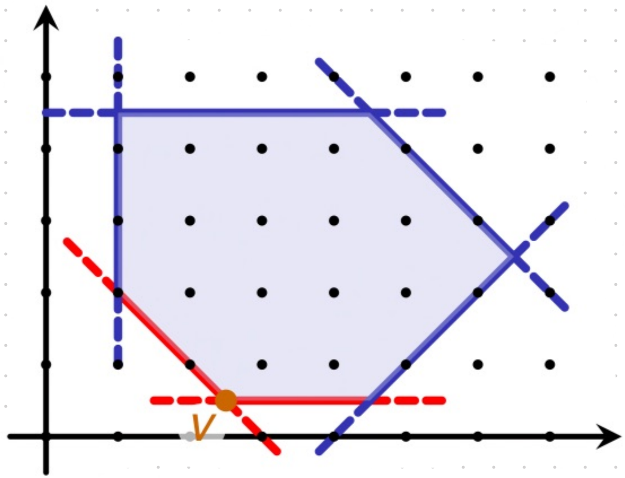
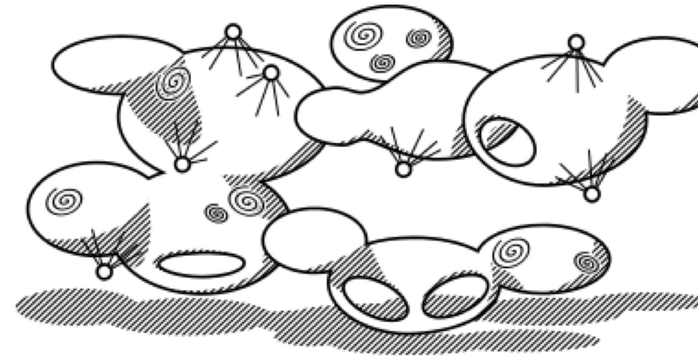




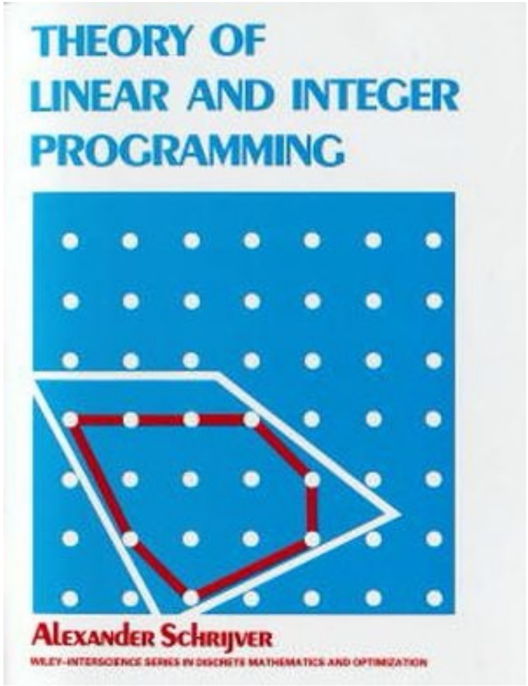
Integer Programs with Bounded Subdeterminants

S. Fiorini LNMB '24

Integer Programming

(IP) $\max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}$

where $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$,
$c \in \mathbb{Z}^n$



Q: What parameter(s) drive the complexity of (IP)?

Parameters of Interest

① $n = \#\text{columns}(A)$ [Lenstra '83]

② $m = \#\text{constraints}(A)$ [Papadimitriou '81]

③ tree-depth(A)

④ dual-tree-depth(A)

[EHKKLO '19] (+ others)

This talk:

$$\Delta(A) := \max\{|\det A'| : A' \text{ sq. submtx } A\}$$

DEF: $A \in \mathbb{Z}^{m \times n}$ is

- totally unimodular (TU) if $\Delta(A) \leq 1$
- totally Δ-modular if $\Delta(A) \leq \Delta$

WHY $\Delta(A)$?

- all vertices x^* of $P := \{x : Ax \le b\}$ have coordinates in $\mathbb{Z} \cup \frac{1}{2}\mathbb{Z} \cup \dots \cup \frac{1}{\Delta}\mathbb{Z}$ (whenever $\Delta(A) \le \Delta$)

- $\mathrm{diam}(P) = \mathrm{poly}(n, \Delta)$ [BSEHN '14]

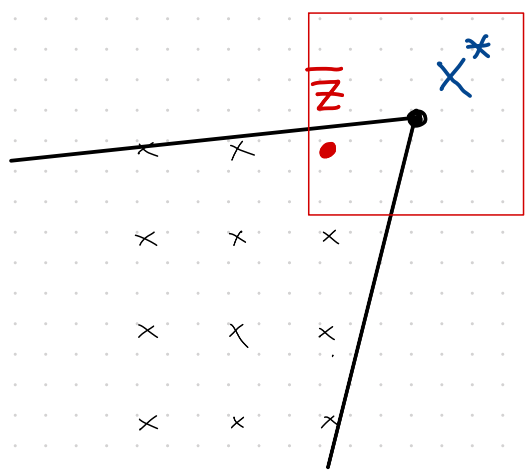
- If $\Delta(A) = 1$ then all vertices of P are integral! And solving (IP) is equivalent to solving:

$$(LP) \quad \max\{c^T x : Ax \leq b\}$$

- (LP) can be solved in strongly polynomial time if $\Delta(A) = O(1)$ [Tardos' 86]

THM [CGST '86] Suppose (IP) is bounded and feasible. Then $\forall x^*$ OPT sol of (LP), $\exists \bar{z}$ OPT sol of (IP) s.t.

$$\| x^* - \bar{z} \|_\infty \leq n \cdot \Delta$$



CONJECTURE ("Folklore"): $\forall \Delta \in \mathbb{Z}_{\geq 1}$ constant, can solve (IP) in (strongly) polynomial time if $\Delta(A) \leq \Delta$

Conjecture true for $\Delta = 1$: "TU case"

THM [Artmann, Weismantel, Zenklusen '17]:
Conjecture holds for $\Delta = 2$!

