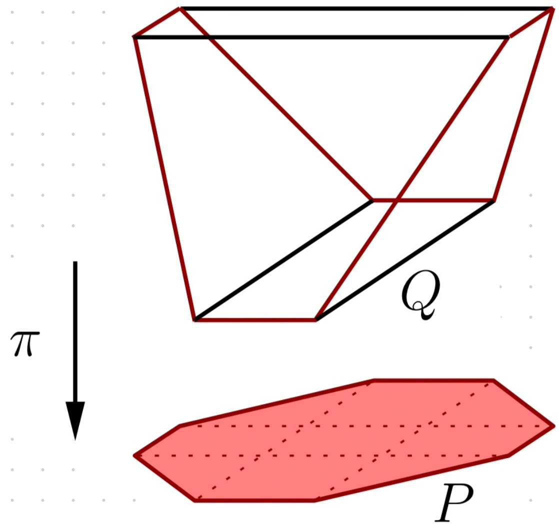
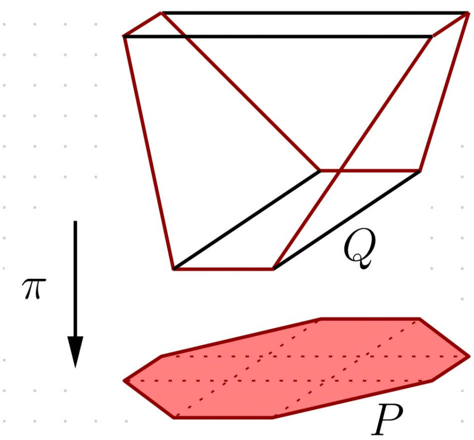


Extended Formulations and Extension Complexity

S. Fiorini LNMB '24

EXTENDED FORMULATIONS



DEF. $P \subseteq \mathbb{R}^d$ polytope

$Q = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^k : Ex + Fy \leq g\}$

is extended formulation of Q if

$Ax \leq b \Leftrightarrow x \in P \Leftrightarrow \exists y : Ex + Fy \leq g$

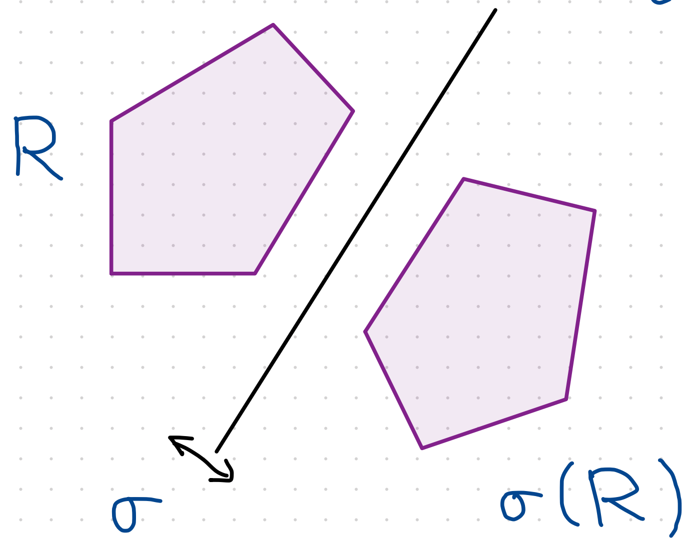
size = # ineqs

EXTENSION COMPLEXITY

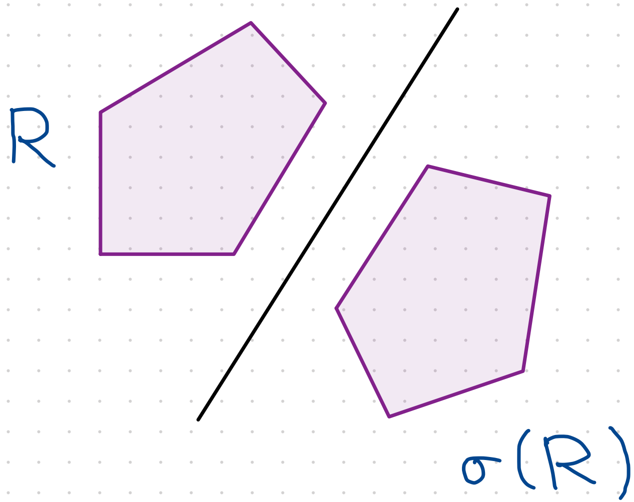
DEF. $xc(P) := \min \text{ size}(\text{EF of } P)$

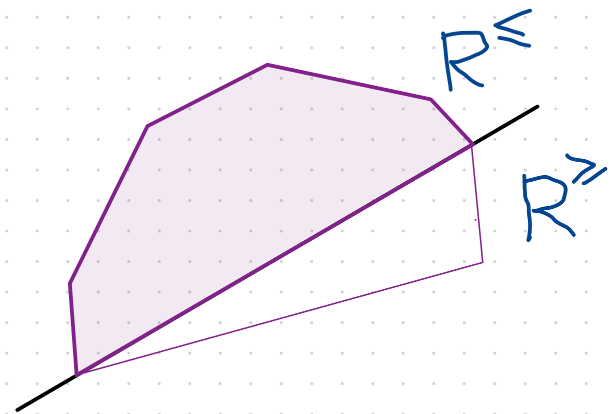
Reflections

$H := \{x : a^T x = \beta\}$



$$\operatorname{conv}(R \cup \sigma(R)) = \{x : \exists\, y, \lambda \text{ s.t. } y \in R,\; x = y + \lambda a,\; a^T y \leq a^T x \leq 2\beta - a^T y\}$$





