# Beating the curse of dimensionality in options pricing and optimal stopping

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LNMB

# Outline

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# Motivation

- Opt. pricing and opt. stopping central to Fin. Math, OR, AP
- Rich history in NL → Amsterdam Bourse in 1600's
- Many real-world comp. problems have ...
  - Full path-dependence
  - High-dim process generating $\mathcal{F}$
- Leads to Curse of dimensionality
  - Cannot solve DP
  - Known ADP, Simulation, dual, PDE, ..., approaches
  - Seems theoretically intractable, vast lit.
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- We prove . . .
  - ∃ "simple and elegant formula" for opt. stop. as $\infty$ sum
  - Even better, if you truncate the sum after k terms . . .
    - Error bounded by $\frac{1}{k}$
    - Easy to simulate!
  - Yields efficient rand. $\epsilon$-optimal algorithms
    - Runtime $T^{poly(\frac{1}{\epsilon})}$
    - Data-driven : $T^{poly(\frac{1}{\epsilon})}$ samples
    - Even with high-dim and path-dependence
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    - (sample-based) PTAS for gen. opt. stop
  - New connection to net. flows
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1. Punchline
2. Model + Problem
3. Intuition + Main results
4. Future Research

# Model

- Discrete time : T periods
- $\mathcal{F} = \{\mathcal{F}_t, t \in [1, T]\}$ gen. by $D$-dim process $\mathbf{Y} = Y_1, \ldots, Y_T$
- Cost functions $\{g_t, t \in [1, T]\}$
  - $Z_t \triangleq g_t(Y_{[t]})$ = cost of stop. at $t$
  - Non-neg. + integrable
- $\mathcal{T}$: set of $\mathcal{F}$-adapted stop.times in $[1, T]$
- Note: disc. time, won't dwell on pathologies
  - Assume all conditionings etc. well defined
- Note: Sometimes assume $Z_t \in [0, 1]$ or $[0, U]$
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  - Could state in terms of u.b., frame, etc.

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- Discrete time : T periods
- $\mathcal{F} = \{\mathcal{F}_t, t \in [1, T]\}$ gen. by $D$-dim process $\mathbf{Y} = Y_1, \ldots, Y_T$
- Cost functions $\{g_t, t \in [1, T]\}$
  - $Z_t \triangleq g_t(Y_{[t]}) = \text{cost of stop. at t}$
  - Non-neg. $+$ integrable
- $\mathcal{T}$: set of $\mathcal{F}$-adapted stop.times in $[1, T]$
- Note: disc. time, won't dwell on pathologies
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# Problem statement

- $\inf_{\tau \in \mathcal{T}} E[Z_\tau]$
  - Full path-dependence
  - High dimensionality
  - Massive state space
  - Gen. disc-time optimal stopping
  - Pricing Bermudan Options
  - Fundamental problem in control theory
  - WLOG generally discuss the min. problem
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# Outline

1. Punchline
2. Model + Problem
3. **Intuition + Main results**
4. Future Research

# Main intuition

- $\text{OPT} \triangleq \inf_{\tau \in \mathcal{T}} E[Z_\tau]$
- $\text{OPT} = \inf_{\tau \in \mathcal{T}} E[Z_\tau] \geq E\left[\min_{t \in [1,T]} Z_t\right]$
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$$E\Bigg[\min_{i\in[1,T]}\Bigg(Z_i - E\Big[\min_{j\in[1,T]} Z_j|\mathcal{F}_i\Big] - E\Bigg[\min_{j\in[1,T]}\Bigg(Z_j - E\Big[\min_{k\in[1,T]} Z_k|\mathcal{F}_j\Big]\Bigg)|\mathcal{F}_i\Bigg]\Bigg)\Bigg]+$$

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- $E[\min_{t \in [1,T]} Z_t^k]$ can be computed by sim!
- No curse of dimensionality!
- Completely data-driven
- Recursive, complexity ↑ in k
- But only need a few terms!
- Explicit runtime depends on assumptions + type of approx.
- In general $\epsilon$-approx for OPT in $T^{\text{poly}(\frac{1}{\epsilon})}$ time! (w.h.p.)
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# Algorithmic implications (cont.)

## "Canonical" Theorem

- Suppose $P(Z_t \in [0, 1]) = 1$ for all $t$.
- Then for all $\epsilon, \delta \in (0, 1)$, $\exists$ a rand. alg. $\mathcal{A}_{\epsilon,\delta}$ s.t. ...
  - In time
  - With only calls to a simulator for $Y$ (cond. on hist.)
  - Returns r.v. $X$ s.t.

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- A correspondence between opt. stop and min-cut
- Illustrate in trivial 3-stage problem
  - Driving process $Y$ is 1-d , supp. $\{1, 2\}$
  - $Z_t$ = payout if stop at time t = most recent Y
- Complicated duality lit $\rightarrow$ max-flow min-cut
- Expansion = iterative flow alg.
- Won't give all details of reduction
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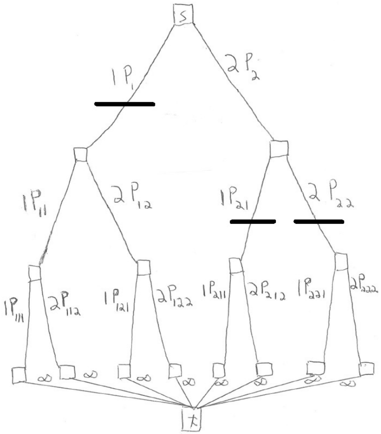

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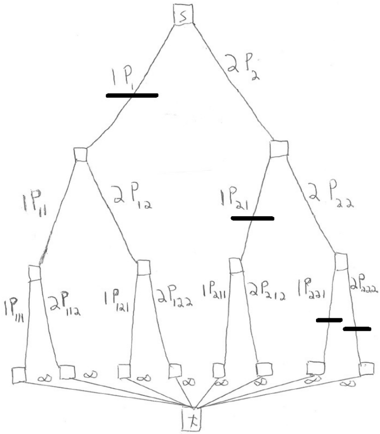

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# The max-flow connection (cont.)

- THEOREM : Solving opt.stop equal to solving min-cut
- OBS : Novel unification of many duality results for opt.stop
  - Karatzas, Kogan and Haugh, Rogers, Glasserman, ...
  - Opt. dual martingale $\leftrightarrow$ max-flow
- OBS : Simple proofs and intuition about many past results
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# Outline

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- Recall : $Z_t^k = Z_t^{k-1} - E[\min_{i \in [1,T]} Z_i^{k-1} | \mathcal{F}_t]$
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