Two Lectures on Fair Division

Hervé Moulin

University of Glasgow and HSE St Petersburg

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formal modeling of fair division starts 70 years ago

- *mathematicians:* the cake-division model; Steinhaus 1948
- *game theorists:* axiomatic bargaining: Nash 1950; cooperative games: Shapley 1953
- *economists:* No Envy and fair competitive trade; Foley 1967, Varian 1974

standard examples

family heirlooms: silverware, paintings

seats in overdemanded classes

family chores

work shifts, teaching loads

divorce, dissolution of a partnership

more recent examples: *peer to peer* Fair Division on the Internet

to share computing resources

to share memory space

online barter for goods and services

FD inspires an active field of research

at the interface of microeconomics and internet science

- around 1997: Algorithmic Game Theory is born
- about 2000: emergence of Algorithmic Mechanism Design and Computational Social Choice
- the ACM *Electronic Commerce* conferences (EC) start in 1999, followed by the *Web and Internet Economics* conferences (WINE) in 2004, and by the *Computational Social Choice* conferences (COMSOC) in 2010
- in 2014 EC becomes the 15th *Economics and Computation* conference
- and the ACM Society launches a new journal: *Transactions on Economics and Computation*

the fair division problem

- a closed economy: agents cannot find multiple copies of these items in the competitive market; the manna can include cash, but out of pocket payments are ruled out

- efficiency $\Longrightarrow$ unequal shares, so *defining* fairness is not straighforward

- indivisible items $\Longrightarrow$ unequal shares, so *approximating* fairness or efficiency is inevitable also desired

- incentive compatibility
- the division rule must be scalable, its computational complexity must be low

what distinguishes the new research from the classic results of the 60s and 70s:

elicitation of preferences must remain simple for practical implementation

$\Longrightarrow$ the mechanisms work for specific restricted domains of preferences, **much** smaller than the full-fledged Arrow Debreu domain with multiple divisible commodities

each domain suits a certain class of division problems we review four models

$\longrightarrow$ one dimensional manna with convex preferences

$\longrightarrow$ divisible goods, perfect complements (Leontief preferences)

$\longrightarrow$ divisible or indivisible goods, perfect substitute (linear preferences)

$\longrightarrow$ divisible or indivisible bads (or mixed manna) with linear preferences

Model 1: One dimensional manna

ω: amount of a non disposable item, hours of baby-sitting, shares of a stock, teaching loads,..

agent i has single-peaked (i. e., convex) preferences over his share z_i with peak π_i

rule of interest: the *uniform division rule Sprumont (1991)* start from equal split $\overline{z}_i = \frac{1}{n}\omega$ for all i

stay there if the peaks π_i are all on the same side of $\frac{1}{n}\omega$

if not $N^{under} = \{i | \pi_i < \frac{1}{n}\omega\}$, $N^{over} = \{i | \pi_j > \frac{1}{n}\omega\}$, $N^{fs} = \{i | \pi_k = \frac{1}{n}\omega\}$

agent in N^{fs} get $\frac{1}{n}\omega$

if $\sum_N \pi_i > \omega$ each $i \in N^{under}$ gets $z_i = \pi_i$, each j in N^{over} is uniformly rationed: equalizes the gains $(z_j - \frac{1}{n}\omega)$ under the efficiency constraint $z_j \leq \pi_j$

if $\sum_N \pi_i < \omega$ ...

this rule is miraculous Sprumont (1991)

- unique, continuous in the parameters, easy to compute

- simple messages: report own peak

- efficient and Envy-Free (I prefer my share to yours)

- strategyproof: reporting wrong peak does not pay

- ditto if a group of agents try a coordinated misreport

these properties are characteristic

Moulin (2017) generalizes this result to any problem where

individual allocations are single dimensional

preferences are single-peaked

the feasibility constraints are convex

examples: voting over an interval

trading between two suppliers and three demanders: $z_1 + z_2 = z_3 + z_4 + z_5$

Model 2: Perfect Complements goods

users sharing cloud computing need resources (CPUs, memory, bandwidth, etc.) in fixed proportions

entrepreneurs need labor, capital, and raw material, in fixed proportions

A the set of goods, $\omega \in \mathbb{R}_+^A$ the manna, agent i's utility

$$u_i(z_i) = \min_{a \in A}\{\frac{z_{ia}}{w_{ia}}\}$$

rule of interest: the *Egalitarian rule EE(ω) (Pazner and Schmeidler (1979))*

find the largest parameter λ such that for some feasible allocation z

$$u_i(z_i) \geq u_i(\lambda\omega) \text{ for all } i$$

$\longrightarrow$ implement z

equivalently: find an efficient allocation equalizing the *relative utilities*

$$\frac{\text{utility of my share } u_i(z_i)}{\text{my utility of all the resources}}$$