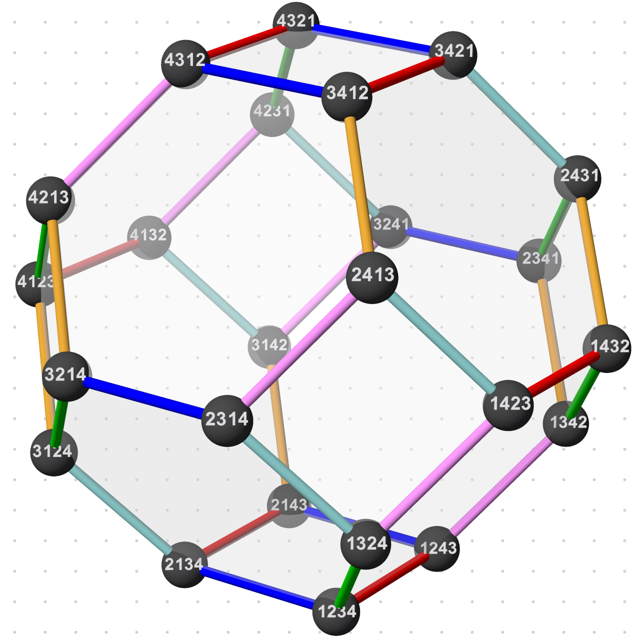
$$\Rightarrow \operatorname{xc}(\operatorname{conv}(R^{\leq} \cup \sigma(R^{\leq}))) \leq \operatorname{xc}(R) + 2$$

REGULAR POLYGONS

$xc\,(\text{regular } 2^k\text{-gon}) \leq 2k$

[Ben-Tal & Nemirovski '01]

Permutahedron

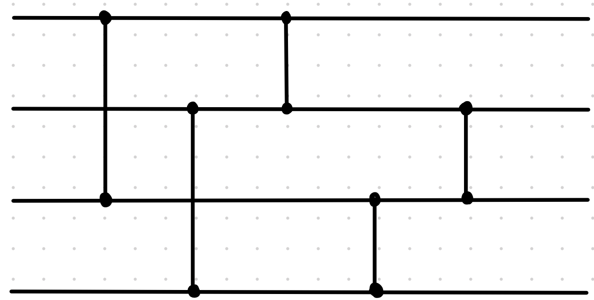


Def: n-permutahedron

$$:= \operatorname{conv}\left\{ \begin{pmatrix} \alpha(1) \\ \vdots \\ \alpha(n) \end{pmatrix} : \alpha \in \mathrm{Sym}(n) \right\}$$

Use hyperplanes $H_{ij} := \{ x \in \mathbb{R}^n : x_i = x_j \}$

$\exists$ sorting network with $O(n \log n)$ comparators (AKS '83)



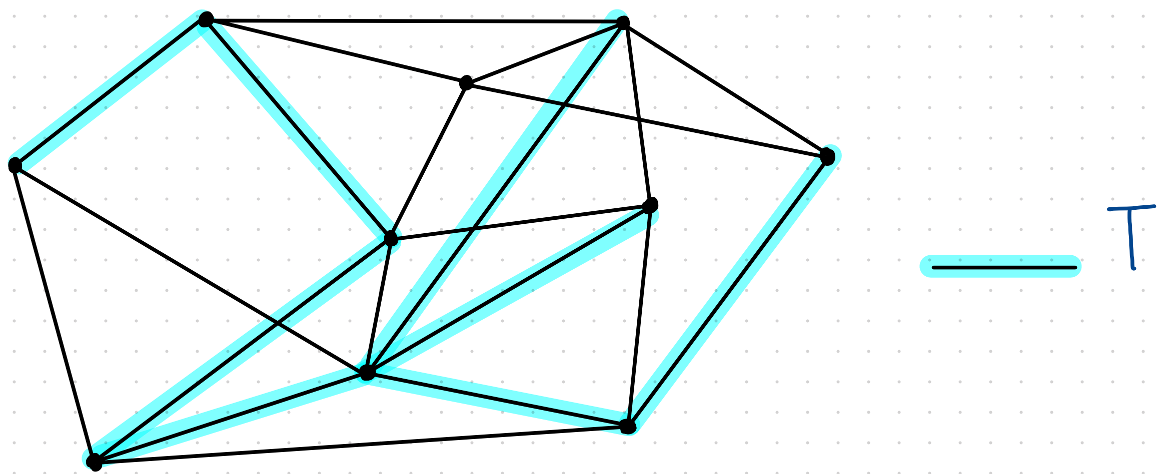
$\Longrightarrow$ $xc(n\text{-permutahedron}) = O(n \log n)$

[Goemans '14]

SPANNING TREE POLYTOPE

$G = (V, E)$ graph

$STP(G) := \text{conv}\{\chi^T : T \subseteq E \text{ spanning tree of } G\}$



$$STP(G) = \left\{ x \in \mathbb{R}^E : \begin{array}{ll} \sum_{e \in E(S)} x_e \leq |S| - 1 & \forall S \neq \emptyset \\ \sum_{e \in E} x_e = |V| - 1 & \\ x_e \geq 0 & \forall e \end{array} \right\}$$

EXTENDED FORMULATION FOR STP(G)

$$x_{vw} = z_{v,w,u} + z_{w,v,u} \qquad \forall \text{ edge } vw,\ u \neq v,w$$

$$x_{vw} = 1 - \sum_{u \neq v,w} z_{v,u,w} \qquad \forall \text{ edge } vw$$

$$z \geq 0$$

[Martin '91]

$$\Longrightarrow \quad xc(STP(G)) \leq |E| \cdot |V|$$

PLANAR GRAPHS : SPANNING TREES

[Williams '01]

