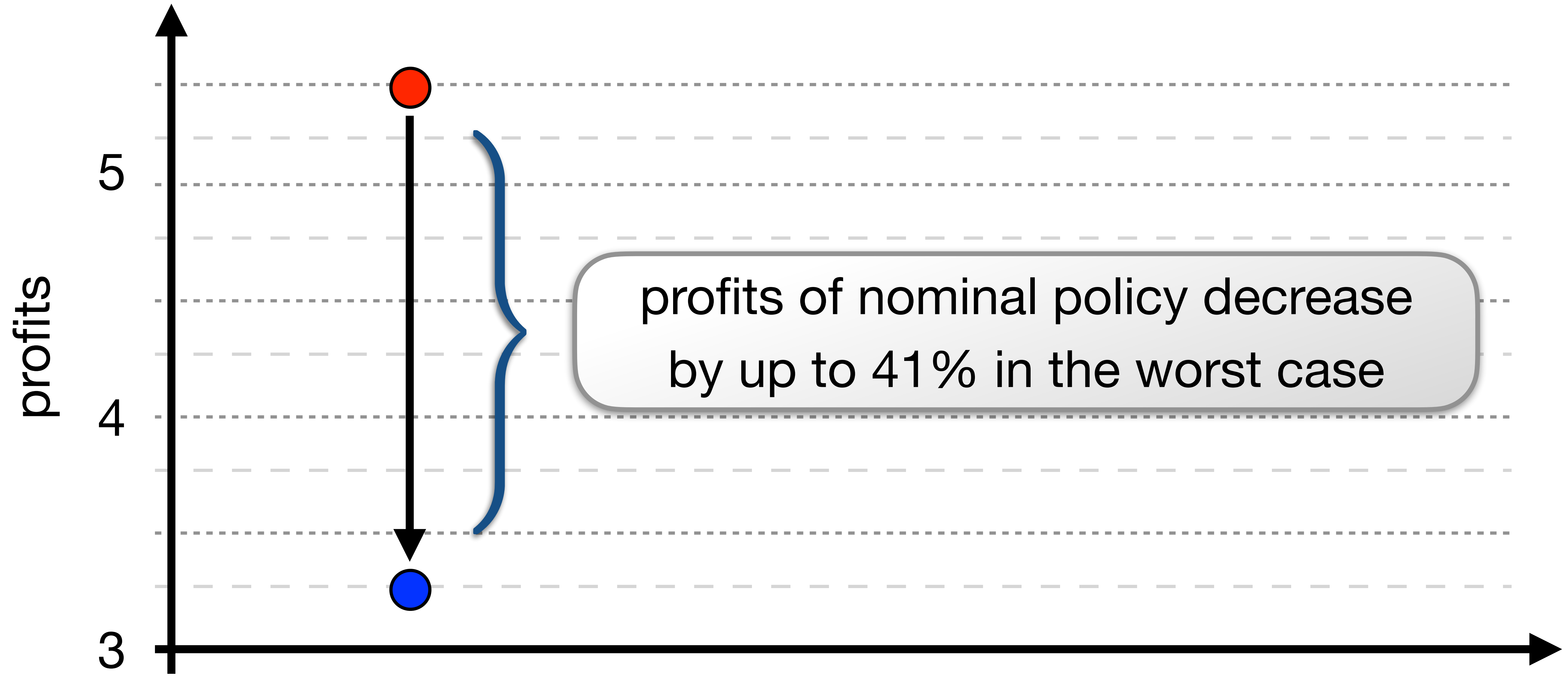
(£5.42 nominal)

