Size-Based Scheduling in Service Systems

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Joint work with:

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In a nutshell

We study scheduling policies based on (noisy) service-time information in service systems.

A motivational example: Scheduling in a bank's call center

Data set from SEE Lab (Technion).

- Individual-level call data (April 2007 - June 2009)
- We can track customers using their unique ID's
- Callers contact 12 times on average
- 1,835 agents in total, 400-450 agents on a day
- Customer abandonment: 4.5%

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Can we use data from the past transactions of a customer to predict their service times, and use these predictions to schedule more efficiently?

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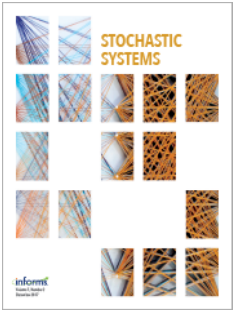
What if we have a noisy estimate of the service time, and the model is $M/GI/k + GI$?

A difficult problem, even in $M/G/1$



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Open Problem—Size-Based Scheduling with Estimation Errors

Douglas G. Down

Published Online: 18 Sep 2019 | https://doi.org/10.1287/stsy.2019.0041

Lu D, Sheng H, Dinda P (2004); Wierman and Nuyens (2008); Dell'Amico M, Carra D, Pastorelli M, Michiardi P (2014); Mailach and Down (2017); Scully, Grosof and Harchol-Balter (2020); Mitzenmacher (2021); Scully and Harchol-Balter (2021); Scully, Grosof, and Mitzenmacher (2022); Chen and Dong (2022).

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In $M/G/1$, we know that SRPT is optimal.

In $M/G/k$, we know that SRPT is asymptotically optimal.

Question: Assuming service times are perfectly known, what do we know about SRPT in multi-server queues with abandonment?

Answer: Nothing.

Plan for the talk

1. Service times perfectly known: SRPT in $M/GI/s + GI$

   Dong and Ibrahim. 2021. *SRPT scheduling in many-server queues with impatient customers.* Management Science.

2. Noisy service-time information: SJF in $M/GI/s + GI$

   Dong and Ibrahim. 2023. *SJF scheduling in many-server queues with impatient customers and noisy service-time estimates.*

Perfect Service-Time Information: SRPT in $M/GI/s + GI$

Snapshot of main results

Here, we will consider an asymptotic many-server overloaded regime, and we will focus on throughput.

(Atar, Giat, Shimkin 2010; Atar, Kaspi, Shimkin 2014; Bassamboo and Randhawa 2016; Puha and Ward 2021; ...)

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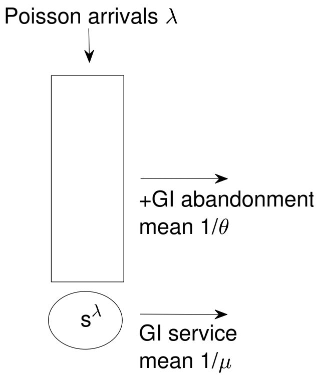
We will derive limits for steady-state performance measures.

We will demonstrate that SRPT asymptotically maximizes the throughput.

We will show that the SRPT system is well-approximated asymptotically by a two-priority system.

Modelling framework



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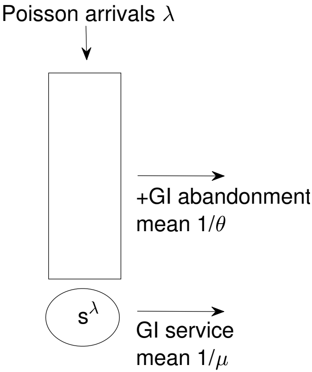

- Keep traffic intensity $\rho = \lambda / s^\lambda \mu > 1$ fixed
- Made stable by abandonments
- Let $\lambda \uparrow \infty$ and $s^\lambda \uparrow \infty$
- Abandonment and service-time distributions fixed
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- If all servers busy:
  - Update remaining processing times of all jobs in service
  - If service time of arrival $<$ longest remaining processing time in service $\Rightarrow$ preempt the longest remaining processing time
  - Else, join queue.