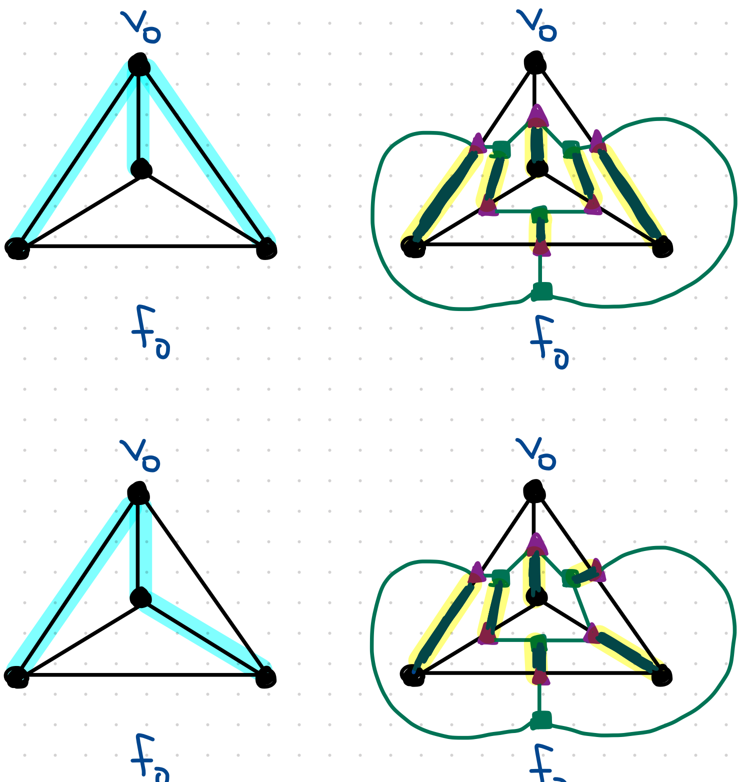
Assume $G=(V,E)$ is planar

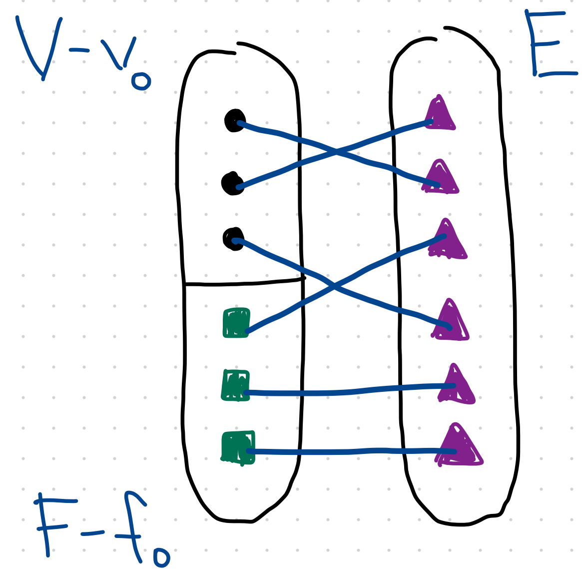
$$F := \{\text{faces of } G\}$$

Take $v_0 \in V$, $f_0 \in F$ incident

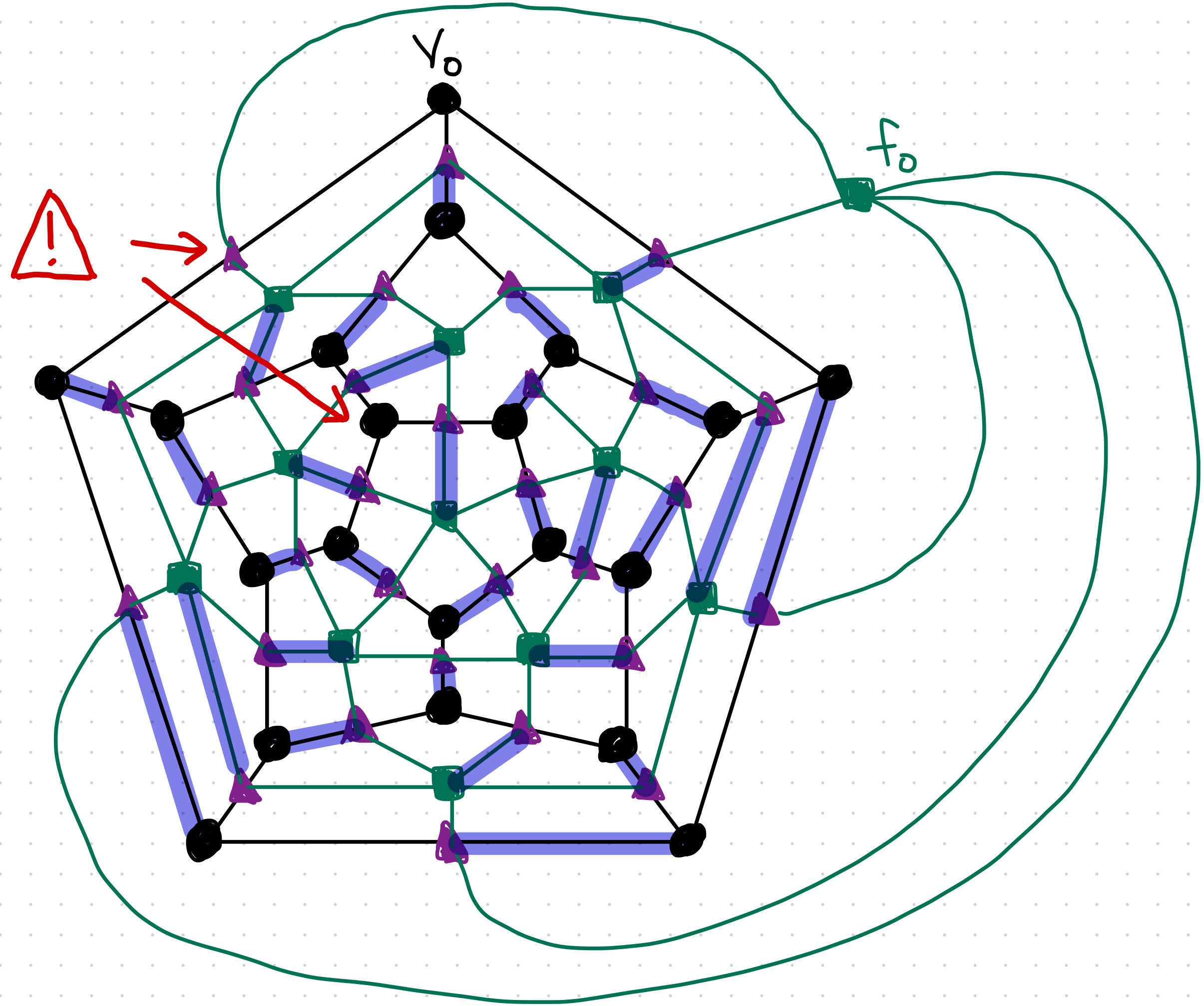
By Euler's formula, $|V-v_0| + |F-f_0| = |E|$