easy to compute: find the critical *overdemanded* commodity (typically only one) and use it to measure utilities

example: $\omega = (\omega_a, \omega_b) = (6, 4)$

Ann: $u_1(a_1, b_1) = \min\{\frac{a_1}{3}, \frac{b_1}{5}\}$; Bob: $u_2(a_2, b_2) = \min\{\frac{a_2}{2}, b_2\}$; Charles: $u_3(a_3, b_3$

good A is critical:

$$\frac{\frac{a_1}{3}}{\min\{\frac{6}{3}, 4\}} = \frac{\frac{a_2}{2}}{\min\{\frac{6}{2}, 4\}} = \frac{a_3}{\min\{6, \frac{4}{2}\}} = \frac{3}{7}$$

$$\Longrightarrow \text{Ann: } (\frac{18}{7}, \frac{6}{7}) \text{ ; Bob: } (\frac{18}{7}, \frac{9}{7}) \text{ ; Charles: } (\frac{6}{7}, \frac{12}{7})$$

with some good B to spare

this rule is miraculous (Ghodsi et al. 2011)

- welfarist, unique, continuous in the parameters, easy to compute
- efficient and Envy-Free
- strategyproof
- groupstrategyproof

note:unused goods must be discarded !

properties are not characteristic *(Xue and Li 2013)*

Model 3: Perfect substitutes goods

agents report *additive utilities* by dividing 1000 points over the goods: siblings $\longrightarrow$ family heirlooms, enthusisatic teachers $\longrightarrow$ classes, ..

A the set of goods, $\omega \in \mathbb{R}_+^A$ the manna, agent i's utility

$$u_i(z_i) = \sum_{a \subset A} w_{ia} \cdot z_{ia}$$

the *Egalitarian* rule is still meaningful but challenged by the *Competitive* rule:

find a price $p \in \mathbb{R}_+^A$ and a feasible allocation $(z_i)_{i \in N}$ such that z_i maximizes i's utility under budget constraint

$$p \cdot z_i \leq 1$$

example: $\omega = (\omega_a, \omega_b) = (40, 80)$

$$\text{Ann: } u_1(a_1, b_1) = a_1 + 3b_1 \text{ ; Bob: } u_2(a_2, b_2) = a_2 + 2b_2$$

$$\text{Charles: } u_3(a_3, b_3) = a_3 + b_3$$

good B is popular, good A is not

Competitive allocation

utilities

	a (40)	b(80)
Ann	1	3
Bob	1	2
Charles	1	1

allocation

price	1	1
Ann	0	40
Bob	0	40
Charles	40	0

budget 40

Egalitarian allocation

utilities

	a (40)	b(80)
Ann	1	3
Bob	1	2
Charles	1	1

(rounded) allocation

	a	b
Ann	0	36
Bob	0	38
Charles	40	6

$\longrightarrow$ Ann envies Bob

$\longrightarrow$ easy misreport for Ann or Bob: increase the relative worth of the good you do not get (nobody can misreport at the Competitive allocation, *for this particular example*) *the Competitive rule is miraculous (Eisenberg Gale 1960; Chipman and Moore 1976, Bogomolnaia et al. 2017)*

the competitive utility profile maximizes the *Nash product of utilities* $\Longrightarrow$

- welfarist, unique utilities and price, continuous in the parameters, (relatively) easy to compute

- efficient and Envy-Free

- everyone benefits when the pile of goods increases (not true for EE(ω)

- if a good becomes more attractive to me, I receive (weakly) more of this good (not true for EE(ω))

not strategyproof but no minimally fair efficient rule can be

Model 3 bis: Perfect substitutes indivisible goods

furniture, artsy objects, workers ..

No Envy no longer compatible with efficiency

a much weaker test of fairness, dating back to *Steinhaus (1948)*

A the set of all goods, X^i agent i's share

$$\textbf{Fair Share}: u_i(X^i) \geq \frac{1}{n} u_i(A) \text{ for all } i$$

Question: how to adapt it when goods are indivisible?

first proposal

Fair ShareX: for all i for all $a \in A \setminus X^i$ we have $u_i(X^i \cup \{a\}) \geq \frac{1}{n} u_i(A)$

bad news: it cannot be always guaranteed

# of objects	1	1	1	1	8
$u_1 = u_2 = u_3$	2	2	2	2	$\frac{1}{4}$

a better proposal by *Budish (2011)*: make each agent play an adversarial Divide and Choose game

write $\mathcal{P} = (Y^i)_{i \in N}$ for a $|N|$-partition of A:

$$\textbf{MaxMin Share}:\ u_i(X^i) \geq \max_{\mathcal{P}} \min_{j \in N} u_i(Y^j)$$

Surprising fact *(Procaccia and Wang, 2014)*:

There are (very rare!) problems with three or more agents where we cannot give his maxmin share to each agent.

