Inefficiency of Multi-Unit Auctions

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Overview

Part I

Back to School: Standard Single-Item Auctions

Single-Item Auction

Setting:

- single item to be sold
- set of players (bidders) $N = [n]$
- every player $i \in N$:
  - valuation v_i: i's worth for receiving the item (private!)
  - bid b_i: i's bid for the item

Auctioneer receives bids $\mathbf{b} = (b_i)_{i \in N}$ and determines:

1. winner i^* in N who receives the item
   ($x_i = 1$ if player $i \in N$ wins, $x_i = 0$ otherwise)
2. price p that player i^* has to pay for the item

utility of player i: $u_i(\mathbf{b}) = x_i(v_i - p)$ (quasi-linear)

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Design Goals

Goals:

1. Strategyproofness: every player i maximizes his utility by bidding *truthfully*, i.e., $b_i = v_i$ is a dominant strategy
2. Efficiency: under truthful bidding, the computed outcome maximizes *social welfare* $\sum_{i \in N} x_i v_i$
3. Polynomial-time computability: outcome is computable in polynomial time

Standard Single-Item Auctions

First Price Auction:

1: Collect the bids $(b_i)_{i \in N}$ of all players
2: Determine a player $i^* \in N$ whose bid is highest (break ties arbitrarily)
3: Charge i^* the highest bid $p = \max_i b_i$

Problem: players have an incentive do underbid
$\rightarrow$ First Price Auction is not strategyproof

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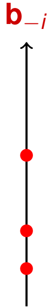

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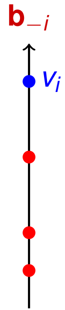

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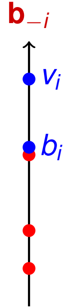

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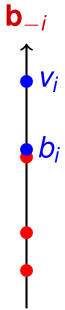

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Vickrey Auction:

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2: Determine a player $i^* \in N$ whose bid is highest (break ties arbitrarily)
3: Charge i^* the second highest bid $p = \max_{i \neq i^*} b_i$

Theorem

The Vickrey Auction is strategyproof, efficient and runs in polynomial time.

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Proof: Fix a player i and consider arbitrary bids $\mathbf{b}_{-i}$ of the other players. Let B be the highest bid of $\mathbf{b}_{-i}$.

Case 1: $v_i > B$

$$u_i(v_i, \mathbf{b}_{-i}) \geq u_i(b_i, \mathbf{b}_{-i}) \qquad \forall b_i$$

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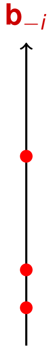

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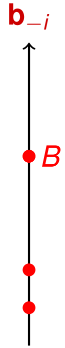

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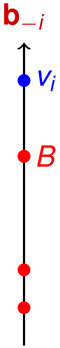

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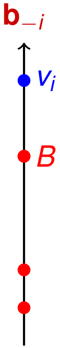

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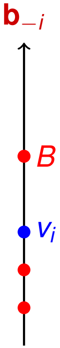

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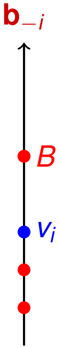

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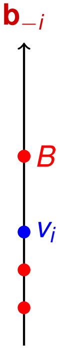

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Part II

Multi-Unit Auctions: Introduction and Motivation

Multi-Unit Auctions

Setting:

- want to sell k identical units (items) of a single good
- n multi-demand players (bidders):
  - $v_i(j)$ = player i's valuation for receiving j items
  - $b_i(j)$ = player i's marginal bid for the jth item
- run an auction to allocate items and determine payments
  - $x_i(\mathbf{b})$ = number of items allocated to player i
  - $p_i(\mathbf{b})$ = price to be paid by player i
- player i's utility $u_i(\mathbf{b}) = v_i(x_i(\mathbf{b})) - p_i(\mathbf{b})$
- social welfare

$$SW(\mathbf{b}) = \sum_{i=1}^{n} v_i(x_i(\mathbf{b}))$$

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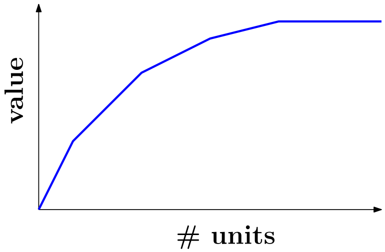
Valuation Functions