Spanning trees in G $\equiv$ perfect matchings in incidence graph





$$\Rightarrow xc(STP(G)) = O(|V|)$$



v_0
f_0

YANNAKAKIS' FACTORIZATION THM

THM (YANNAKAKIS '91): $\forall P$ s.t. $\dim(P) \geq 1$

$$xc(P) = rk_+(S)$$

where S is any "slack matrix" of P

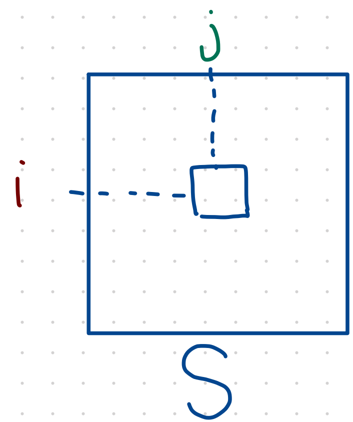
Slack Matrices

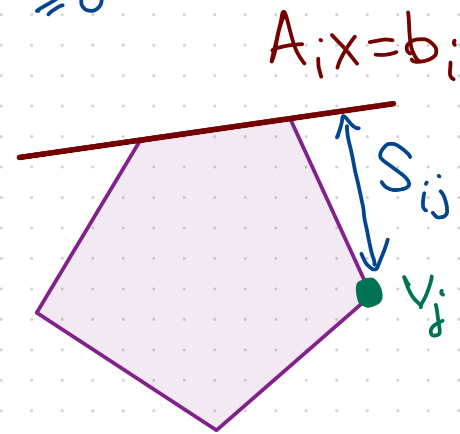
$P = \{x : A_1 x \leq b_1, \dots, A_m x \leq b_m\}$ H-description

$P = \text{conv}\{v_1, \dots, v_n\}$ V-description

Def: slack matrix $S = (S_{ij}) \in \mathbb{R}^{m \times n}_{\geq 0}$

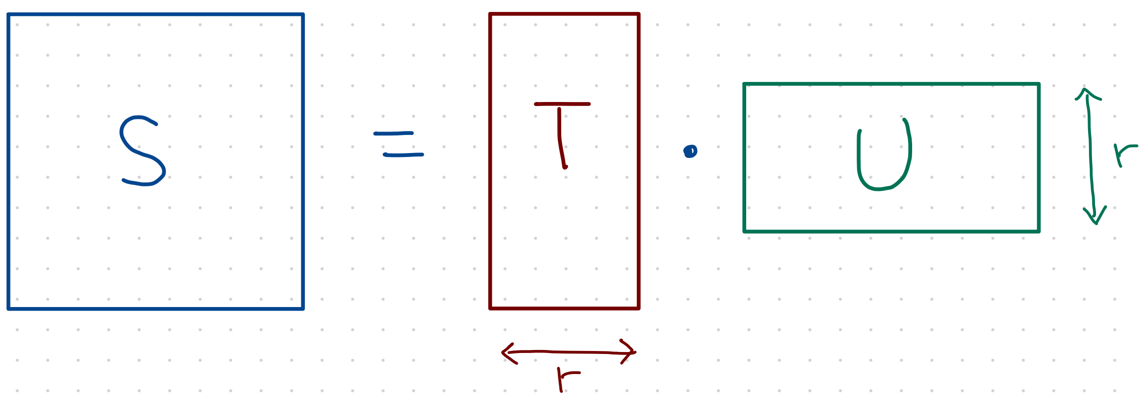
$S_{ij} := b_i - A_i v_j$





NONNEGATIVE RANK

DEF: $\mathrm{rk}_+(S) := \min\left\{ r : \exists\, T \in \mathbb{R}^{m\times r}_{\geq 0},\ U \in \mathbb{R}^{r\times n}_{\geq 0} \ \text{ s.t. } \ S = TU \right\}$