# The *Randomized* *Robust* Assortment Optimization Problem

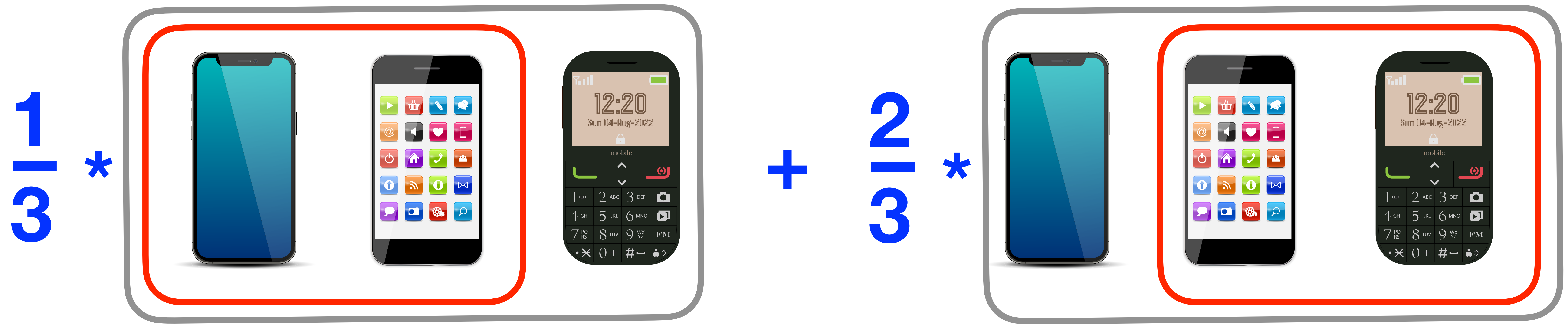
$\frac{1}{3}$ *

\+ $\frac{2}{3}$ *

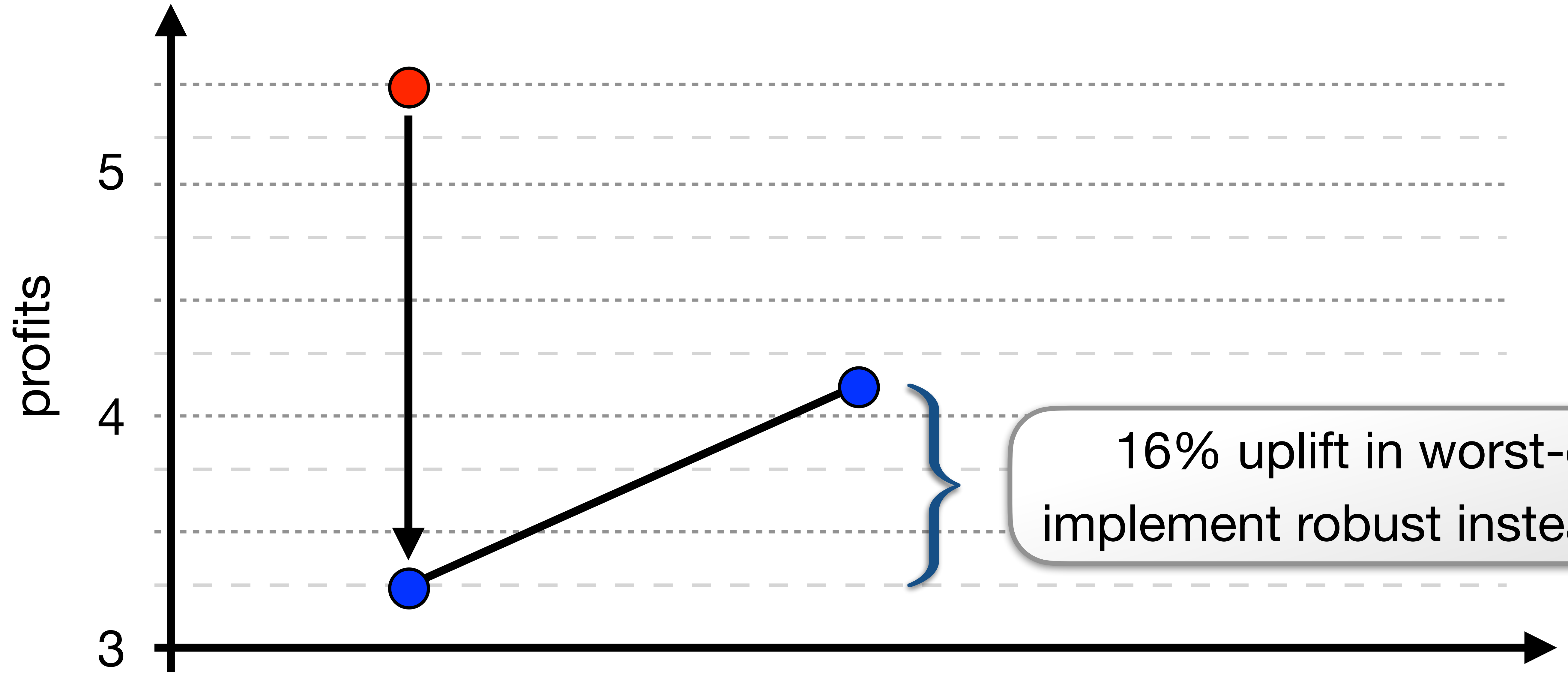
**£5.33** *worst-case* expected profit



# The Randomized Robust Assortment Optimization Problem

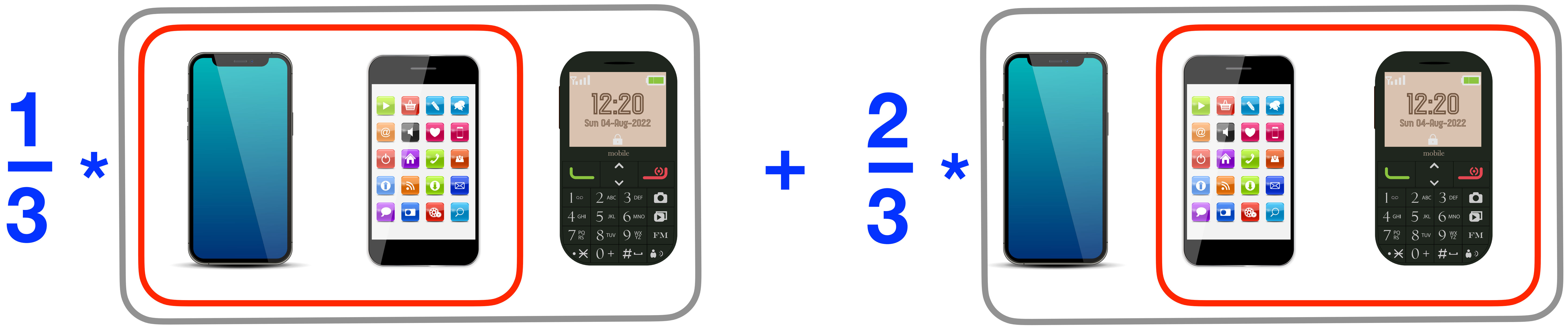



**£5.33** *worst-case* expected profit

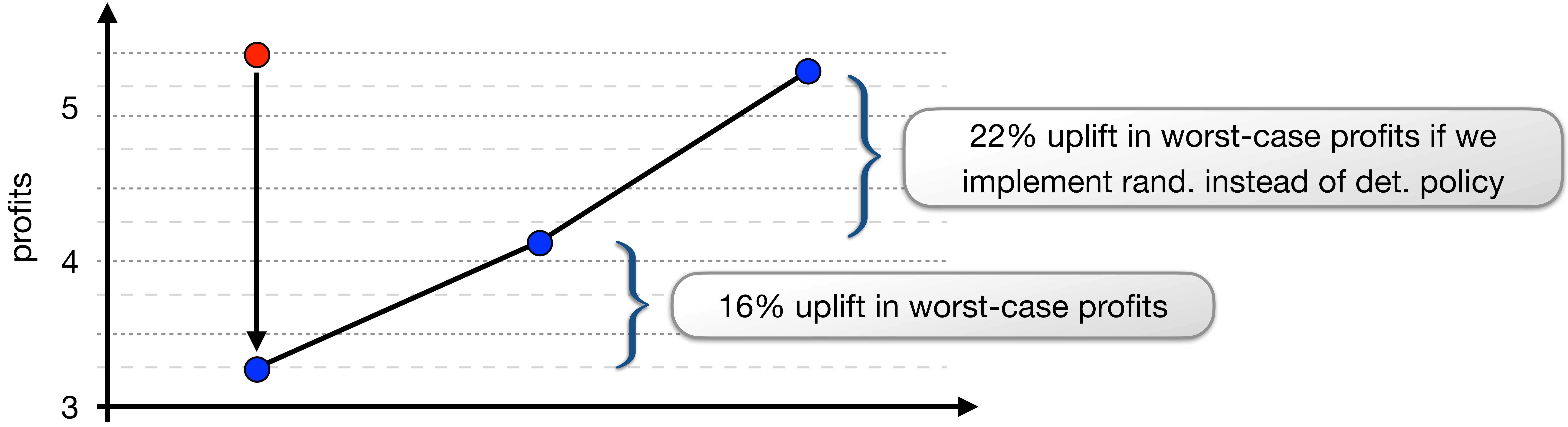



16% uplift in worst-case profits if we implement robust instead of nominal policy

# The Randomized Robust Assortment Optimization Problem




**£5.33** *worst-case* expected profit

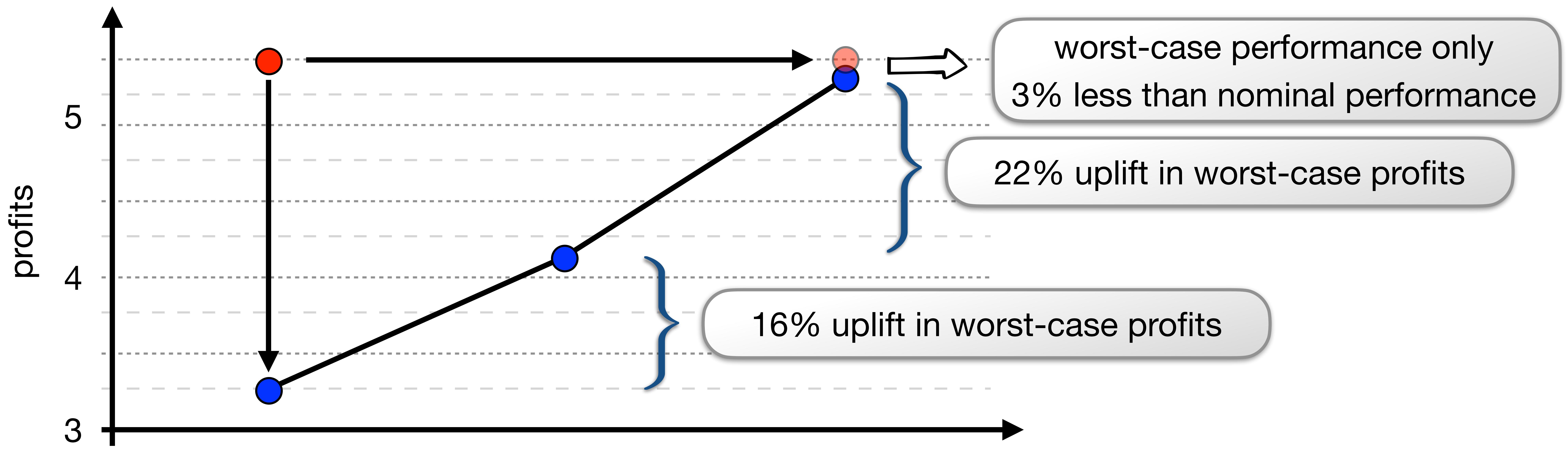


# The Randomized Robust Assortment Optimization Problem

$\frac{1}{3}$ *


+ $\frac{2}{3}$ *


**£5.33** *worst-case* expected profit



# The Randomized Robust Assortment Optimization Problem

**Why does randomization help?**

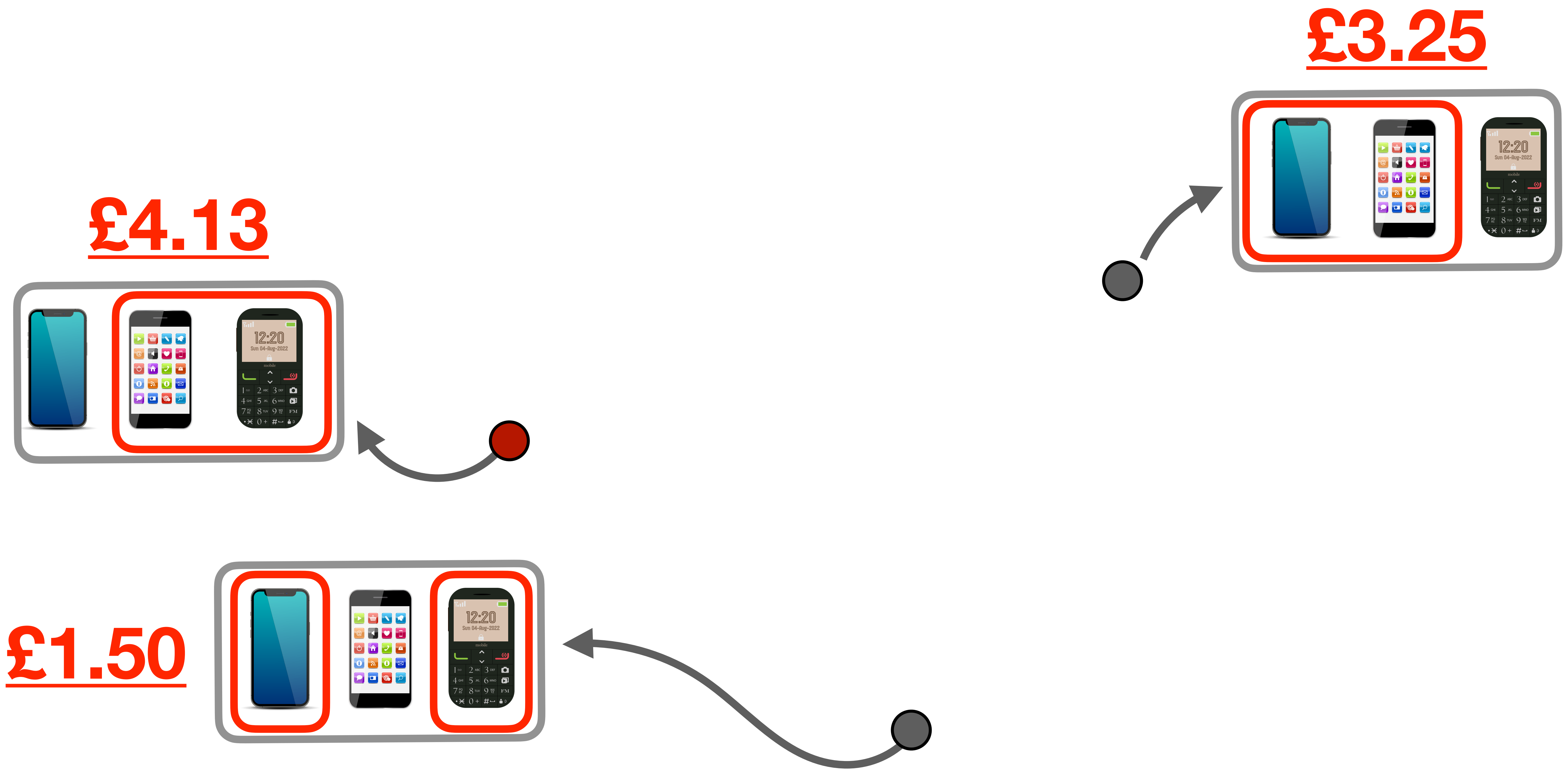
## Why does randomization help?