REM: Still open for all constants $\Delta \geq 3$

DECOMPOSITION OF TU MATRICES

THM [Seymour '80]: all TU matrices can be obtained, starting from

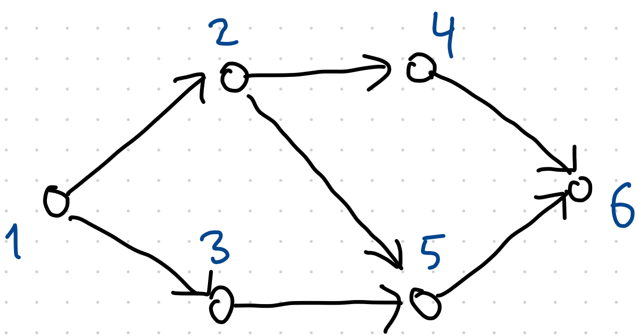
- network matrices
- transposed network matrices
- two sporadic TU matrices

by operations that preserve TUness

Network Matrices

Consider a min cost flow problem

$$\min \{ c^T x : Ax = b, \ \ell \leq x \leq u \}$$



$$A = \begin{bmatrix} +1 & +1 & & & & & \\ -1 & & +1 & +1 & & & \\ & -1 & & & +1 & & \\ & & -1 & & & +1 & \\ & & & -1 & -1 & & +1 \\ & & & & & -1 & -1 \end{bmatrix}$$

Pick basis and pivot

$$\begin{bmatrix} +1 & +1 & & & & & \\ -1 & & +1 & +1 & & & \\ & -1 & & & +1 & & \\ & & -1 & & & +1 & \\ & & & -1 & -1 & & +1 \\ & & & & & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} +1 & & +1 & & & & \\ -1 & +1 & & +1 & & & \\ & & -1 & & +1 & & \\ & -1 & & & & +1 & \\ & & & -1 & -1 & & +1 \\ & & & & & -1 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} +1 & & +1 & & & & \\ -1 & +1 & & +1 & & & \\ +1 & & & & +1 & & \\ & -1 & & & & +1 & \\ & +1 & & & & & +1 \\ & & & & & & \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{\begin{matrix} +1 & \\ -1 & +1 \\ +1 & \\ & -1 \\ & +1 \end{matrix}} & I \\ 0 & 0 \end{bmatrix}$$

D network mtx

The Sporadic Matrices

$$R_{10} = \begin{bmatrix} -1 & 1 & & & 1 \\ 1 & -1 & 1 & & \\ & 1 & -1 & 1 & \\ & & 1 & -1 & 1 \\ 1 & & & 1 & -1 \end{bmatrix}$$

$$R_{12} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & & \\ 1 & & 1 & 1 & \\ 1 & & & 1 & 1 \\ 1 & 1 & & & 1 \end{bmatrix}$$

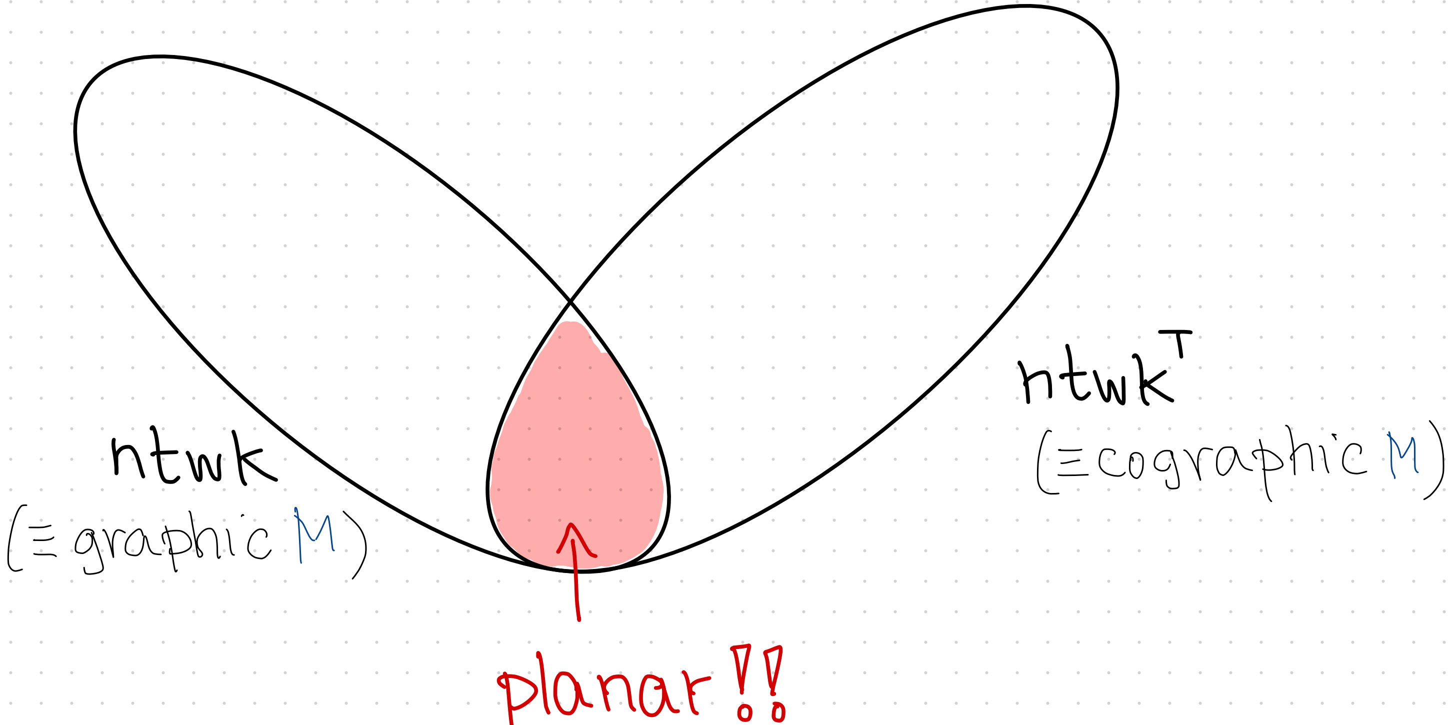
The Operations

1-sum: $A_1 \oplus_1 A_2 = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$

2-sum: $\left[\begin{array}{c|c} A_1 & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline a_1 & 1 \end{array}\right] \oplus_2 \left[\begin{array}{c|c} 1 & a_2 \\ \hline \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & A_2 \end{array}\right] = \left[\begin{array}{c|c} A_1 & 0 \\ \hline a_1 & a_2 \\ \hline 0 & A_2 \end{array}\right]$

and also 3-sum, pivoting, permuting $\begin{cases} \text{rows} \\ \text{cols} \end{cases}$ resigning $\begin{cases} \text{row} \\ \text{col} \end{cases}$, duplicating $\begin{cases} \text{row} \\ \text{col} \end{cases}$, adding $\begin{cases} \text{row} \\ \text{col} \end{cases}$ with ≤ 1 nonzero