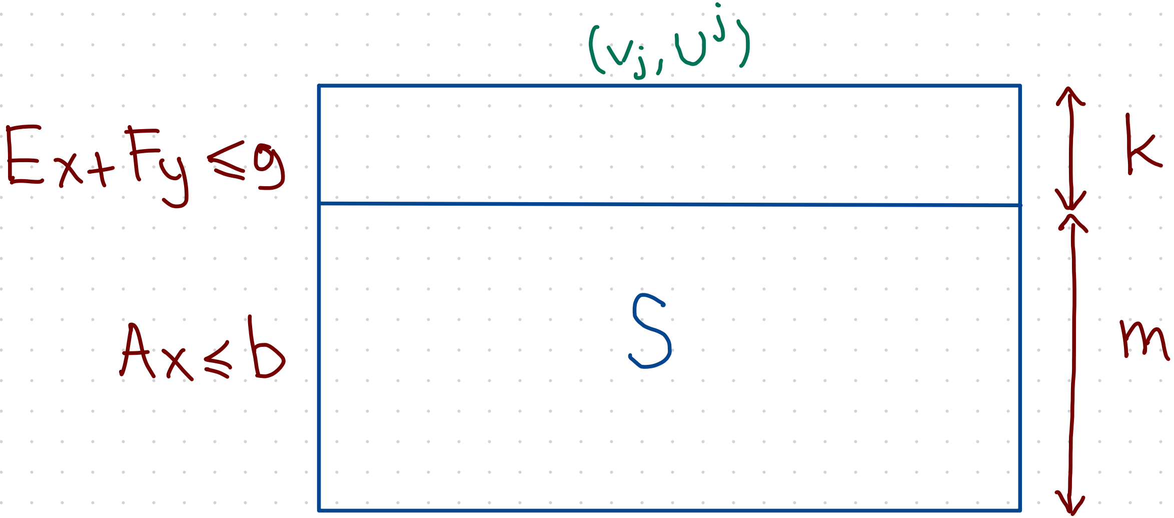
PROOF OF FACTORIZATION THM

① $xc(P) \leq rk_+(S)$: If $S = TU$, let

$$Q := \{(x,y) : Ax + Ty = b,\ y \geq 0\}$$

Then $\text{proj}_x(Q) = P$

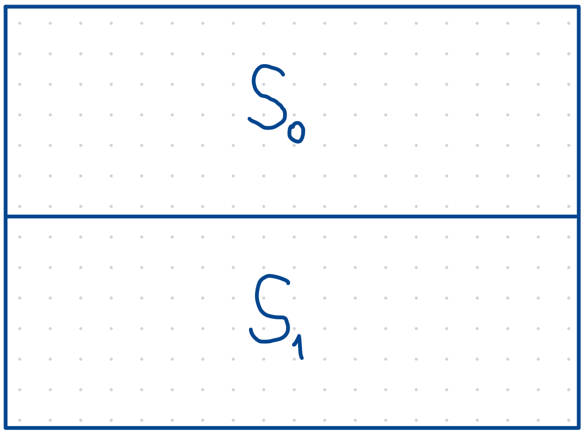
② $rk_+(S) \leq xc(P)$:



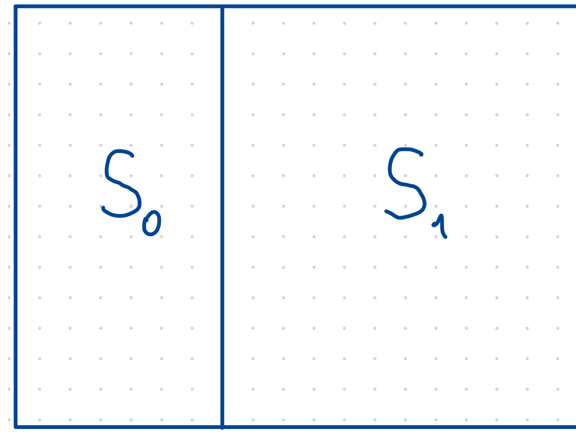
get k nneg row vectors "generating" the m rows of S (via Farkas' Lemma)

$\square$

Factorizing by splitting S



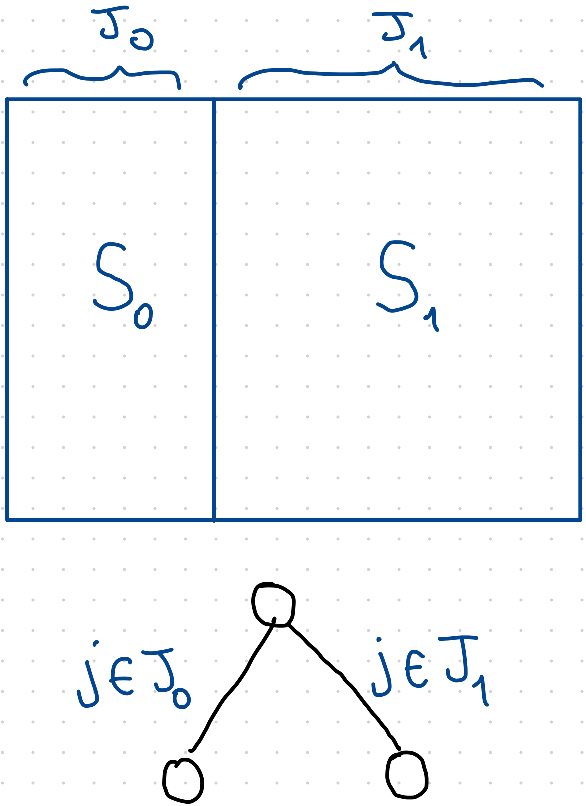
$P = P_0 \cap P_1$

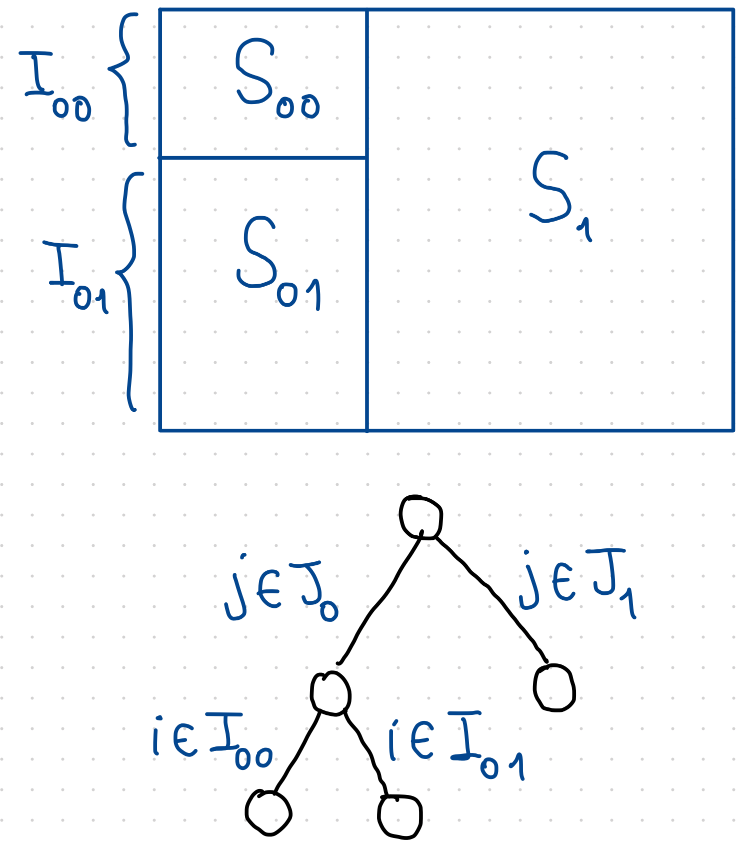


$P = \mathrm{conv}(P_0 \cup P_1)$

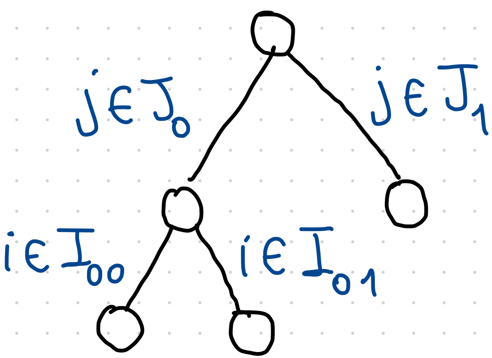
in both cases: $\mathrm{rk}_+(S) \leq \mathrm{rk}_+(S_0) + \mathrm{rk}_+(S_1)$