Valuation Functions: for every player i

- $v_i(0) = 0$
- v_i is non-decreasing

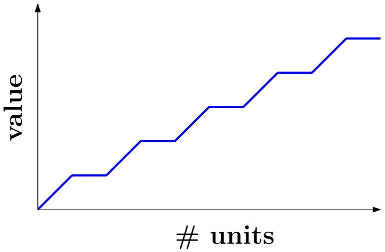
Submodular Valuation Functions

$$v_i(x) - v_i(x-1) \geq v_i(x+1) - v_i(x)$$



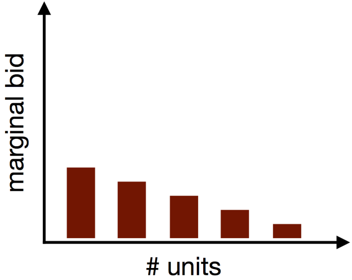
Subadditive Valuation Functions

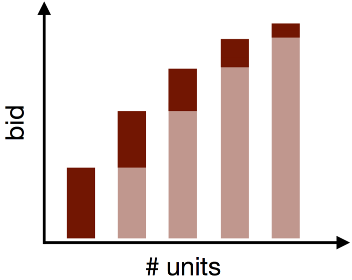
$$v_i(x+y) \leq v_i(x) + v_i(y)$$



Bidding Format

Marginal Bids: every player i specifies a vector of marginal bids $\mathbf{b}_i = (b_i(1), \ldots, b_i(k))$





$\rightarrow$ player i's bid for receiving x items is

$$\hat{b}_i(x) = \sum_{j=1}^{x} b_i(j)$$

Bidding Format

Standard Bidding Format: player *i*'s marginal bids are required to be non-increasing:

$$b_i(1) \geq b_i(2) \geq \cdots \geq b_i(k)$$

[Krishna '02, Milgrom '04]

Uniform Bidding Format: player *i*'s marginal bid is $\bar{b}_i$ for the first q_i items and zero for the remaining ones:

$$b_i(1) = \cdots = b_i(q_i) = \bar{b}_i \quad \text{and} \quad b_i(q_i + 1) = \ldots b_i(k) = 0$$

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Standard Multi-Unit Auctions

Allocation Rule: k items are allocated to the players that issued the k highest marginal bids (ties are broken arbitrarily)

Different Pricing Rules:

1. **Discriminatory Auction:** every player pays for each item the corresponding winning marginal bid (aka pay-as-bid)
2. **Uniform Price Auction:** every player pays for each item the highest losing marginal bid
3. **Vickrey Auction:** every player pays for his jth item the jth highest losing bid of the *other* players

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Example ($k = 4$ items)

b_1	b_2	Payments: Discriminatory	Uniform	Vickrey
10		10	3	1
	9	9	3	3
8		8	3	1
	7	7	3	2
3				
2				
	1			
	1			

State of Affairs

Vickrey Auction is well-understood: strategyproof, individual rational and efficient [Vickrey '61]

Discriminatory Auction: not strategyproof, ...