Simulation results under SRPT

We consider the $M/M/s + E_2$ system with $\rho = 1.4$ and $1/\theta = 1$.

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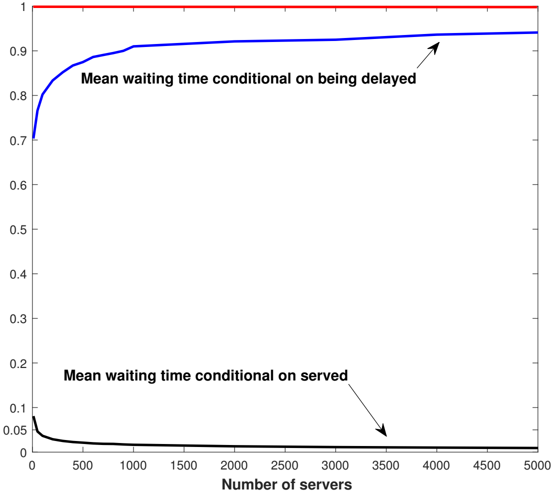

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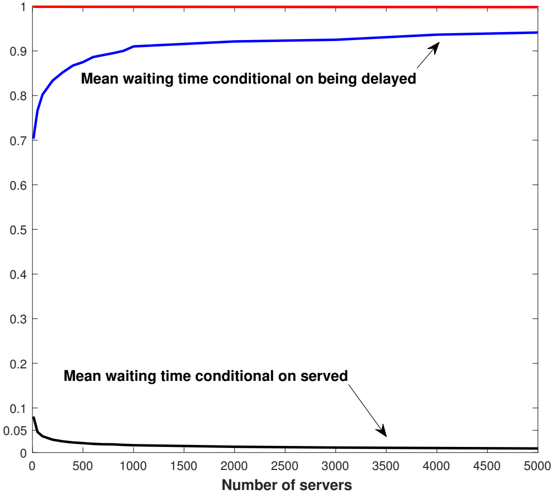

State-space collapse.

Threshold

Define the threshold τ:

$$\lambda \cdot \mathbb{P}(S \leq \tau) \cdot \mathbb{E}[S|S \leq \tau] = \lambda \mathbb{E}[S\mathbf{1}(S \leq \tau)] = s,$$

where we have:

- λ: arrival rate
- S: service time
- s: number of servers

See Chen and Dong (2022) for a similar idea in $GI/GI/1$.

Main Theorem

For the sequence of $M/GI/s^{\lambda} + GI$ queues under SRPT with $\rho^{\lambda} = \lambda/s_{\lambda}\mu > 1$ held fixed, in steady state:

1. $\lim_{\lambda\to\infty} \mathbb{P}(Served|S \leq \tau) = 1$.
2. $\lim_{\lambda\to\infty} \mathbb{P}(Served|S > \tau) = 0$.
3. $\lim_{\lambda\to\infty} \mathbb{E}[W|Served] = 0$.
4. $\lim_{\lambda\to\infty} \mathbb{E}[W|Abandon] =$ Mean time to abandon.
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4. $\lim_{\lambda \to \infty} \mathbb{E}[W | Abandon] =$ Mean time to abandon.
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That is, customers with short service times (below threshold) are served immediately, and customers with long service times eventually abandon.

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Use a coupling proof with a loss queue with two priority classes.

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3. Conclude, by induction, that you serve more customers in the SRPT system.
4. In loss system, all class 1 customers are served asymptotically and throughput is maximal.
5. Conclude that throughput is maximal in SRPT system, and derive limits for remaining performance measures.

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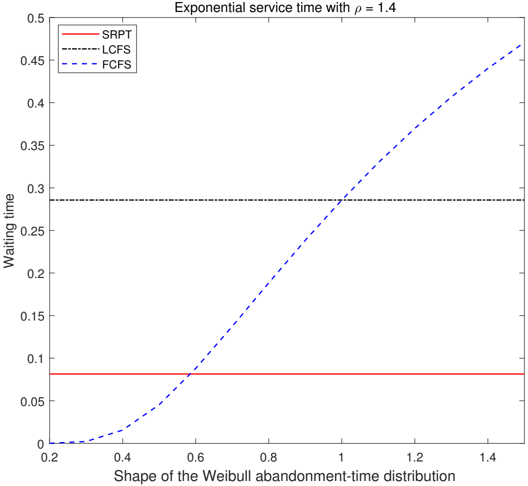
- SRPT minimizes the expected waiting time conditional on being served.
- SRPT maximizes the expected waiting time conditional on abandoning.
- Performance under SRPT is insensitive to the abandonment distribution, beyond the mean.