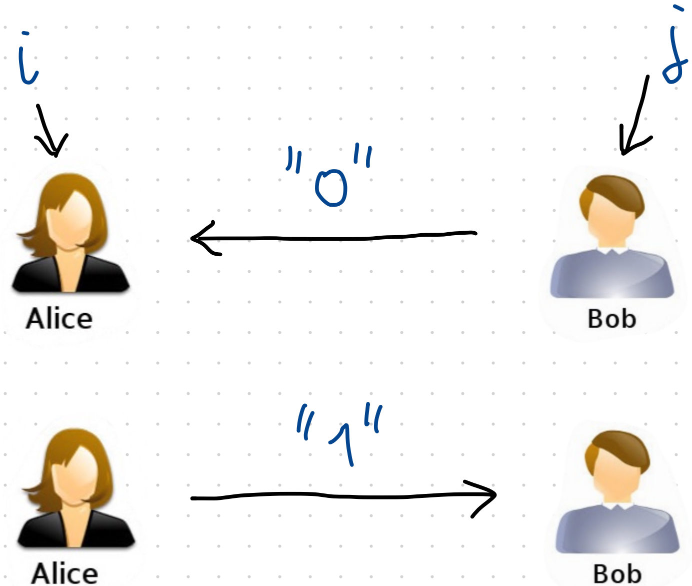
Protocol Trees





Communication Complexity



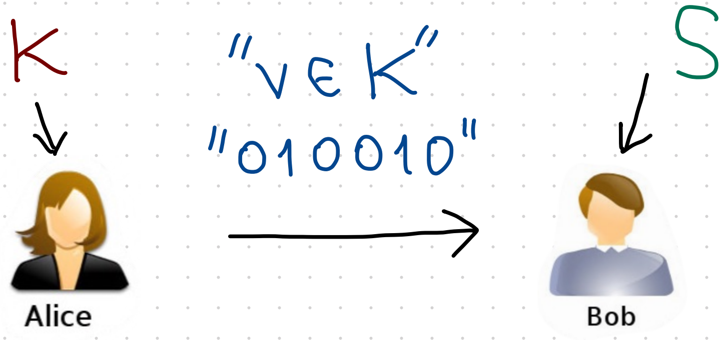


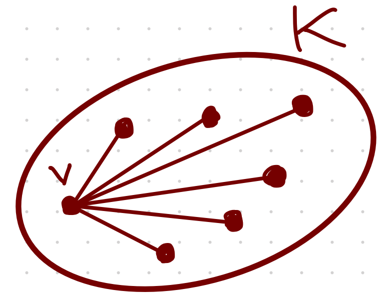
Def: $CC_{det}(S) :=$ min # bits exchanged to compute $S_{ij} = S(i,j)$

CLIQUE VS STABLE SET [Y '91]

G graph $\quad M(K,S) := \begin{cases} 0 & \text{if } K \cap S \neq \emptyset \\ 1 & \text{if } K \cap S = \emptyset \end{cases}$

$cc_{det}(M) \leq$???





good if $\exists\, v \in K : d(v) \leq \frac{|V|}{2}$

$\Longrightarrow \quad cc_{det}(M) = O(\log^2 n)$

$\Longrightarrow \quad rk_+(M) = 2^{O(\log^2 n)}$

Polyhedral Pairs

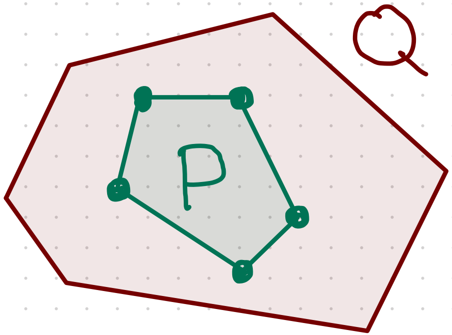
M is not a slack matrix

But can find P, Q with $P \subseteq Q$

s.t. slacks of pts of P w.r.t. ineqs of Q are $M(K,S)$

$$P := \operatorname{conv}\{\chi^S : S \text{ stable set of } G\} = \mathrm{STAB}(G)$$

$$Q := \{x : \sum_{v \in K} x_v \leq 1 \ \forall K \text{ clique of } G\}$$



M slack matrix of pair (P, Q)

$$\mathrm{rk}_+(M) = \min\{\mathrm{xc}(R) : P \subseteq R \subseteq Q\}$$

LOWER BOUNDS ON rk_+

Nneg factorization $S = TU$

$$\Leftrightarrow \quad S = T^1 U_1 + \underbrace{T^2 U_2}_{\text{nneg, rk} \leq 1} + \dots + T^r U_r$$

Normalizations: ① $\|S\|_\infty = 1 \to \mathrm{HSB}(S)$

② $\|S\|_1 = 1 \to \mathbb{C}[S]$

Taking Supports

If $S = T^1 U_1 + T^2 U_2 + \cdots + T^q U_q$ then

$$\mathrm{supp}(S) = \underbrace{\mathrm{supp}(T^1 U_1)}_{\text{rectangle } R_1} \cup \cdots \cup \underbrace{\mathrm{supp}(T^q U_q)}_{R_q}$$

$\Rightarrow \mathrm{rk}_+(S) \geq \mathrm{rc}(S)$ rectangle covering number of S

For example,

$$S = \begin{pmatrix} 0 & 0 & * & * & * \\ * & 0 & 0 & * & * \\ * & * & 0 & 0 & * \\ * & * & * & 0 & 0 \\ 0 & * & * & * & 0 \end{pmatrix}$$