INTUITION FOR SEYMOUR'S DECOMP



Circuits of Matrices

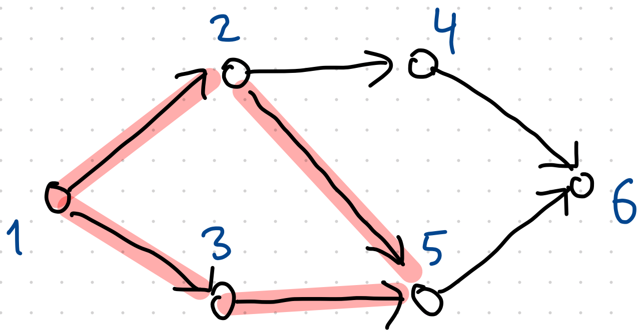
Def: $x \in \mathbb{Z}^n$ is a circuit of $A \in \mathbb{Z}^{m \times n}$ if

(i) $Ax = 0$

(ii) $x \neq 0$ and is support-min in $\ker A \setminus \{0\}$

(iii) $\gcd(x_1, \dots, x_n) = 1$

EXAMPLE:

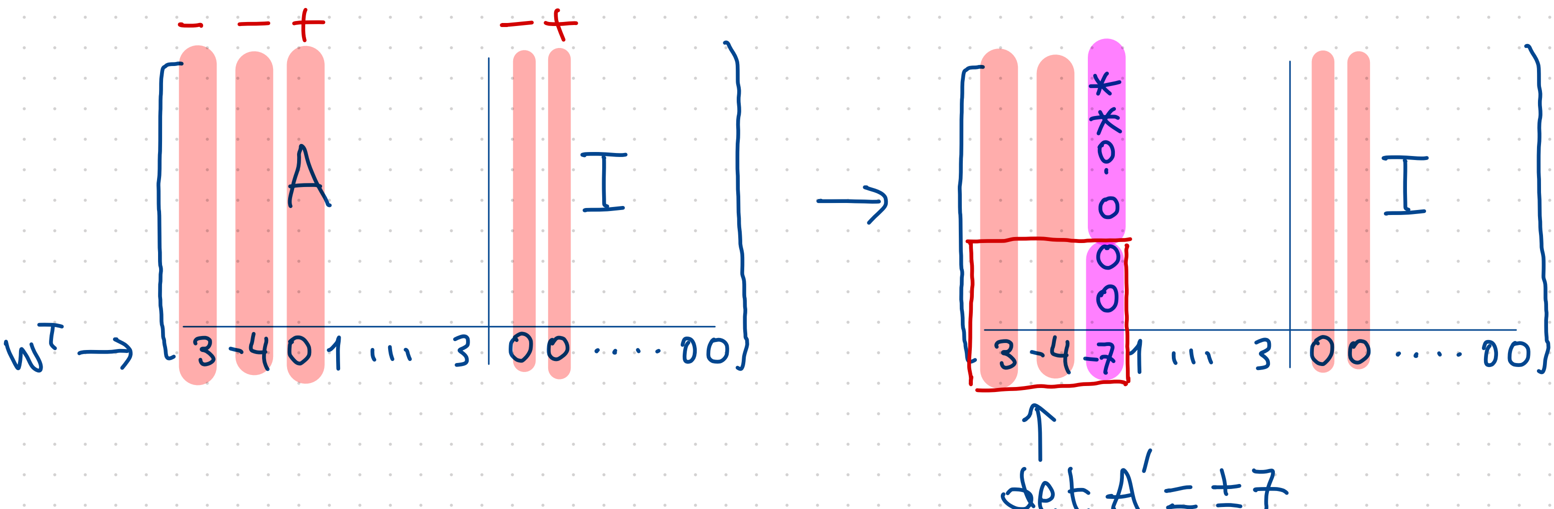


$$A = \begin{array}{c} \begin{array}{ccccccc} + & - & + & - & & & \end{array} \\ \begin{bmatrix} +1 & +1 & & & & & \\ -1 & & +1 & +1 & & & \\ & -1 & & & +1 & & \\ & & -1 & & & +1 & \\ & & & -1 & -1 & & +1 \\ & & & & & -1 & -1 \end{bmatrix} \end{array}$$

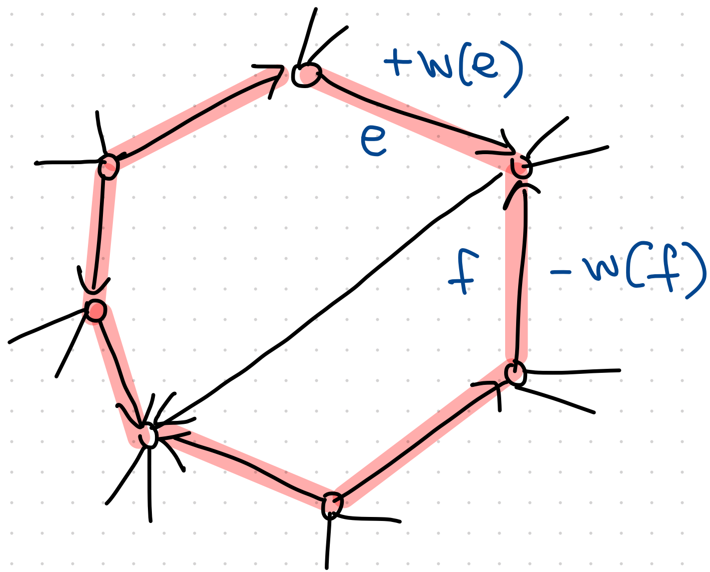
$$x = \begin{bmatrix} +1 \\ -1 \\ +1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ is a circuit of A

LEMMA: Suppose A is TU, take $w \in \mathbb{Z}^n$

Then $\Delta\begin{pmatrix} A \\ w^T \end{pmatrix} = \max\left\{ w^T x : \begin{pmatrix} x \\ y \end{pmatrix} \text{ circuit of } (A\ I) \right\}$



① If A is network, then $\Delta\begin{pmatrix} A \\ w^T \end{pmatrix}$ is max weight of circuit in a directed graph G