Effect of the abandonment distribution



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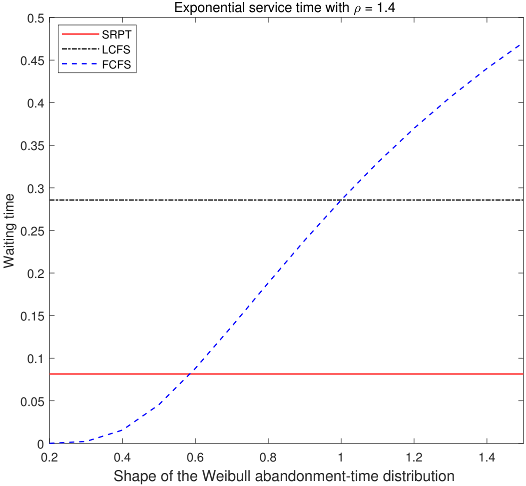

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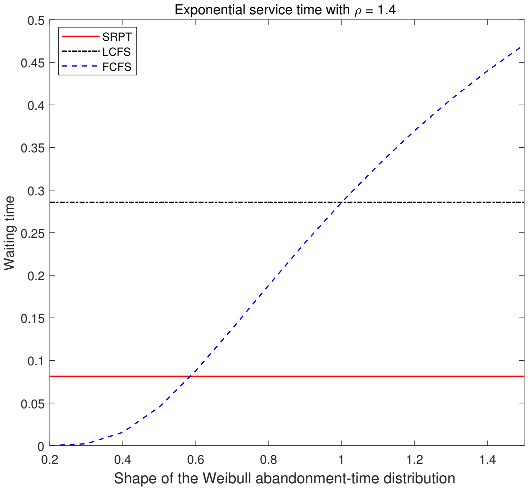
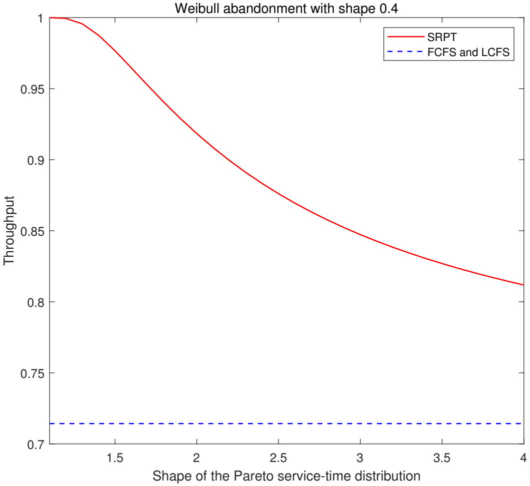


Recall that, among blind policies:

- Weibull shape $\alpha < 1 \Rightarrow$ DHR $\Rightarrow$ FCFS minimizes fluid waiting time
- Weibull shape $\alpha > 1 \Rightarrow$ IHR $\Rightarrow$ LCFS minimizes fluid waiting time

Bassamboo and Randhawa (2015).

Effect of the service-time distribution



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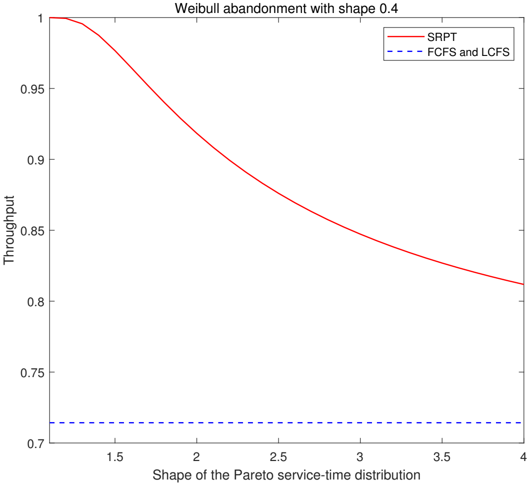

- $\uparrow$ shape Pareto service time $\Rightarrow$ lighter tail
- SRPT has stronger advantage under heavier tails

Lin, Wierman, Zwart (2011), Chen and Dong (2022).