1. **Mathematical Interpretation:**

# The Randomized Robust Assortment Optimization Problem

## Why does randomization help?

1 **Mathematical Interpretation:**



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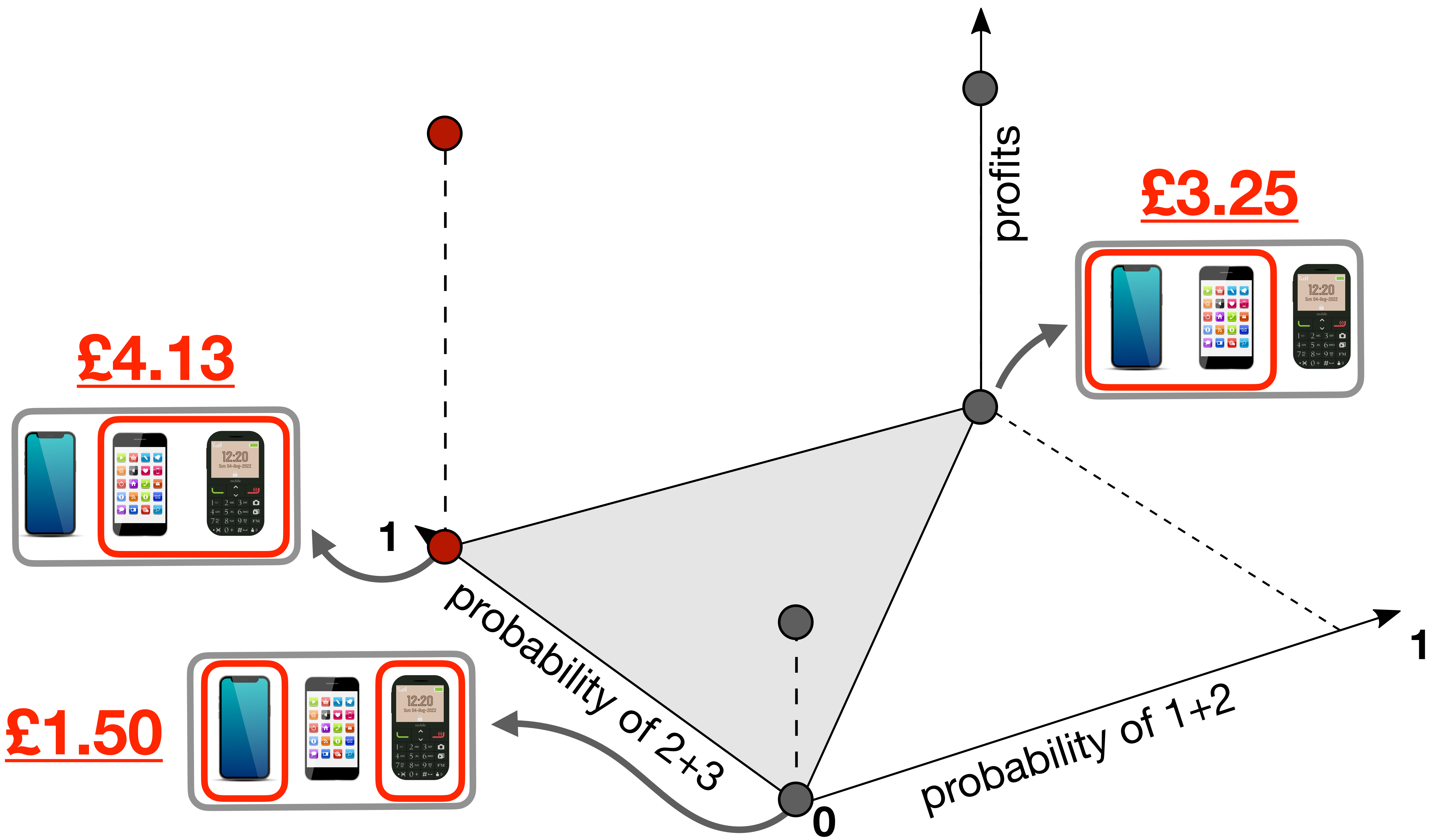

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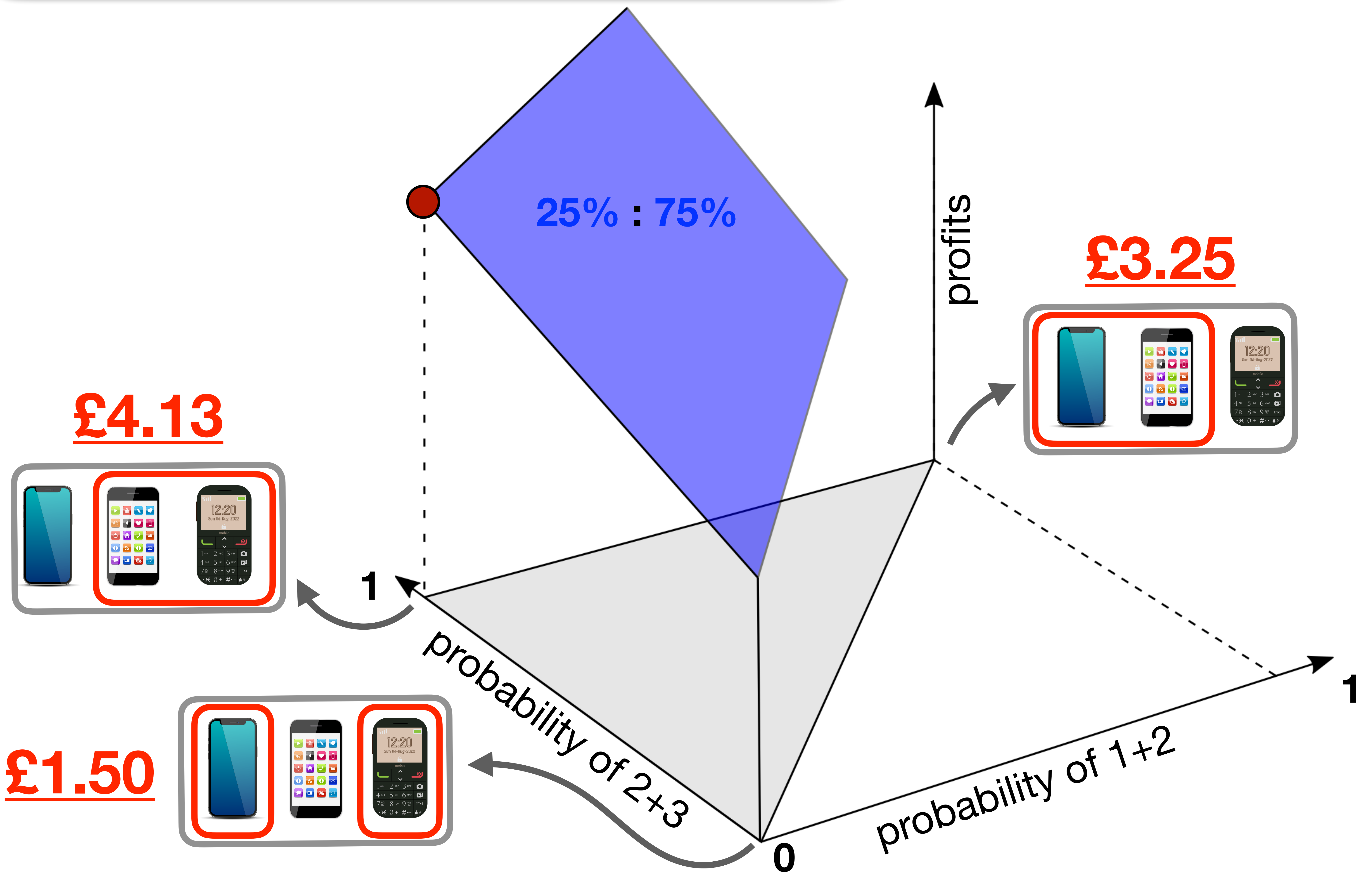