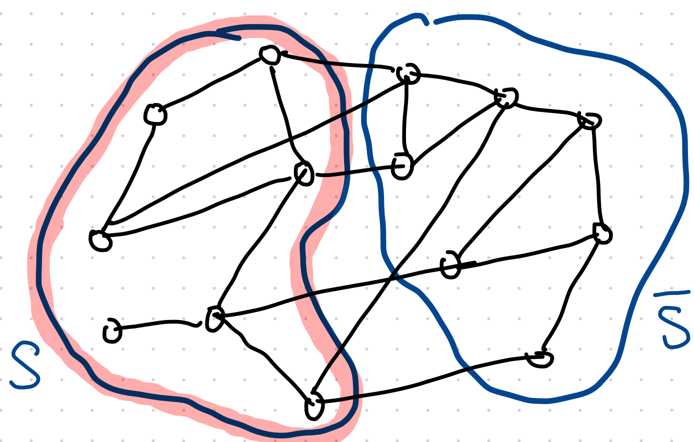


② If A is transposed network then

$$\Delta\begin{pmatrix} A \\ w^T \end{pmatrix} = \max\left\{ \bar{w}(S) : S \subseteq V(G),\ S \text{ and } \bar{S} \text{ are connected} \right\}$$

where $\bar{w}(v)$ are weights on vertices of G

s.t. $\sum_{v \in V} \bar{w}(v) = 0$

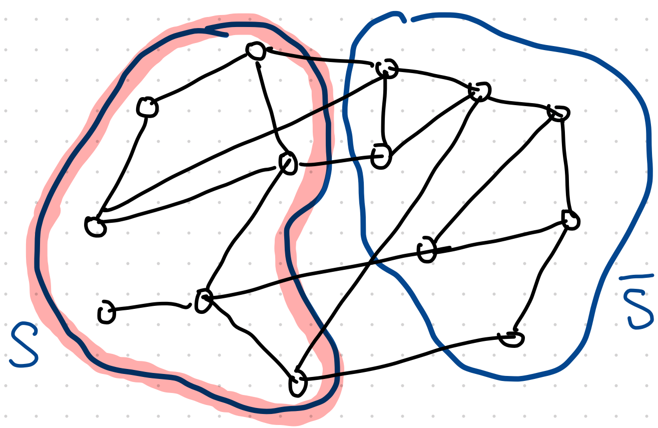


SOME OF THE "MAGIC"

G graph, 4-connected

$\overline{w} \in \mathbb{Z}^{V(G)}$ s.t. $\sum_{v \in V(G)} \overline{w}(v) = 0$

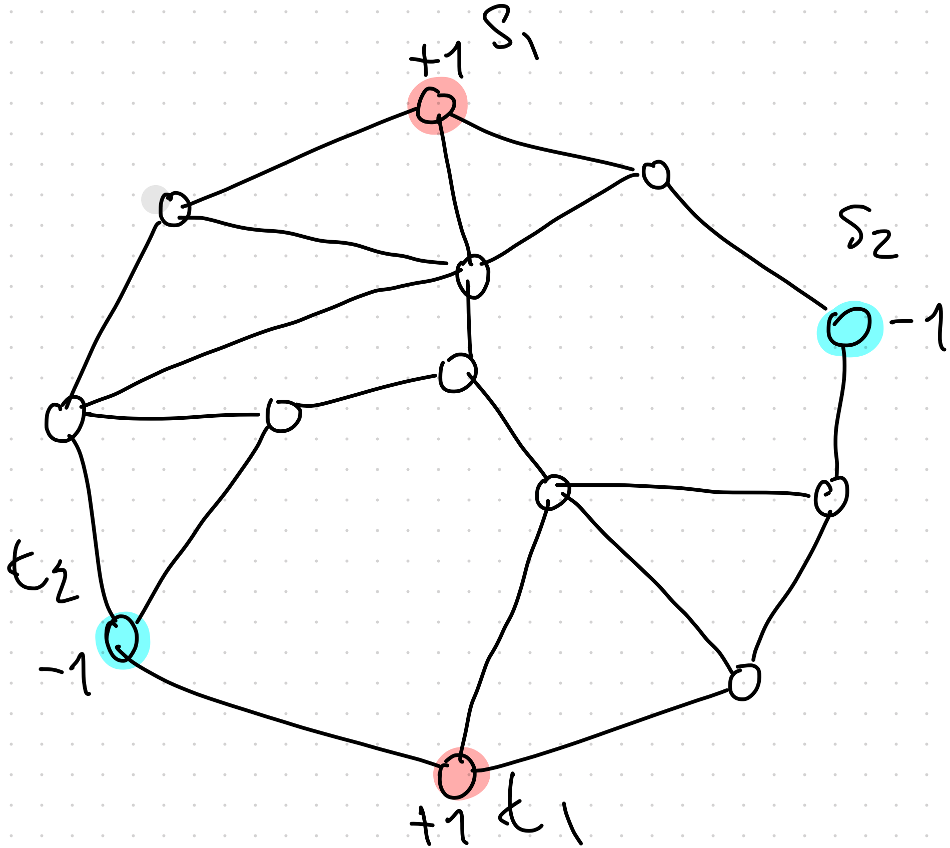
$\sum_{v \in S} \overline{w}(v) \leq 1 \quad \forall$ "docset" S



REM: $\overline{w}(v) \in \{-1, 0, +1\} \ \forall v \in V(G)$

DEF: v terminal if $\overline{w}(v) \neq 0$

Either #terminals $\leqslant 2$, and always OK
or #terminals $\geqslant 4$, and ...



+1
s_1
s_2
−1
t_2
−1
+1
t_1

THE BIMODULAR ALGORITHM

Assuming $\Delta(A) \leq 2$, [AWZ '17] solve

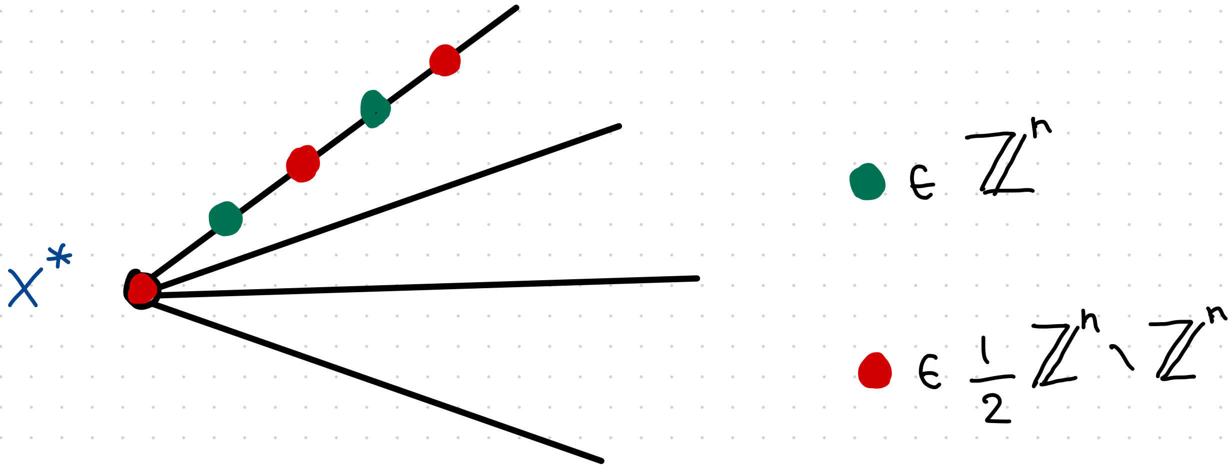
$$\text{(IP)} \quad \min\{c^T x : Ax \geq b,\ x \in \mathbb{Z}^n\}$$