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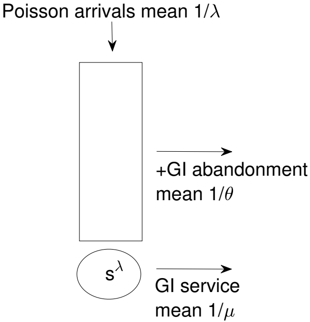
What if service-time predictions are noisy?

What if preemptions are not allowed?

Imperfect Service-Time Information: SJF in $M/GI/s + GI$

Modelling framework



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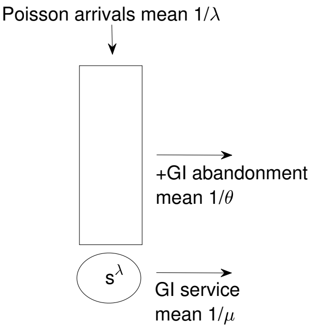

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- Let $\lambda \uparrow \infty$ and $s^\lambda \uparrow \infty$
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SJF scheduling with service-time predictions

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- Preemptions are not allowed.
- Arrival who finds empty server: begins service immediately.
- If all servers busy, join queue.
- When there is a service completion, schedule shortest predicted service time from queue first.

Imperfect service-time information

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We assume that $\mathbb{E}[S_i|\hat{S}_i = y]$ increases in y, for any y.

For example, this is satisfied in a regression model:

$$S_i = \hat{S}_i + \epsilon_i,$$

where $\hat{S}_i$ and ϵ_i are independent.

An updated threshold

Recall how we defined the threshold τ earlier:

$$\lambda \cdot \mathbb{P}(S \leq \tau) \cdot \mathbb{E}[S | S \leq \tau] = \lambda \mathbb{E}[S \mathbf{1}(S \leq \tau)] = s.$$

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Now, we define the threshold $\hat{\tau}$ as follows:

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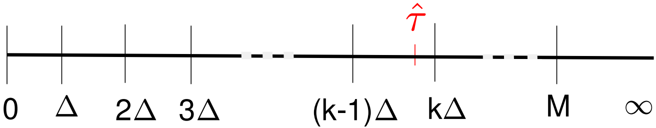
We show that, asymptotically, prioritizing customers with $\hat{S} < \hat{\tau}$ over customers with $\hat{S} \geq \hat{\tau}$ maximizes the throughput.

"Discretized" SJF Policy: SJF^{Δ}



- Class 1: $\hat{S} \in [0, \Delta)$
- Class 2: $\hat{S} \in [\Delta, 2\Delta)$
- Class 3: $\hat{S} \in [2\Delta, 3\Delta)$
- $\ldots$
- Class k: $\hat{S} \in [(k-1)\Delta, k\Delta)$
- $\ldots$
- Class $\lfloor M/\Delta \rfloor$: $\hat{S} \in [(\lfloor M/\Delta \rfloor - 1)\Delta, M)$
- Class $\lfloor M/\Delta \rfloor + 1$: $\hat{S} \in [M, \infty)$

Class has lower index $\Rightarrow$ Higher non-preemptive priority.

Main Theorem

For the sequence of $M/GI/s^{\lambda} + GI$ queues under SJF^{Δ} with $\rho^{\lambda} = \lambda / s_{\lambda}\mu > 1$ held fixed, in steady state:

1. $\lim_{\Delta \downarrow 0} \lim_{\lambda \to \infty} \mathbb{P}(\mathit{Served} | \hat{S} \leq \hat{\tau}) = 1.$
2. $\lim_{\Delta \downarrow 0} \lim_{\lambda \to \infty} \mathbb{P}(\mathit{Served} | \hat{S} > \hat{\tau}) = 0.$
3. $\lim_{\Delta \downarrow 0} \lim_{\lambda \to \infty} \mathbb{E}[W | \mathit{Served}] = 0.$
4. $\lim_{\Delta \downarrow 0} \lim_{\lambda \to \infty} \mathbb{E}[W | \mathit{Abandon}] =$ Mean time to abandon.
5. SJF^{Δ} maximizes throughput among non-preemptive policies that use the noisy service-time information.
6. SJF^{Δ} and SJF have asymptotically the same performance.