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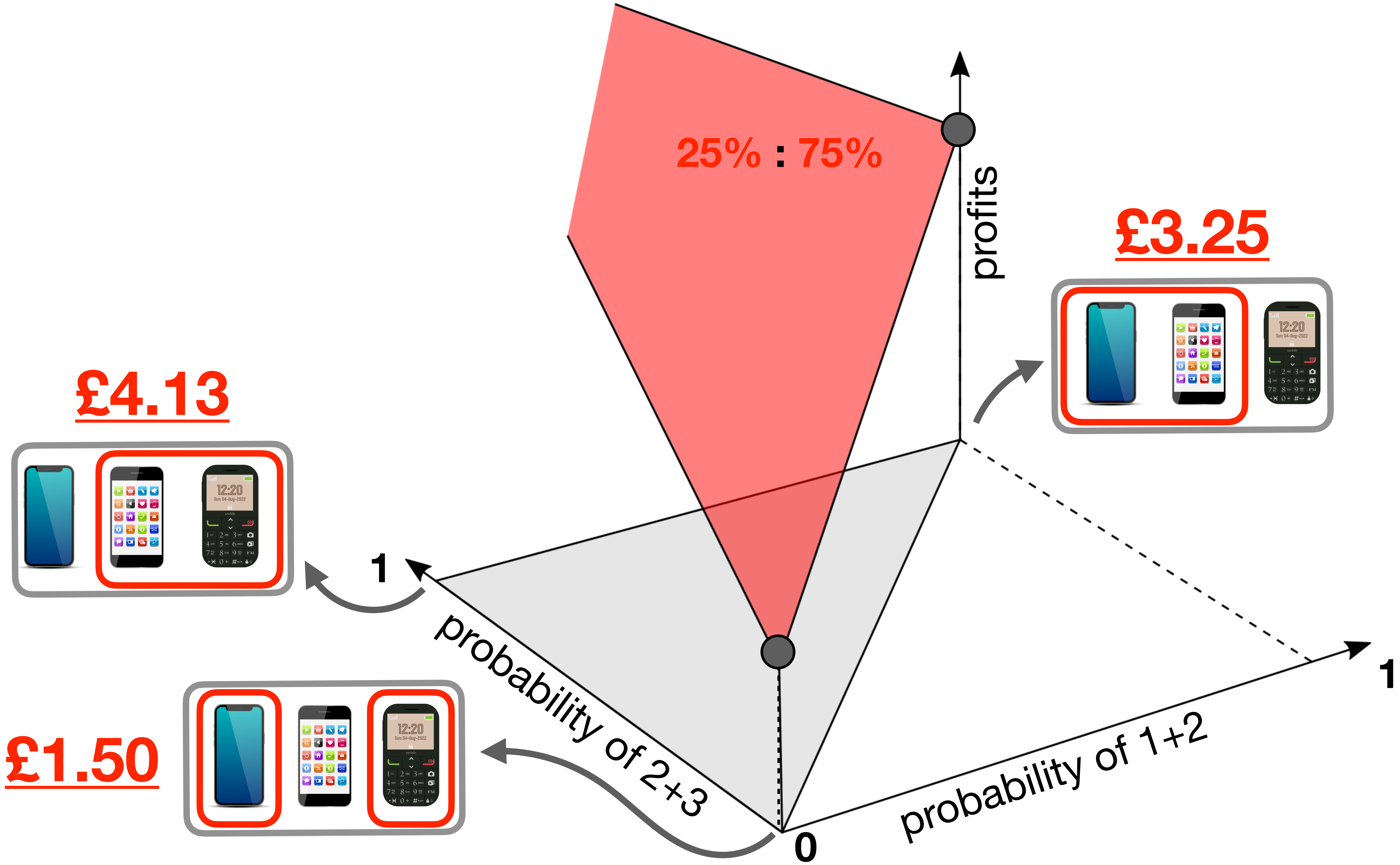

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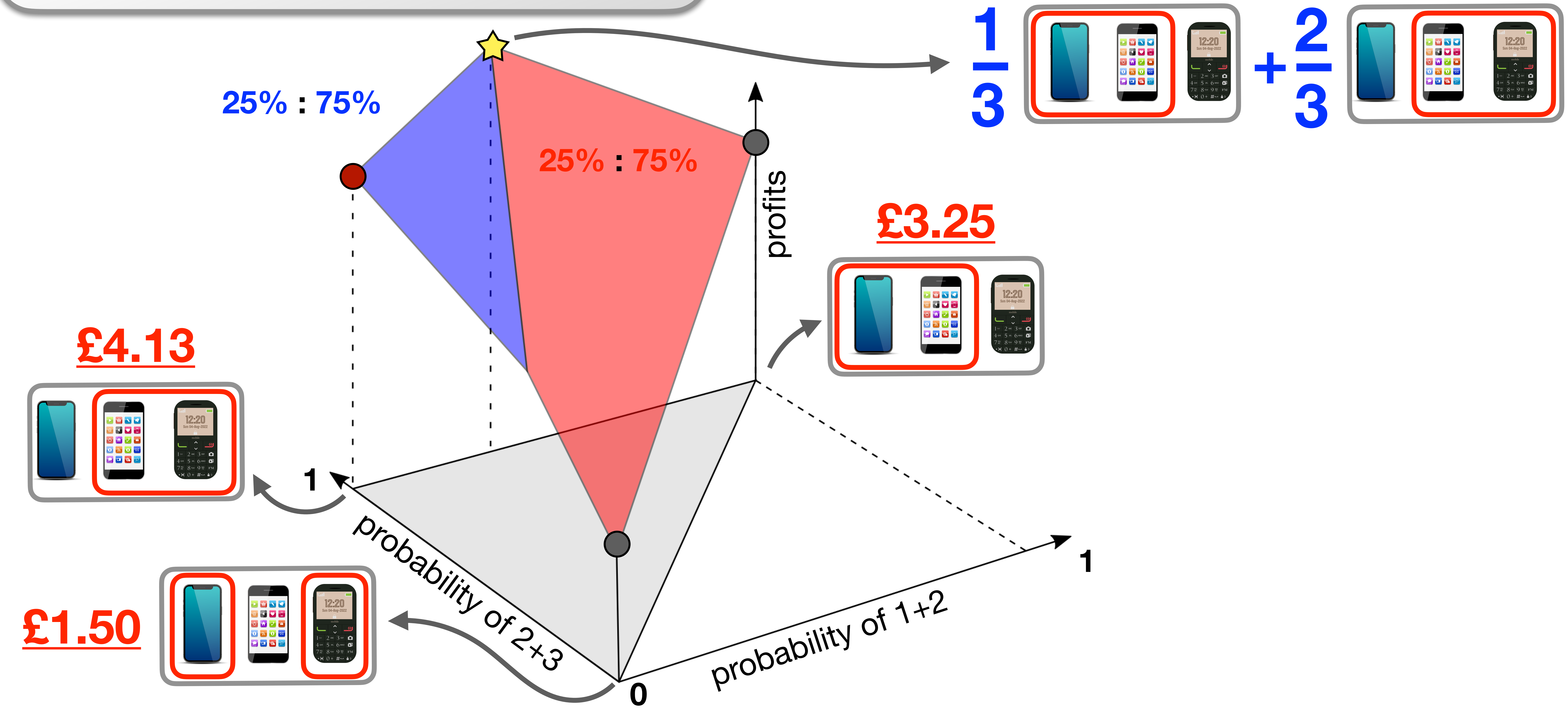


# The Randomized Robust Assortment Optimization Problem

**Why does randomization help?**

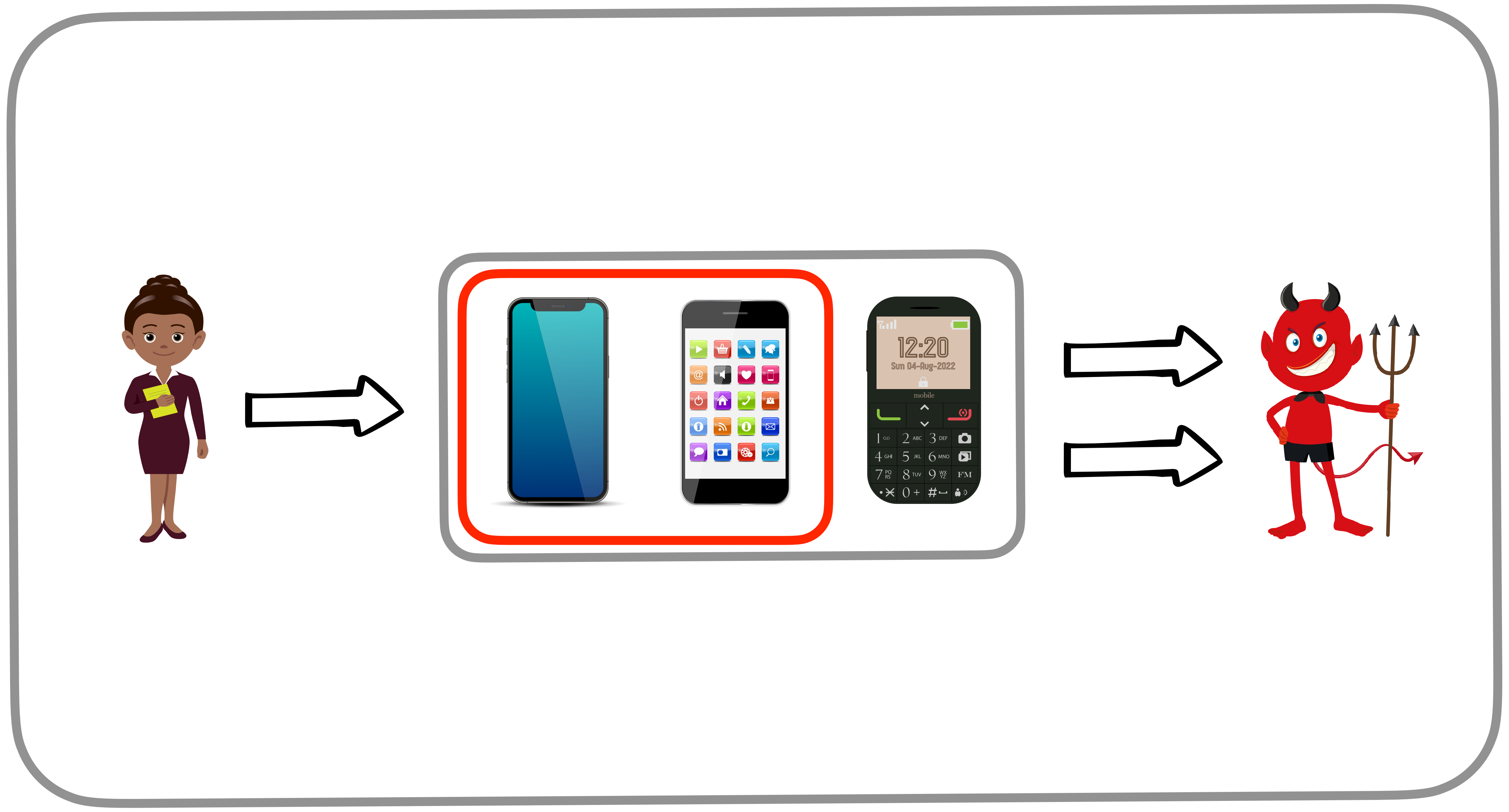
2 **"Game-Theoretic" Interpretation:**

# The Randomized Robust Assortment Optimization Problem

**Why does randomization help?**

2 "Game-Theoretic" Interpretation:

deterministic robust



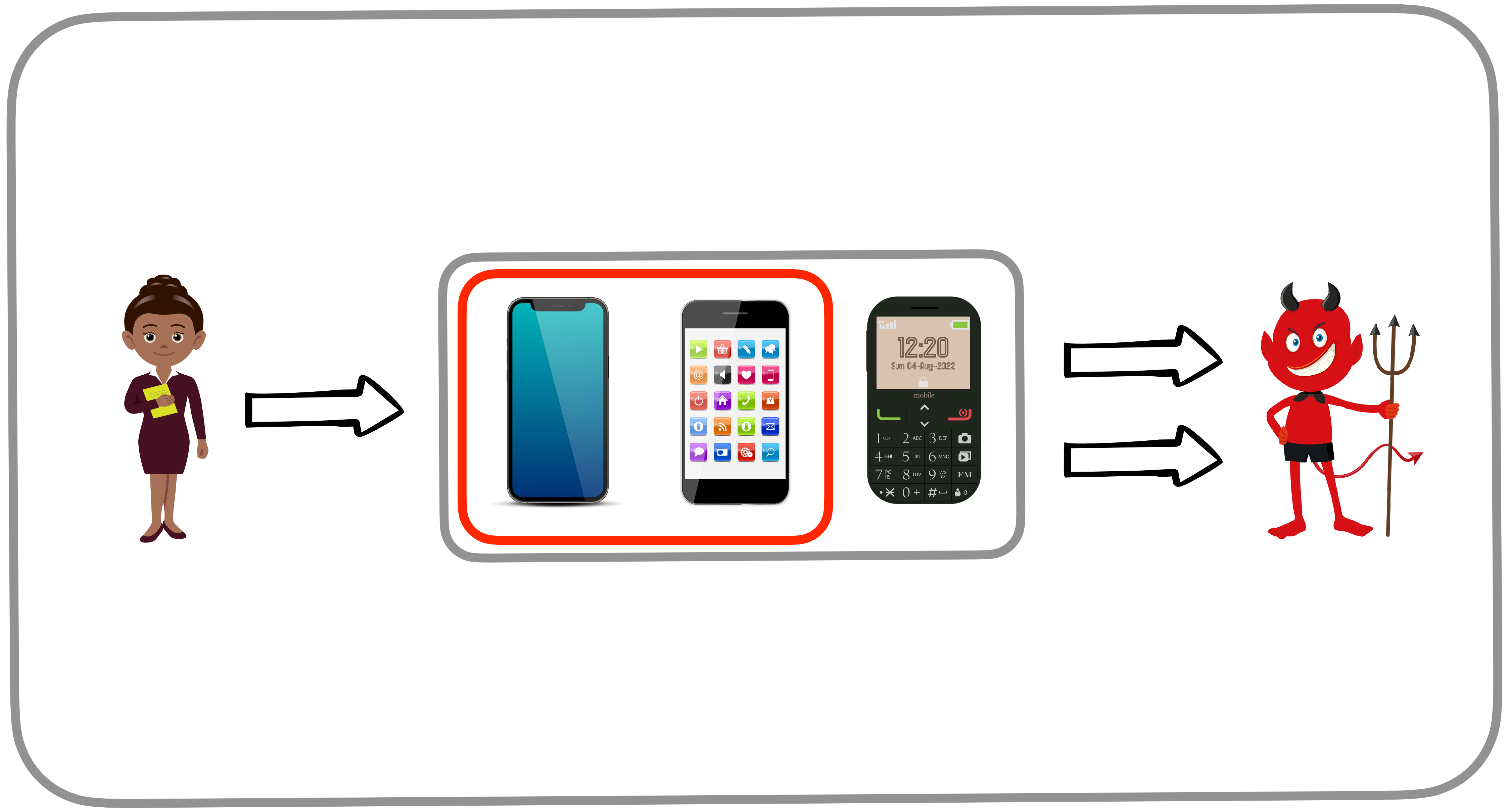
# The Randomized Robust Assortment Optimization Problem

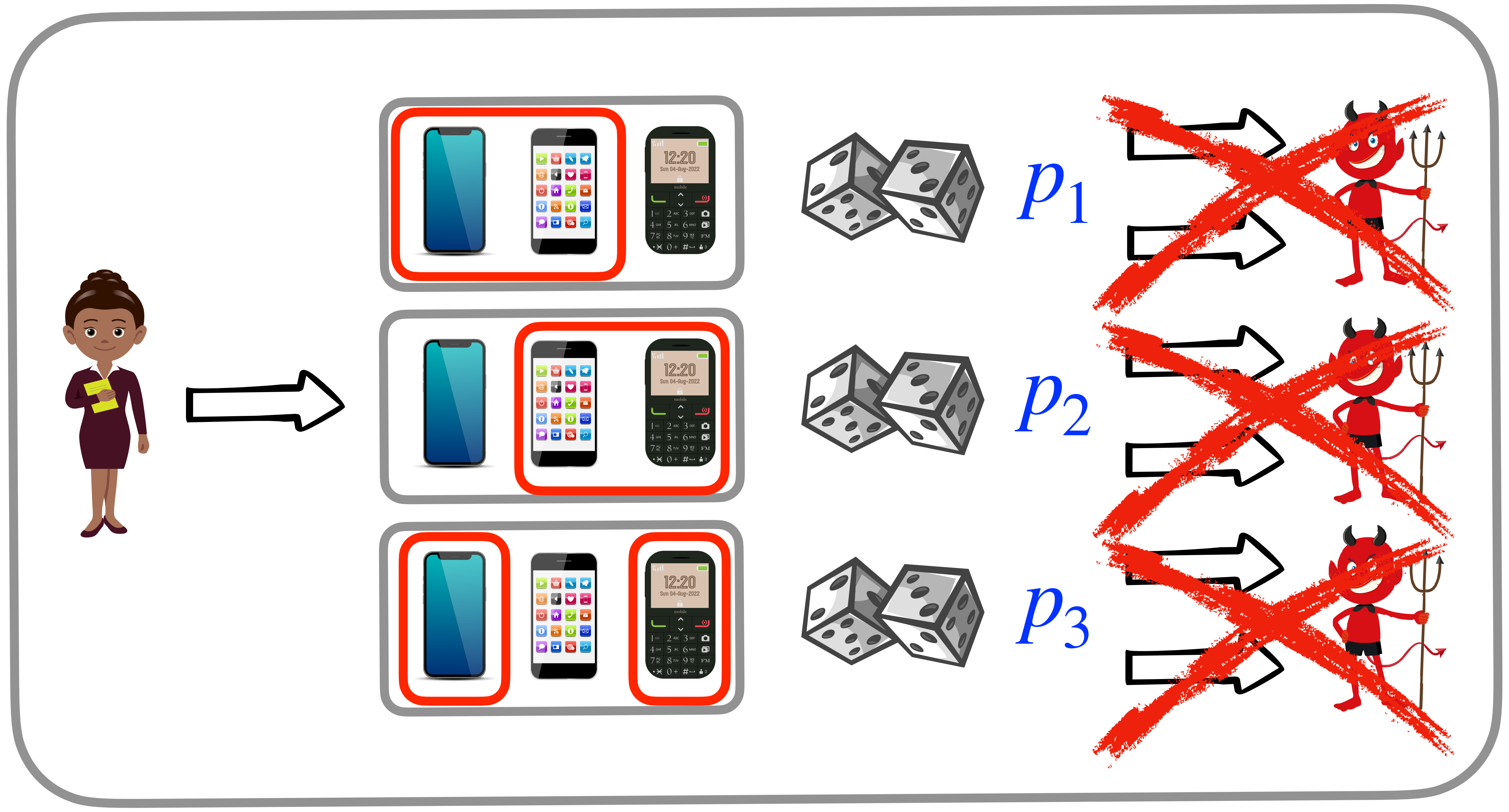
## Why does randomization help?