Steps: ① solve (LP) $\rightarrow$ get x^*

② tight constraints $\bar{A}x = \bar{b}$

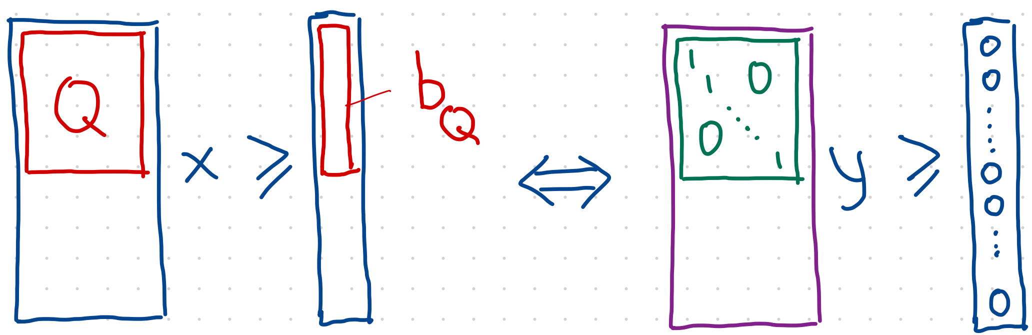
$$\boxed{\bar{A}}\, x \geq \boxed{\bar{b}}$$

[Veselov & Chirkov '09]



③ Drop constraints not tight at x^*

④ Pick basis Q and let $y := Q(x - x^*)$



$$\text{(IP)} \quad \min\{d^T y : Ty \geq 0,\ Q^{-1}y + x^* \in \mathbb{Z}^n\}$$

$$\Updownarrow$$

$$y \in \mathbb{Z}^n,\ \sum_{i \in S} y_i \text{ odd}$$

where T is TU mtx

$Ty \geq 0 \iff Ty - z = 0, \; z \geq 0$

Every sol is sum of circuits of $(T - I)$

$$\begin{bmatrix} y \\ z \end{bmatrix} = c_1 + c_2 + \cdots + c_t$$

An odd number of c_i's are odd

$\Longrightarrow t = 1$ wlog !!

Final problem is min cost odd circuit

Given TU mtx $A \in \{0, \pm 1\}^{m \times n}$

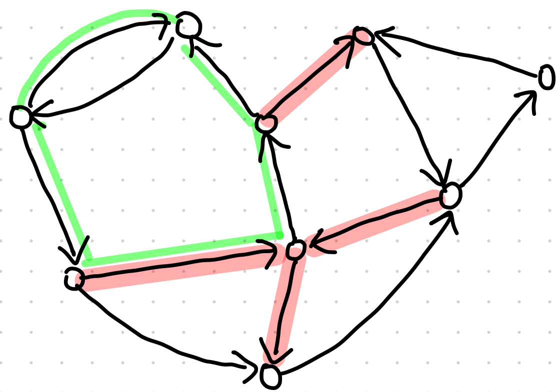
cost vector $c \in \mathbb{Z}^n$

subsets $I, J \subseteq [n]$

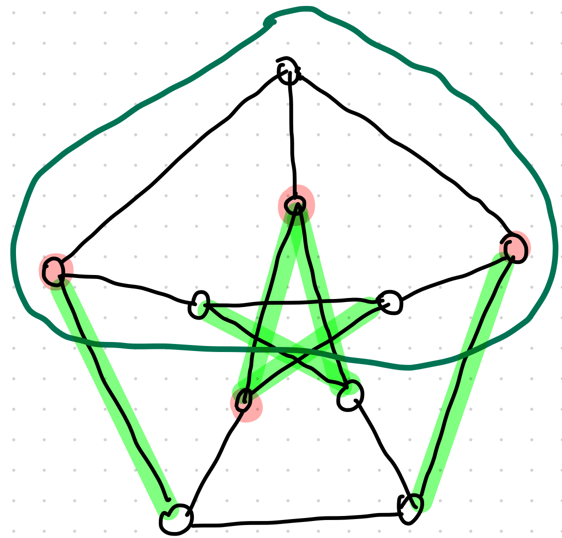
Find $\min \{ c^T x : Ax = 0,\ x_i \geq 0\ \forall i \in I,\ \sum_{j \in J} x_j \equiv 1 \pmod 2 \}$

BASE CASES FOR MCOC

<u>Graphic case</u> : min odd cycle
(=ntwk)



Co-graphic case : min odd cut
(= ntwk^T)



TΔM IP's with $\leqslant 2$ Nonzeros per Row

<u>Thm</u> [FJWY'21] For every fixed $\Delta \in \mathbb{Z}_{\geqslant 1}$ there is a (strongly) polynomial time algorithm for