But we can always guarantee at least $\frac{2}{3}$ *of his maxmin share to everyone.*

$\longrightarrow$ the first statement does not raise an important practical issue the No Envy test is the next property toward the competitive approach

it can be weakened as follows:

No Envy1: for all i, j there is $a \in X^j$ s.t. $u_i(X^i) \geq u_i(X^j - a)$

I do not envy your share if I can reduce it by one object

$\longrightarrow$ No Envy1 is easy to achieve: give objects in round robin fashion with a fixed priority order (the *NFL draft* mechanism)

$\longrightarrow$ but this mechanism is not efficient *the Nash product strikes again !*

Caragianis et al. (2016) show:

A partition $\mathcal{P}$ maximizing the product of the utilities is efficient and meets No Envy1

the maximization above is NP-hard in the number of agents and objects

but can compute an efficient and No Envy1 partition in time polynomial in the number of agents and objects strenghtening No Envy1

No EnvyX: for all i, j for all $a \in X^j$ we have $u_i(X^i) \geq u_i(X^j - a)$

Open question: *does a No EnvyX partition always exist ? even a possibly inefficient one?*

Model 4: Perfect substitutes bads (or mixed manna)

we divide undesirable tasks**:** family chores $\longrightarrow$ cleaning, baby sitting; discouraged teachers $\longrightarrow$ classes,..

disutilities

$$\text{Ann: } u_1(a_1, b_1) = a_1 + 3b_1 \text{ ; Bob: } u_2(a_2, b_2) = a_2 + 2b_2$$

$$\text{Charles: } u_3(a_3, b_3) = a_3 + b_3$$

$$\text{to divide } (\omega_a, \omega_b) = (80, 40)$$

Bad A is popular, bad B is not

same definitions of the Egalitarian and Competitive rules

Egalitarian rule

		a (80)	b(40)
disutilities	Ann	1	3
	Bob	1	2
	Charles	1	1

		a	b
(rounded) allocation	Ann	53	0
	Bob	27	8
	Charles	0	32

where again Ann envies Bob and agents have easy misreporting strategies *there are now two Competitive divisions !!*

disutilities

	a (80)	b(40)
Ann	1	3
Bob	1	2
Charles	1	1

allocation 1

price	1	1
Ann	40	0
Bob	40	0
Charles	0	40

budget 40

(rounded) allocation 2

price	1	2
Ann	53	0
Bob	27	13
Charles	0	27

budget 53

when we divide bads the multiplicity issue is not an anomaly, and can be very severe

$\Longrightarrow$ *we do not know a normatively compelling single-valued competitive division of chores*

in fact *every single-valued efficient and envy-free division rule will be discontinuous in the utility parameters* (Bogomolnaia et al.2017)

so the Egalitarian rule (unique, continuous, Fairt Share) is a plausible choice to divide bads

Model 4 bis: Perfect substitutes indivisible bads (and mixed manna)

divorce, dissolution of a partnership produce assets and liabilities; so do family estates

the approximation

Fair ShareX: for all i and all $a \in X^i$ we have $u_i(X^i \diagdown \{a\}) \geq \frac{1}{n} u_i(Y)$

is now always feasible, and achieved by a relatively simple, but inefficient mechanism

Open question: *is Fair ShareX compatible with efficiency?*

No Envy1: for all i, j there is $a \in X^j$ s.t. $u_i(X^i \setminus \{a\}) \leq u_i(X^j)$

I do not envy your share if I can reduce my share by one chore

$\longrightarrow$ No Envy-1 is easy to achieve: give objects in round robin fashion with a fixed priority order (the *NFL draft* mechanism)

$\longrightarrow$ but this mechanism is not efficient

Open question: *is No Envy1 compatible with efficiency?*

fair division mechanisms eschew the need to define precise property rights, or to use direct bargaining or markets; they are centralized allocation rules with very small transaction costs

the power of theoretical rules is in their normative properties, but they are vindicated only if real participants in real problems adopt them

their (soft) implementation on free websites like SPLIDDIT, Adjusted Winner is currently limited to a handful of "iconic" division problems such as the ones described above, but also sharing the rent between flatmates, sharing a taxi ride, or distributing credit in a joint project

Thank You