2 “Game-Theoretic” Interpretation:

**deterministic robust**

**randomized robust**



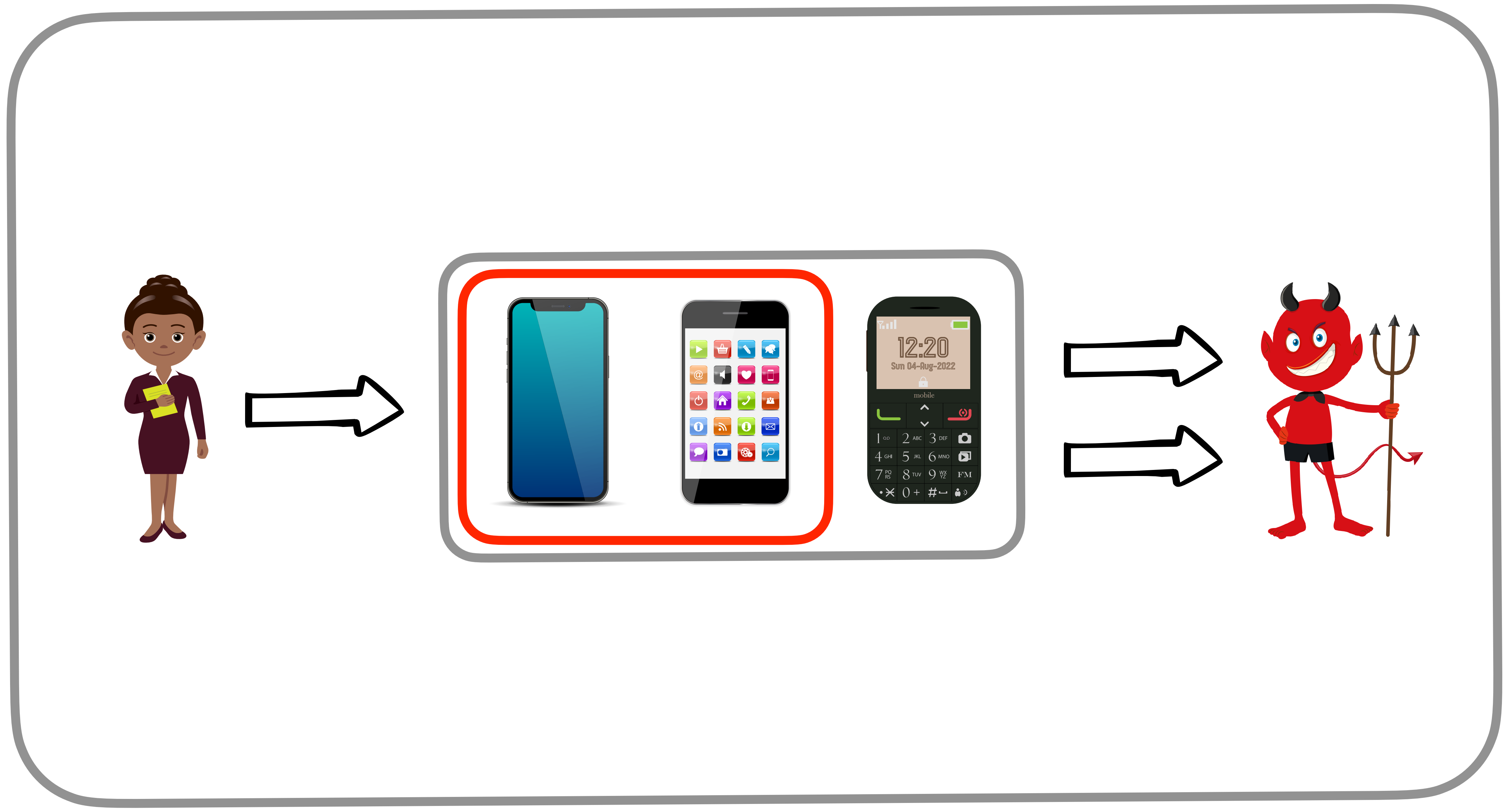


# The Randomized Robust Assortment Optimization Problem

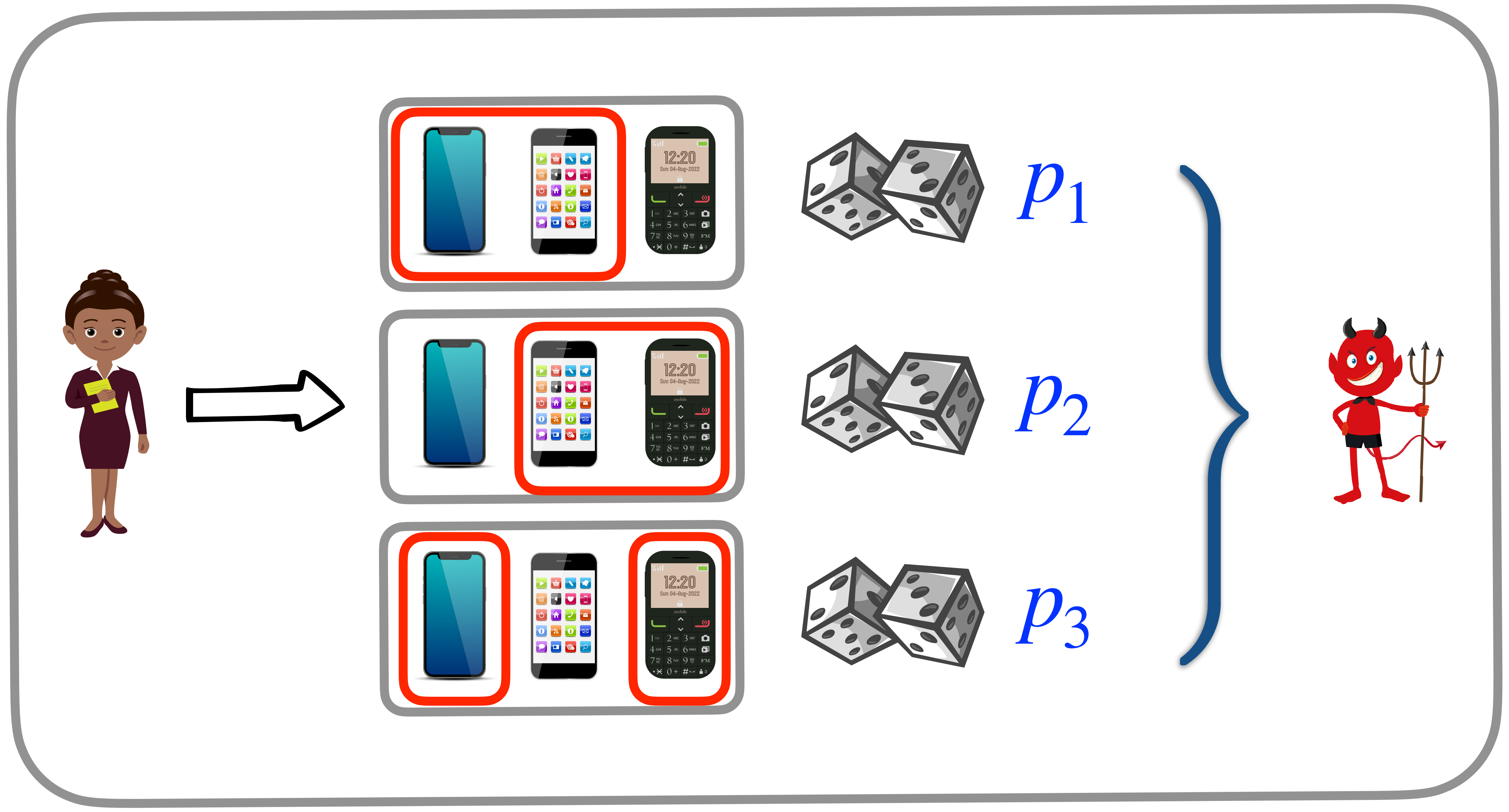
## Why does **randomization** **help**?

**2** "Game-Theoretic" Interpretation:

**deterministic robust**



**randomized robust**



# The Randomized Robust Assortment Optimization Problem

**Why does randomization help?**

3 **“Managerial” Interpretation:**

# The Randomized Robust Assortment Optimization Problem

## Why does randomization help?

### 3 “Managerial” Interpretation:


*randomization = diversification*

# The Randomized Robust Assortment Optimization Problem

## Why does randomization help?

3 “Managerial” Interpretation:


*randomization = diversification*

*But divide your investments among many places, for you do not know what risks might lie ahead.*

(Book of Ecclesiastes)

*My ventures are not in one bottom trusted,*
*Nor to one place; nor is my whole estate*
*Upon the fortune of this present year:*
*Therefore, my merchandise makes me not sad.*

(Merchant of Venice)

1. **When does randomization improve the worst-case profit?**

# ???

1. When does randomization improve the worst-case profit?

2. Is in-sample improvement = out-of-sample improvement?

1. When does randomization improve the worst-case profit?

2. Is in-sample improvement = out-of-sample improvement?

3. How can we compute optimal randomized assortments?

# ???

1. When does randomization improve the worst-case profit?
2. Is in-sample improvement = out-of-sample improvement?
3. How can we compute optimal randomized assortments?
4. How can we implement optimal randomized assortments?