$$(\mathrm{IP}) \quad \max\{c^T x : Ax \leqslant b,\ x \in \mathbb{Z}^n\}$$

where A has $\Delta(A) \leqslant \Delta$ and

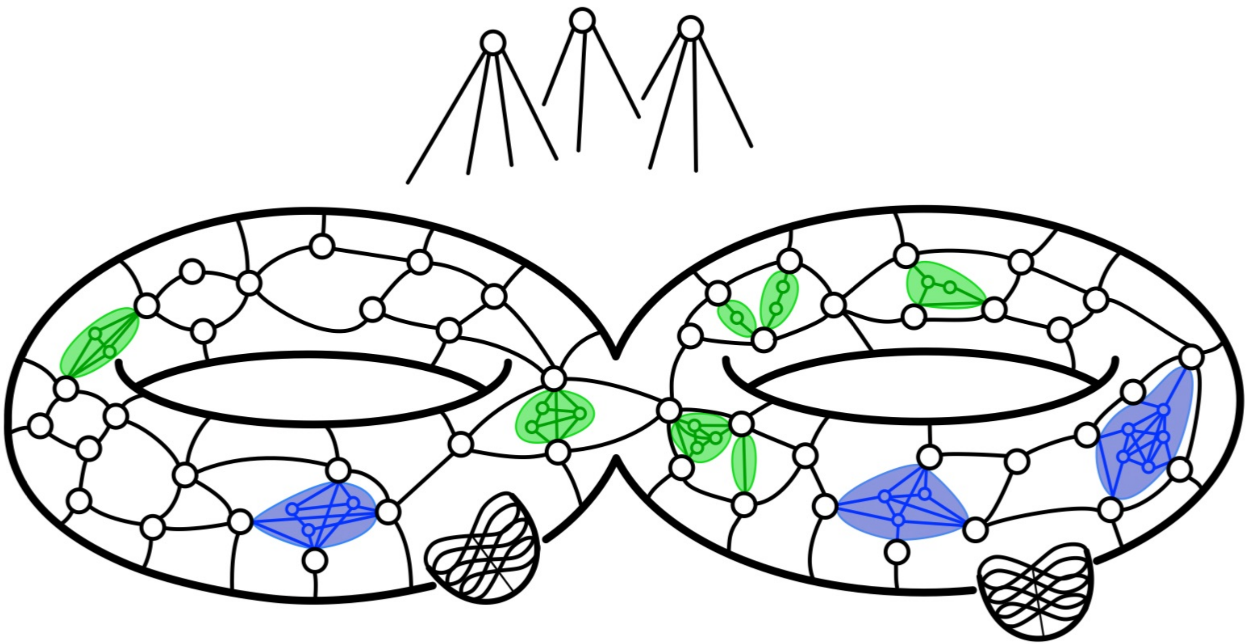
$\leqslant 2$ nonzeros per row

(or per col)

Approach has three parts:

1. Little IP theory to reduce to stable set problem on graphs with odd cycle packing number $O(\log \Delta)$
2. Heavy graph (minor) theory to "control" structure of such graphs
3. Intricated Dynamic Programming algorithm using structure

STRUCTURE OF GRAPHS WITH SMALL OCP



TU + k Rows

Since September '22, I am working on solving (IP) $\max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}$ where A has $\Delta(A) \leq \Delta$ and

$$A = \left[\begin{array}{c} T \\ \hline W \end{array}\right] \begin{array}{l} \left.\vphantom{T}\right\} \text{ TU} \\ \left.\vphantom{W}\right\} k \text{ extra rows} \end{array}$$

(k constant)

together with:

- Manuel Aprile (U Padova)
- Stefan Kober
- Gwenaël Joret } since July '23
- Michał Seweryn
- Stefan Weltge (TU Munich)
- Lena Yuditsky

THM [AFKJSWY'24+]

For every constants $\Delta, k \in \mathbb{Z}_{\geq 1}$ there is a (strongly) polynomial time algorithm for solving

$$(\text{IP}) \quad \max\{c^T x : Ax \leq b, x \in \mathbb{Z}^n\}$$

where A is totally Δ-modular and becomes ntwk^T after deleting $\leq k$ rows and columns

OUR APPROACH

① IP theory to

- relate $\Delta(A)$ to circuits of $\begin{pmatrix} T & I \end{pmatrix}$
- bound #augmentations on circuits

② Graph theory to

- control graphs with no rooted $K_{2,t}$ minor

③ Dynamic Program

About $\Delta(A)$

Recall:

LEMMA: Let $T \in \{0, \pm 1\}$ be TU, $w \in \mathbb{Z}^n$

Then $\Delta\begin{pmatrix} T \\ w^T \end{pmatrix} = \max\{w^T x : \begin{pmatrix} x \\ y \end{pmatrix}$ circuit of $(T\ I)\}$

Proximity Result

$$(IP_2) \quad \max\{c^T x : Ux = b,\ \ell \le x \le u,\ Wx \le d,\ x \in \mathbb{Z}^n\}$$

LEMMA (informal): Can efficiently compute integer point z satisfying all "TU constraints", s.t. $\exists$ opt. solution

$$\bar{z} = z + \sum_{i=1}^{t} c_i$$

where $c_1, \dots, c_t$ are circuits and $t \le f(k, \Delta)$

Final Problem

$$\max\left\{ c^T x \;:\; x_i - x_j \leq b(i,j) \;\; \forall (i,j) \in E(G) \right.$$
$$\left. Wx \leq d, \;\; x \in \mathbb{Z}^n \right\}$$

s.t. G directed graph

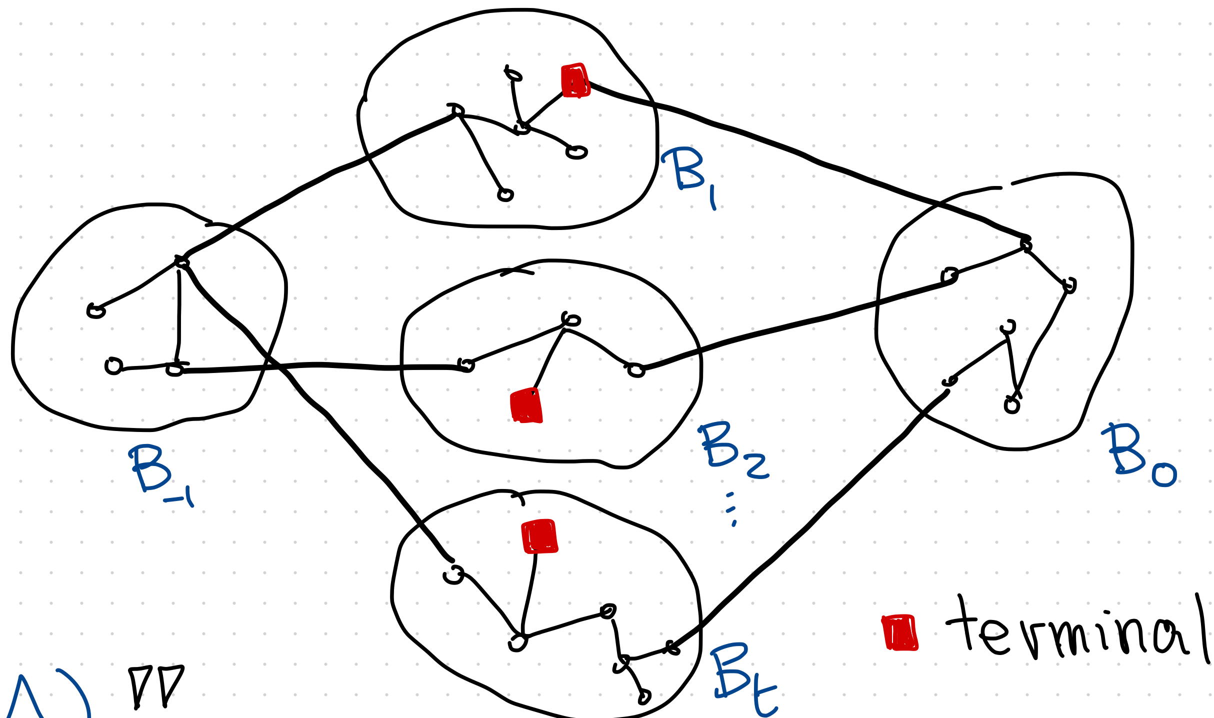
$c^T \mathbb{1} = 0$, $\;W \mathbb{1} = 0$,