Use Atar, Kaspi, Shimkin (2014).

Accuracy of the approximation: SJF vs. Two-class priority rule

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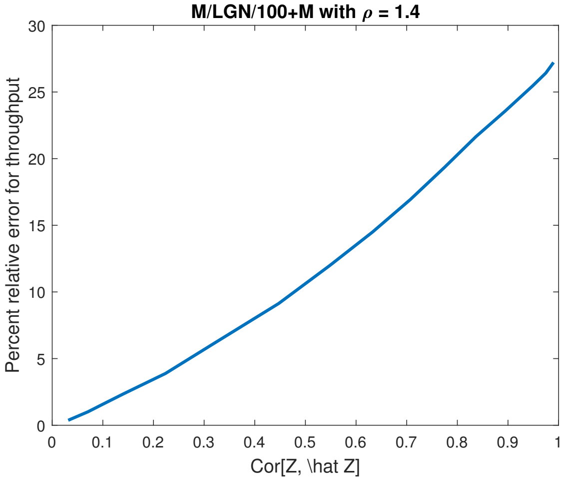
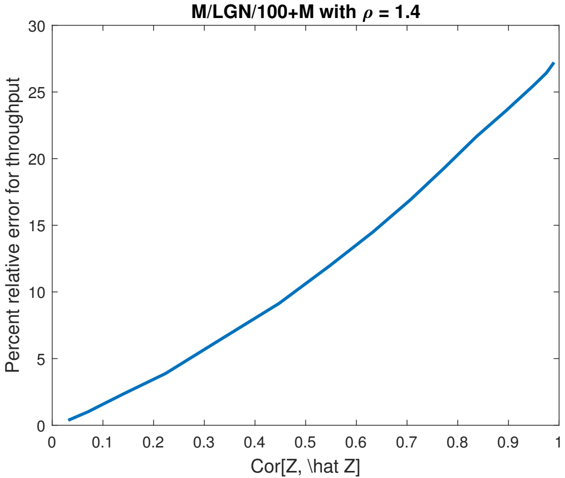


Accuracy of the approximation: SJF vs. Two-class priority rule



- The noisier the service times, the better the two-class approximation
- Recall that service-time predictions can be very noisy

Number of classes: Effect on throughput

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Consider the $M/LGN/100+M$ with $\rho = 1.4$.

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$r[Z, \hat{Z}]$	SJF	2 classes	3 classes	5 classes	10 classes
0.03	0.723	0.722	0.722	0.722	0.722
0.2	0.7677	0.7587	0.7477	0.7657	0.7674
0.5	0.8286	0.807	0.8201	0.8239	0.8243
0.7	0.8593	0.8314	0.8472	0.8487	0.8488
0.95	0.8764	0.8449	0.8595	0.8605	0.8606
0.99	0.8802	0.8474	0.8618	0.8626	0.8626

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Divide the high class ($< \tau$) into equally-sized classes.

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0.5	0.8286	0.807	0.8201	0.8239	0.8243
0.7	0.8593	0.8314	0.8472	0.8487	0.8488
0.95	0.8764	0.8449	0.8595	0.8605	0.8606
0.99	0.8802	0.8474	0.8618	0.8626	0.8626

- low correlation $\Rightarrow$ two-priority approximation very accurate
- high correlation $\Rightarrow$ some advantage in using 3 classes instead

Takeaways

- Theoretical results about performance of SRPT and SJF in many-server queues with abandonment.
- Implementing SRPT or SJF is hard. Usually, only two or three classes sufficient.
- Accuracy of approximation improves as the noise in the service-time prediction increases.

Thank you!