# Agenda

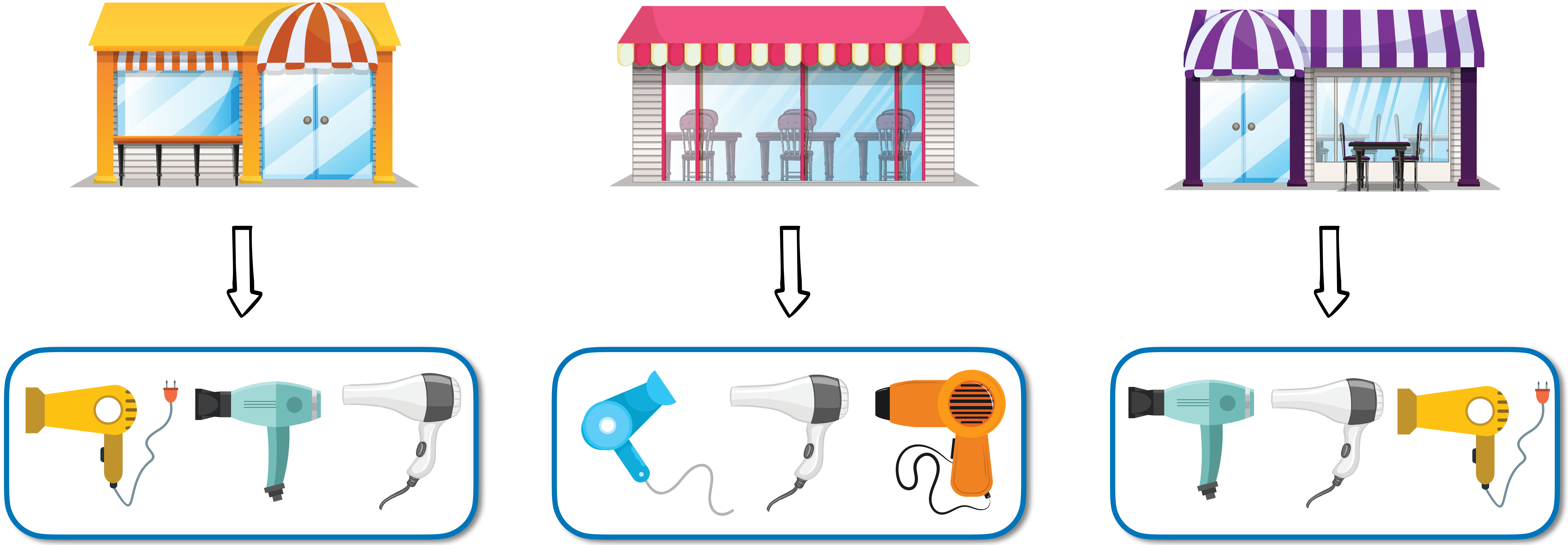
1. **Implementing Randomized Assortments**
2. When Does Randomization Help?
3. Computing Randomized Assortments
4. Numerical Experiments

**Randomization across different users:**


***Each user's experience can be kept consistent via cookies.***

# Implementation: The Brick-and-Mortar Setting

**Randomization across retail stores:**



***Possible for larger chains, not suitable for individual stores.***

https://www.bain.com/insights/successful-a-b-tests-in-retail-hinge-on-these-design-considerations/

1. ~~Implementing Randomized Assortments~~
2. **When Does Randomization Help?**
3. Computing Randomized Assortments
4. Numerical Experiments

**(Nominal) Assortment optimization problem:** $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \boldsymbol{r})$ where

- $\mathcal{N} = \{1, \ldots, N\}$: set of products
- $\mathcal{S} \subseteq \{S : S \subseteq \mathcal{N}\}$: set of admissible assortments
- $\mathfrak{C} : \mathcal{S} \to \Delta(\mathcal{N}_0)$: choice model; $\mathfrak{C}(i \,|\, S) = 0$ if $i \notin S$
- $\boldsymbol{r} = (r_1, \ldots, r_N)$: product prices

**(Nominal) Assortment optimization problem:** $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \boldsymbol{r})$ where

- $\mathcal{N} = \{1, \ldots, N\}$: set of products
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- $\boldsymbol{r} = (r_1, \ldots, r_N)$: product prices

$$R^\star_{\text{nom}} = \max_{S \in \mathcal{S}} R(S)$$

where $R(S) = \sum_{i \in S} r_i \cdot \mathfrak{C}(i \,|\, S)$

**Robust Assortment optimization problem:** $(\mathscr{N}, \mathscr{S}, \mathfrak{C}, \mathscr{U}, r)$ where

- $\mathscr{U}$: uncertainty set
- $\mathfrak{C} : \mathscr{S} \times \mathscr{U} \to \Delta(\mathscr{N}_0)$: choice model; $\mathfrak{C}(i \,|\, S, u) = 0$ if $i \notin S$

**Robust Assortment optimization problem:** $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \mathcal{U}, r)$ where

- $\mathcal{U}$: uncertainty set
- $\mathfrak{C} : \mathcal{S} \times \mathcal{U} \to \Delta(\mathcal{N}_0)$: choice model; $\mathfrak{C}(i \mid S, u) = 0$ if $i \notin S$

$$R^\star_{\text{det}}(\mathcal{U}) = \max_{S \in \mathcal{S}} \min_{u \in \mathcal{U}} R(S, u)$$

where $R(S, u) = \sum_{i \in S} r_i \cdot \mathfrak{C}(i \mid S, u)$

**Robust Assortment optimization problem:** $(\mathcal{N}, \mathcal{S}, \mathfrak{C}, \mathcal{U}, \boldsymbol{r})$ where

- $\mathcal{U}$: uncertainty set
- $\mathfrak{C} : \mathcal{S} \times \mathcal{U} \to \Delta(\mathcal{N}_0)$: choice model; $\mathfrak{C}(i \mid S, u) = 0$ if $i \notin S$

$$R^\star_{\text{det}}(\mathcal{U}) = \max_{S \in \mathcal{S}} \min_{u \in \mathcal{U}} R(S, u)$$

where $R(S, u) = \sum_{i \in S} r_i \cdot \mathfrak{C}(i \mid S, u)$

**Randomized Assortment optimization problem:**

$$R^\star_{\text{rand}}(\mathcal{U}) = \max_{\boldsymbol{p} \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u)$$

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***rand.-proof*** if $R^\star_{\text{rand}}(\mathcal{U}) = R^\star_{\text{det}}(\mathcal{U})$

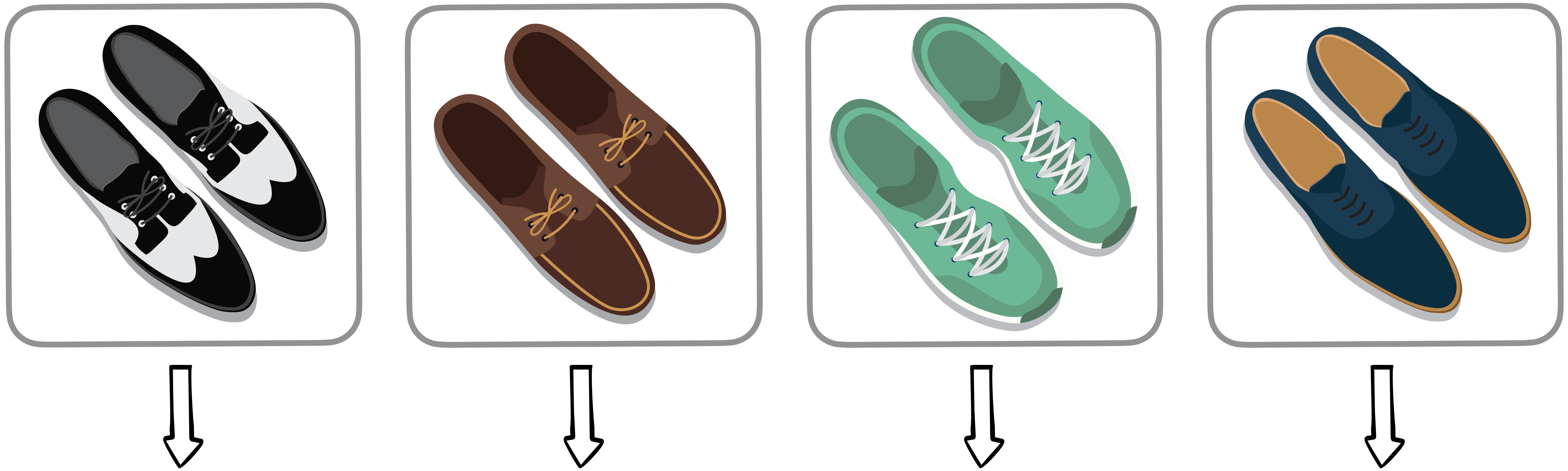
***rand.-receptive*** otherwise

## **The Multinomial Logit Model** (Luce '59, McFadden '80)

Luce (1959), Individual Choice Behavior: A Theoretical Analysis (Wiley, New York)
McFadden (1980), Econometric models for probabilistic choice among products. J. Bus. 53(3):13–29.

# Popular Choice Models

## **The Multinomial Logit Model** (Luce ‘59, McFadden ‘80)



$v_1$ $v_2$ $v_3$ $v_4$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**The Multinomial Logit Model** (Luce ‘59, McFadden ‘80)

$v_1$ $v_2$ $v_3$ $v_4$

**Purchase probability:** $\dfrac{v_i}{v_0 + \sum_{j \in S} v_j}$ if $i \in S$

selected assortment ($S$)

outside option ($v_0$)

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McFadden (1980), Econometric models for probabilistic choice among products. J. Bus. 53(3):13–29.

## Multinomial Logit

Helpful?

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# Popular Choice Models

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Helpful?