$\|W \chi^S\|_\infty \leq \Delta \quad \forall$ docsets $S \subseteq V(G)$

Rooted $K_{2,t}$ Minors

__Def:__ $v \in V(G)$ is terminal if $W^v \neq 0$

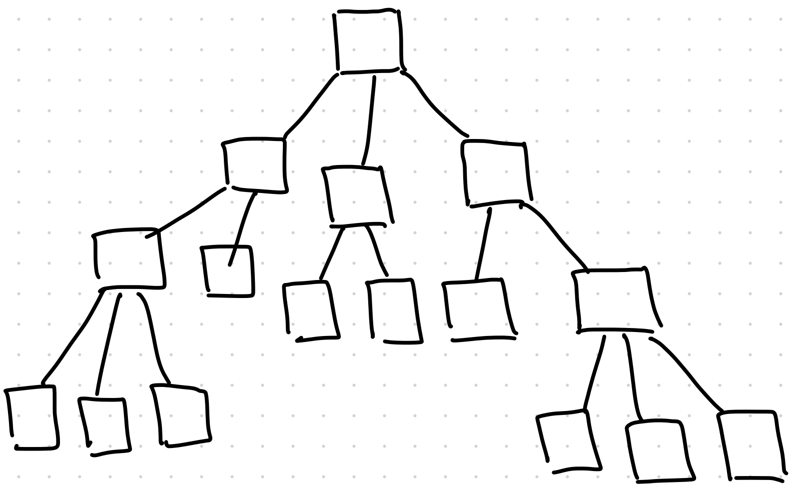
rooted $K_{2,t}$ minor:



Forbidden for $t = g(k, \Delta)$!!

GRAPH STRUCTURE

∃ decomposition tree of G s.t.



- intersection of two adjacent bags is bounded
- each node has bounded #children with terminals

- docsets of G intersect terminals in each bag in "nice" way (poly # ways)

OPEN PROBLEMS

① Generalize thm to all TU matrices (graphic case + 3-sums)

② Further generalize to "binet" matrices and their transposes

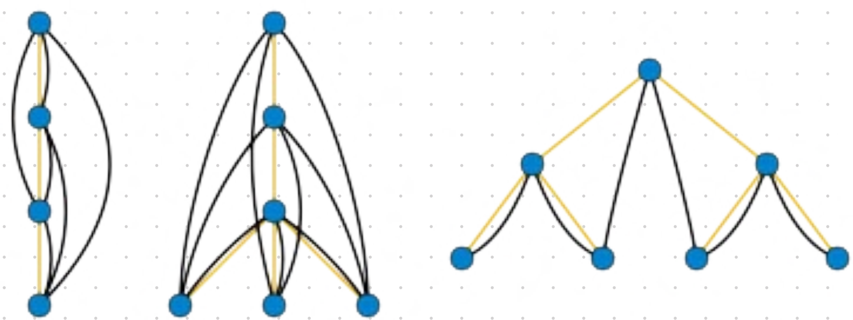
③ Solve whole conjecture???


Dank U

Extra Slides

Tree-Depth

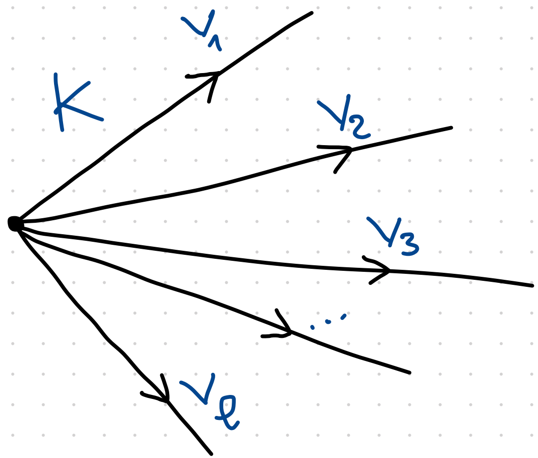
<u>Def</u>: tree-depth(G) = min height of forest F s.t. every edge of G connects pair of vtcs having ancestor-descendant relationship



[Cook et al. '86]

PROOF IDEA: Take any opt. IP sol. $\bar{z}$, look at $y := \bar{z} - x^*$ and partition rows A into $A^{\leq}, A^{\geq}$ s.t. $A^{\leq} y \leq 0$, $A^{\geq} y \geq 0$.

Consider cone $K := \{x : A^{\leq} x \leq 0, A^{\geq} x \geq 0\}$

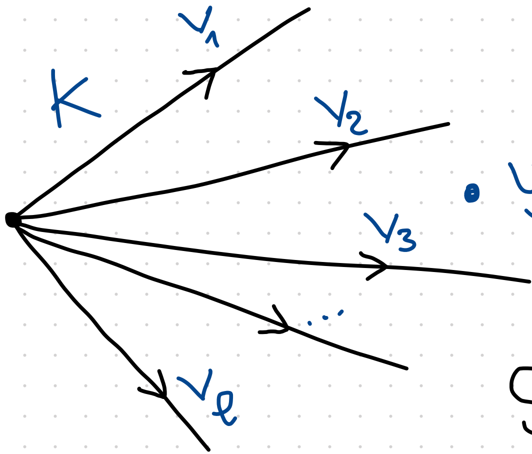


Then $\exists\, v_1, \dots, v_\ell \in \mathbb{Z}^n$:

$K = \mathrm{cone}\{v_1, \dots, v_\ell\}$ and

$\|v_i\|_\infty \leq \Delta \quad \forall i$

(Cramer's rule)



- $y = \bar{z} - x^* = \sum_{i=1}^{k} \lambda_i v_i$ where $k \leq n$ (Caratheodory)

Show $0 \leq \lambda_i < 1$ $\forall i$ wlog

$$\Rightarrow \|\bar{z} - x^*\|_\infty = \|y\|_\infty \leq \sum_{i=1}^{k} \lambda_i \|v_i\|_\infty \leq n \cdot \Delta$$

Δ

MATROIDS

are discrete structures "axiomatizing" linear independence

Given $\mathbb{F}$ field get matroid $M = M(A)$
$A \in \mathbb{F}^{m \times n}$

- elements of M = columns of A
- independent sets of M = sets of linearly independent columns of A
- circuits of M = minimally dependent sets