$\boxed{\begin{matrix} * & * \\ * & * \end{matrix}}$ and $\boxed{\begin{matrix} * \\ * \end{matrix}}$ $\boxed{*}$ are two valid rectangles

Unique Disjointness Matrix

Def: For $a, b \in \{0,1\}^n$

$$M(a,b) = (1 - a^T b)^2$$

is "unique disjointness matrix" since

$$M(a,b) = \begin{cases} 1 & \text{if } a^T b = 0 \\ 0 & \text{if } a^T b = 1 \end{cases}$$

$n=3$

$$
\begin{array}{c}
\\
000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111
\end{array}
\begin{array}{c}
\begin{array}{cccccccc}
000 & 001 & 010 & 011 & 100 & 101 & 110 & 111
\end{array}
\\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & * & 1 & 0 & 0 & * \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & * & 0 & * \\
1 & 1 & 0 & 0 & 0 & 0 & * & * \\
1 & 0 & 0 & * & 0 & * & * & *
\end{bmatrix}
\end{array}
$$

THM [Kaibel & Weltge '15]: Let M be a unique disjointess matrix of param. n then:

$$rc(M) \geq \left(\frac{3}{2}\right)^n$$

this improves on $rc(M) \geq (1+\varepsilon)^n$ following [Razborov '90]

Proof Idea:

Fix valid rectangle $R = A \times B$

Define one-to-one map

$$f_R := \{(a,b) \in R : a^T b = 0\} \longrightarrow \left\{ \begin{bmatrix} 0 \\ * \end{bmatrix}, \begin{bmatrix} * \\ 0 \end{bmatrix} \right\}^n$$

$\Rightarrow$ R covers $\leq 2^n$ disjoint entries

$\Rightarrow$ need $\geq \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$ rectangles

$\triangle$

For instance,

$$\begin{array}{ccc}
\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} & \overset{f_R}{\longmapsto} & \begin{pmatrix} 0 & * & 0 & \color{red}{*} & 0 & * & * & \color{red}{*} \\ * & 0 & \color{red}{*} & 0 & * & 0 & 0 & \color{red}{0} \end{pmatrix} \\
\| & & \| \\
\begin{pmatrix} a^T \\ b^T \end{pmatrix} & & f_R(a,b)
\end{array}$$

Exponential Lower Bounds on xc

M is not a slack matrix of one polytope

But is slack matrix of A PAIR

$$P := \operatorname{conv}\{ xx^T : x \in \{0,1\}^n \}$$

$$Q := \{ y \in \mathbb{R}^{n \times n} : \sum_{i \in a} y_{ii} - \sum_{i,j \in a} y_{ij} \leq 1 \quad \forall a \in \{0,1\}^n \}$$

$$P = COR(n) \cong CUT(K_{n+1})$$

correlation polytope

Computation: for $y := xx^T$, $x \in \{0,1\}^n$

$$0 \leq \Big(1 - \sum_{i \in a} x_i\Big)^2 = 1 + \sum_{i \in a} x_i^2 - 2\sum_{i \in a} x_i + \sum_{i,j \in a} x_i x_j$$

$$= 1 + \sum_{i \in a} x_i^2 - 2\sum_{i \in a} x_i^2 + \sum_{i,j \in a} x_i x_j$$

$$= 1 - \sum_{i \in a} y_{ii} + \sum_{i,j \in a} y_{ij}$$

THM [FMPTW'12]:

$xc(COR(n))$, $xc(CUT(K_n))$ are $2^{\Omega(n)}$

$xc(TSP(n))$ is $2^{\Omega(\sqrt{n})}$

$xc(STAB(G_n))$ is $2^{\Omega(\sqrt{n})}$ for some n-vtx graphs G_n

Prize Winning Thm

- Best Paper Award (STOC'12)
- 10-years TOT Award (STOC'22)
- Gödel Award (2023)

FMPTW:

THM [Rothvoss '14]

$$xc(\text{matching polytope of } K_n) = 2^{\Omega(n)}$$

$$xc(TSP(n)) = 2^{\Omega(n)}$$

Based on HYPERPLANE SEPARATION bound

Open Problems

1. $xc(STP(K_n)) = ???$

2. explicit 0/1 polytope with $xc(P) = 2^{\Omega(d)}$

3. non-constant "inapproximability"