Complexity

# Popular Choice Models

## Multinomial Logit

**Helpful?**

**Complexity**

n/a

# Popular Choice Models

1. ~~Implementing Randomized Assortments~~
2. ~~When Does Randomization Help?~~
3. **Computing Randomized Assortments**
4. Numerical Experiments

**Randomized Assortment optimization problem:**

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**Robust optimization problem**

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**Robust optimization problem with two challenges:**

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**Robust optimization problem with two challenges:**

1. exponentially many decision variables

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**Robust optimization problem with two challenges:**

1. exponentially many decision variables
2. (typically) hard-to-evaluate objective function

## Two-Layer Primal-Dual Solution Approach

**The Randomized RO problem satisfies the following strong duality:**

$$\max_{p \in \Delta(\mathcal{S})} \min_{u \in \mathcal{U}} \sum_{S \in \mathcal{S}} p_S \cdot R(S, u) = \min_{\kappa \in \Delta(\mathcal{U})} \max_{S \in \mathcal{S}} \int_{u \in \mathcal{U}} R(S, u)\, \kappa(\mathrm{d}u)$$

Delage & Saif (2022), The Value of Randomized Solutions in Mixed-Integer Distributionally Robust Optimization Problems, INFORMS Journal on Computing 34(1):333-353

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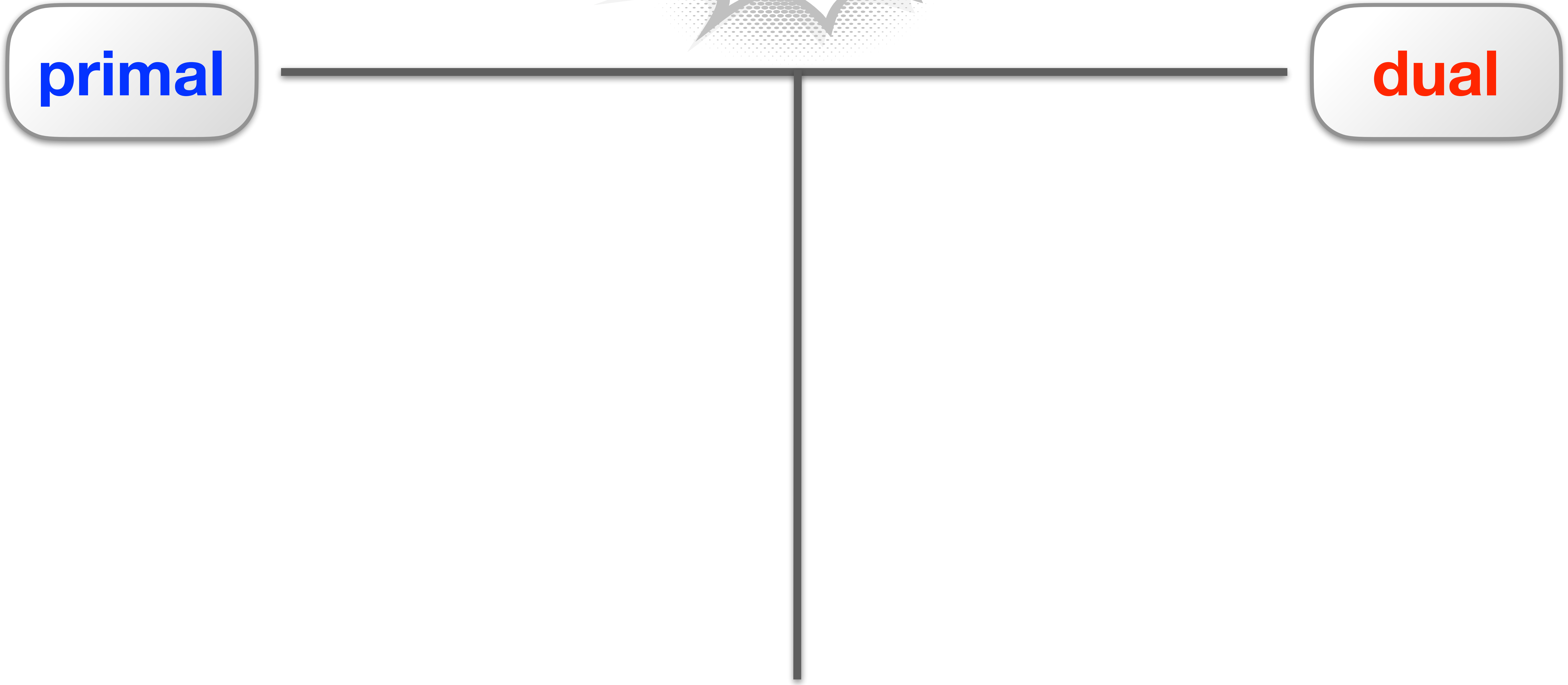

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**We use this strong duality in the outer layer of our solution approach:**

**primal** | **dual**

solve *restricted* primal

$$\max_{p \in \Delta(\hat{\mathcal{S}})} \min_{u \in \mathcal{U}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)$$

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| --- | --- |
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*add worst-case $u^\star$'s to $\hat{\mathcal{U}}$*

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**Theorem:** Finite ε-convergence to optimal $(\boldsymbol{p}^\star, \boldsymbol{\kappa}^\star)$.

# Two-Layer Primal-Dual Solution Approach

**Inner layer**: solve *restricted* primal $\max_{p \in \Delta(\hat{\mathcal{S}})} \min_{u \in \mathcal{U}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)$ with $\hat{\mathcal{S}} \subseteq \mathcal{S}$ "small"

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1. set LB $= -\infty$ and UB $= +\infty$; choose any $p \in \Delta(\hat{\mathcal{S}})$
2. while LB < UB:
   a. solve the evaluation problem

      $$\min_{u \in \mathcal{U}} \sum_{S \in \hat{\mathcal{S}}} p_S \cdot R(S, u)$$

      ⟹ LB $\leftarrow \max\{$LB, obj$\}$

      ⟹ $\hat{\mathcal{U}} \leftarrow \hat{\mathcal{U}} \cup \{u^\star\}$

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# Agenda

1. Implementing Randomized Assortments
2. When Does Randomization Help?
3. Computing Randomized Assortments
4. **Numerical Experiments**

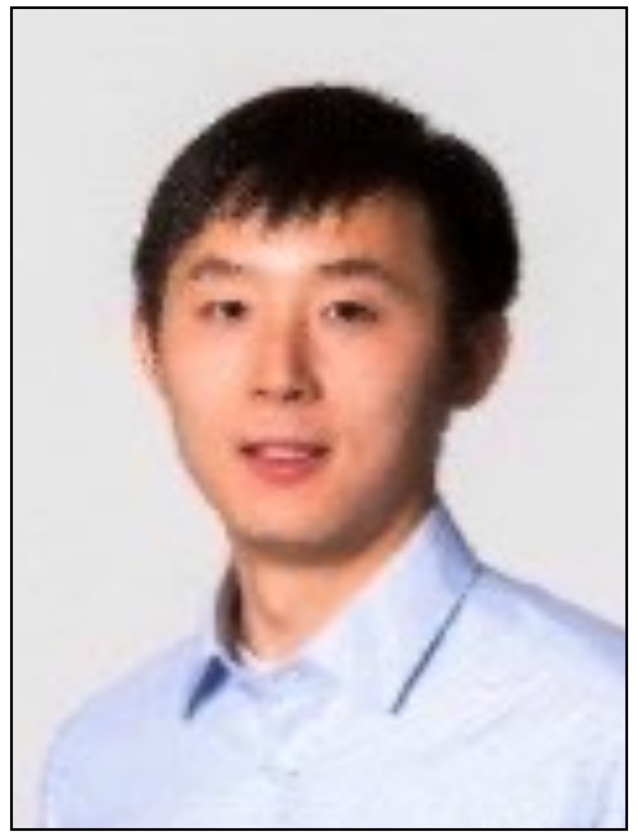
# In-Sample = Out-of-Sample?

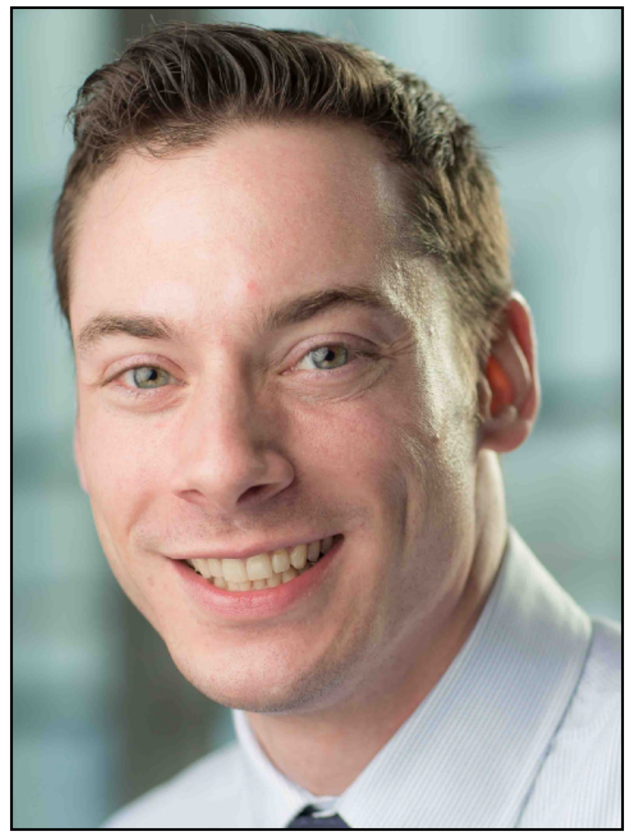
**Data-driven experiment for MNL model:**

- random MNL instances with 10 products
- purchase samples for random assortments under true model
- MLE estimation (with budget uncertainty set for robust approaches)

# This Presentation is Based on...

[1] Z. Wang, H. Peura and WW, Randomized Assortment Optimization, *Forthcoming in Operations Research*, 2024.





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