DEF: Matroid M is regular if

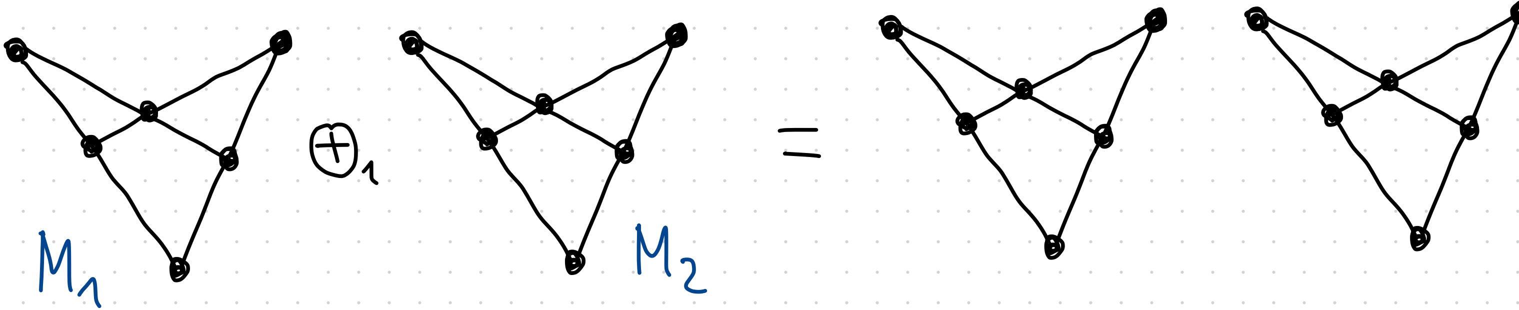
$\exists A$ TU s.t. $M = M(A)$ (over $\mathbb{R}$)

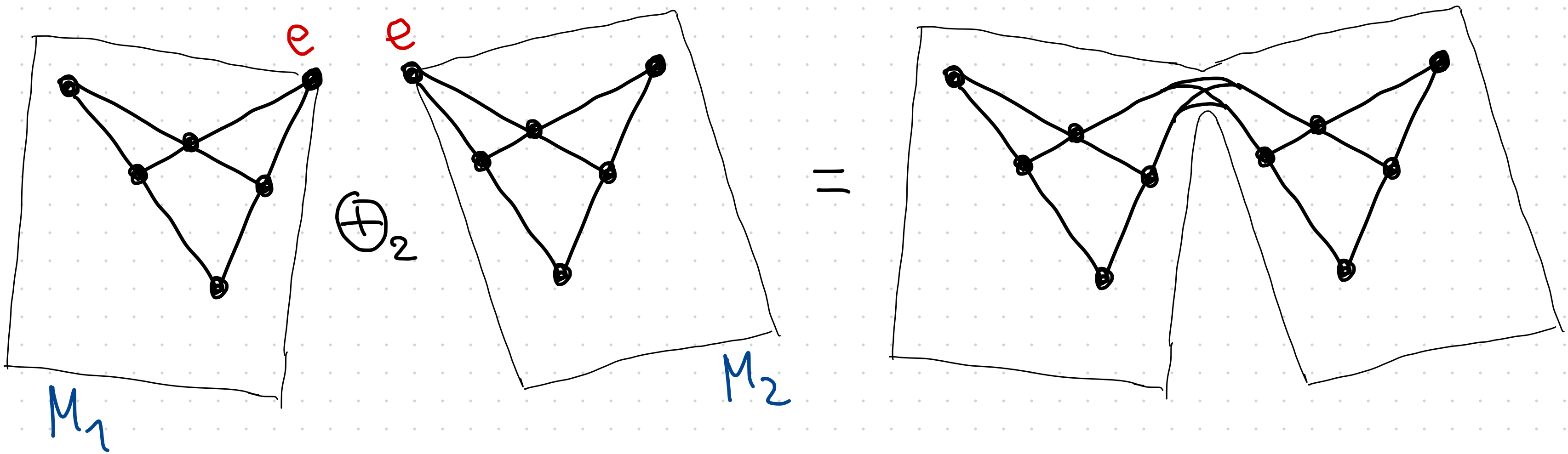
THM: (i) M is regular

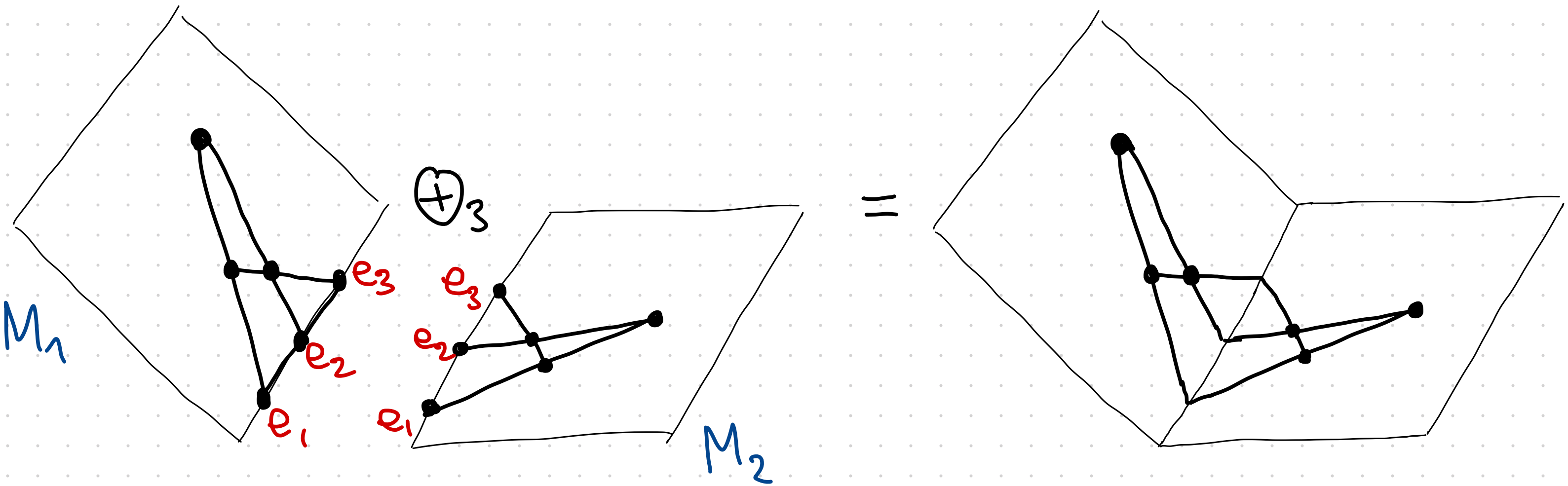
$\Leftrightarrow$ (ii) M can be represented over any field

$\Leftrightarrow$ (iii) M can be represented over $GF(2)$ and $GF(3)$

K-SUMS "MATROIDALLY"







e3
e2
e1
M_1
$\oplus_3$
e3
e2
e1
M_2
=