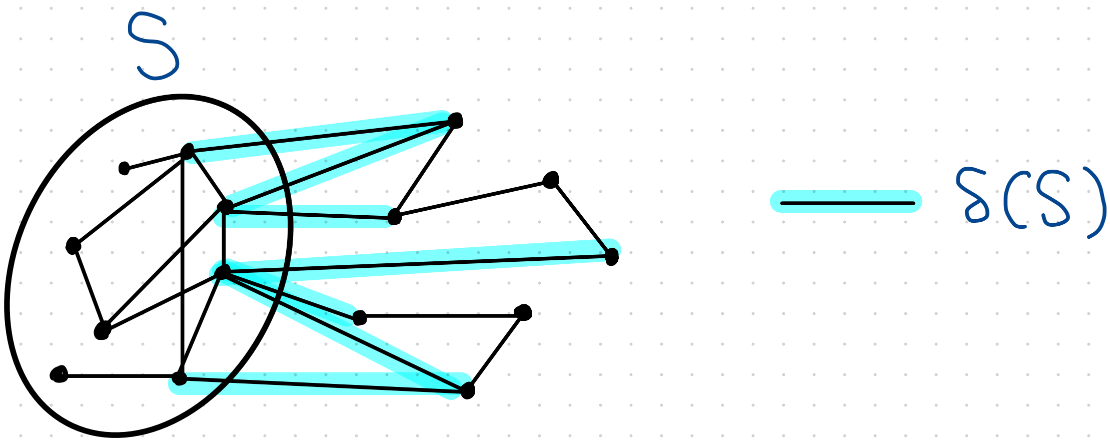
4. $xc_{PSD}(P_{match}(K_n)) = ???$


Dank U

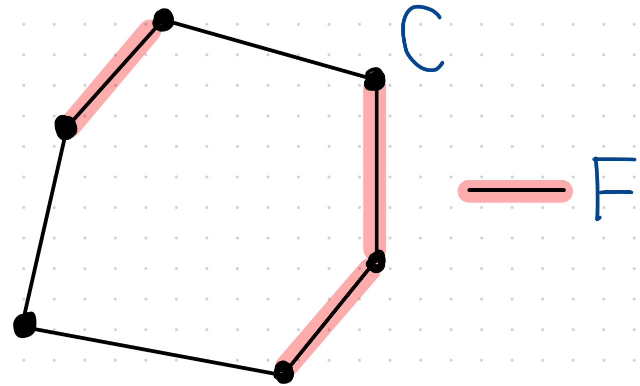
Extra Slides

Relaxation of Cut Polytope

$G = (V(G), E(G))$ graph



$$\mathrm{CUT}(G) := \mathrm{conv}\{\chi^{\delta(S)} : S \subseteq V(G)\}$$

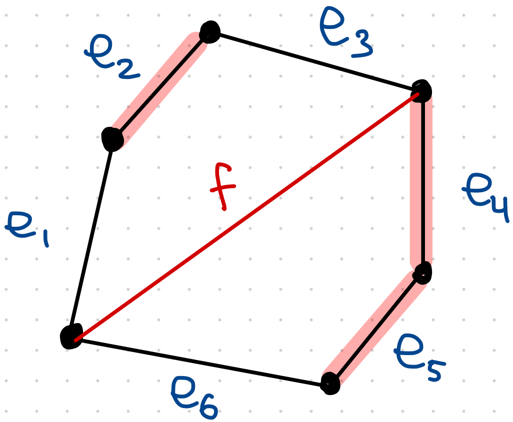


$$R(G) := \left\{ x \in \mathbb{R}^{E(G)} : \right.$$

$$\sum_{e \in F} x_e - \sum_{e \in C \setminus F} x_e \leq |F| - 1 \quad \forall \text{ cycle } C,\ F \subseteq C$$

$$|F| \text{ odd}$$

$$\left. 0 \leq x_e \leq 1 \quad \forall e \in E(G) \right\}$$



$$x_{e_2} - x_{e_1} - x_{e_3} - x_f \leq 0$$

$$x_{e_4} + x_{e_5} + x_f - x_{e_6} \leq 2 \quad [+]$$

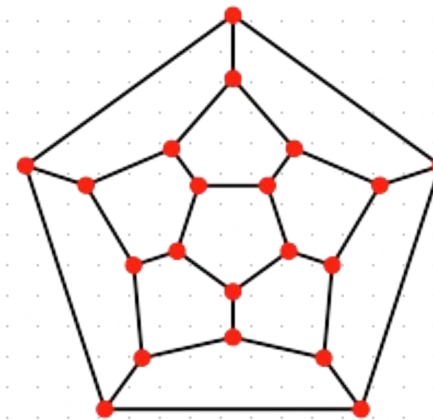
$$x_{e_2} + x_{e_4} + x_{e_5} - x_{e_1} - x_{e_3} - x_{e_6} \leq 2$$

$H \subseteq G \Longrightarrow R(G)$ projects to $R(H)$

$$\Longrightarrow xc\big(R(G)\big) = O(n^3)$$

Planar Graphs : Cut Polytope

Thm [Guenin'01]: If G has no K_5 minor, then $CUT(G) = R(G)$



$\Rightarrow$ if G is planar, then $xc(CUT(G)) \leq O(|V|^3)$

HYPERPLANE SEPARATION

① $\|S\|_\infty = 1 \Longrightarrow \|T^k U_k\|_\infty \leq 1 \quad \forall k$

each $T^k U_k \in \operatorname{conv}\{ xy^T : x \in \{0,1\}^m, y \in \{0,1\}^n \}$
$=: \mathrm{RECT}(m,n)$

$$\Longrightarrow \frac{1}{r} S = \sum_{k=1}^{r} \frac{1}{r} T^k U_k$$

$$rk_+(S) \geq \min\left\{ e \;\middle|\; \frac{1}{e} S \in \mathrm{RECT}(m,n) \right\}$$

Common Information

② $\|S\|_1 = 1 \Longrightarrow$ view S as proba. distribution

$$S = T^1 U_1 + T^2 U_2 + \dots + T^r U_r$$

define random variable K with

$$\mathbb{P}[K = k] := \|T^k U_k\|_1$$

then

$$\mathbb{P}[I = i, J = j \mid K = k] := \frac{T^k U_k(i,j)}{\|T^k U_k\|_1}$$

Get:

$$\log \mathrm{rk}_+(S) \geq \mathbb{H}[K]$$

$$\geq \mathbb{I}[I,J;K]$$

$$\geq \mathbb{C}[S]$$ common information

[Wyner'78]

Where $\mathbb{C}[S] = \mathbb{C}[I;J]$

$$:= \inf_{K:\, I \perp J \mid K} \mathbb{I}[I,J;K]$$

TED SWART '86

P = NP

by

E.R. Swart

Department of Mathematics & Statistics
University of Guelph
Guelph, Ontario, CANADA

Mathematical Series 1986-107

February 1986

$$\max z = \sum_{i} \sum_{j} \sum_{k=1}^{n} x_{ijk}$$

i adjacent to j

subject to

(1) $\sum_{j} x_{0j1} = 1$

j adjacent to 0 = n

(2) $\sum_{i} x_{inn} = 1$

i adjacent to n = 0

(3) $\sum_{j} \sum_{k=1}^{n} x_{ijk} = 1 \quad i = 1, 2, \ldots n - 1$

j adjacent to i

(4) $\sum_{i} (x_{ijk} - x_{j,i,k+1}) = 0 \quad j = 1, 2 \ldots n - 1$

$k = 1, 2 \ldots n - 1$

i adjacent to j

$x_{ijk} \geq 0 \quad \forall i,j,k$

(5) $x_{ijk} - \sum_{t} x_{j,t,k(\mathrm{mod}\, n)} + 1 \leq 0 \quad j = 1, 2 \ldots n$

t(≠i) adjacent to j $\quad \forall$i adjacent to j

$k = 1, 2 \ldots n$