Uniform Price Auction:

> *A [uniform-price] auction proceeds precisely as a [pay-as-bid auction] with one crucial exception: All successful bidders pay the same price, the cut-off price. An apparently minor change, yet it has the major consequence that no one is deterred from bidding by fear of being stuck with an excessively high price. You do not have to be a specialist. You need only know the maximum amount you are willing to pay for different quantities.*
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Discriminatory and Uniform Price Auctions:

- demand reduction: players have an incentive to understate their true valuations in order to obtain items at a better price

  [Ausubel and Crampton '02]

  $\rightarrow$ equilibria are inefficient in general

- still: these auctions are very popular and used in practice

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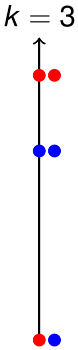

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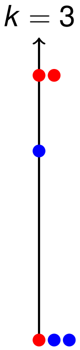

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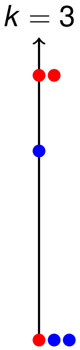

Multi-Unit Auctions in Practice

Inefficiency of Multi-Unit Auctions

Goal of our Studies: gain a precise understanding of the inefficiency of these auction formats

Inefficiency: worst-case ratio of

$$\frac{\text{social welfare of optimal allocation}}{\text{social welfare of equilibrium}}$$

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Our Results: Upper Bounds

Valuation Functions	**Auction Format** (`bidding: standard` | `uniform`) *Discriminatory Auction*	*Uniform Price Auction*
Submodular	$\frac{e}{e-1} \approx 1.58$	3.1462
Subadditive	$2 \mid \frac{2e}{e-1}$	$4 \mid 6.2924$

Remarks:

- improve on previous best bounds by [Syrgkanis, Tardos, STOC'13]
- derive first bounds for subadditive valuation function

Part III

Bounding the Inefficiency of Multi-Unit Auctions

joint work with:

Bart de Keijzer, Vangelis Markakis, Orestis Telelis

Incomplete Information Setting

Setting:

- player i draws his valuation function v_i from a distribution π_i over a finite set V_i
- product distribution $\pi = \prod_i \pi_i$ is public knowledge (**Note:** each player i knows v_i and π, but *not* v_j of $j \neq i$)
- mixed strategy of player i specifies for each valuation function v_i a distribution $\mathbf{B}_i(v_i)$ over marginal bid vectors

Player's Goal: player i determines mixed strategy $\mathbf{B}_i(v_i)$ that maximizes his expected utility *given* v_i:

$$\mathbf{E}_{\substack{\mathbf{v}_{-i} \sim \pi | v_i \\ \mathbf{b} \sim \mathbf{B}(\mathbf{v})}} [u_i(\mathbf{b})]$$

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Bayes-Nash Equilibrium

Solution Concept: a strategy profile $\mathbf{B} = (\mathbf{B}_1, \ldots, \mathbf{B}_n)$ is a (mixed) Bayes-Nash equilibrium if for every player i, every valuation function v_i and every *pure* strategy $\mathbf{b}'_i = \mathbf{b}'_i(v_i)$

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Optimal Allocation: Given a valuation profile $\mathbf{v} = (v_1, \dots, v_n)$, let $\mathbf{x}^{\mathbf{v}} = (x_1^{\mathbf{v}}, \dots, x_n^{\mathbf{v}})$ be an allocation maximizing $SW(\cdot)$.

Bayes-Nash Price of Anarchy:

$$\sup_{\mathbf{B} \in \mathrm{BNE}(\pi)} \frac{\mathbf{E}_{\mathbf{v}\sim\pi}[SW(\mathbf{x}^{\mathbf{v}})]}{\mathbf{E}_{\substack{\mathbf{v}\sim\pi \\ \mathbf{b}\sim\mathbf{B}(\mathbf{v})}}[SW(\mathbf{b})]}$$

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No-Overbidding Assumption for UPA

We make the following assumption for the Uniform Price Auction:

No-Overbidding: given a valuation profile **v**, every player *i* may not overbid his actual valuation

$$\hat{b}_i(x) = \sum_{j=1}^{x} b_i(j) \leq v_i(x) \qquad \forall x \in \{1, \dots, k\}$$

Note:

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Proof Template

Theorem

Suppose that for every player i, every valuation profile $\mathbf{v}$ and every distribution $\mathbf{B}_{-i}$ over bidding profiles $\mathbf{b}_{-i}$, there is a bid vector $\mathbf{b}'_i$ such that

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where $\beta_j(\mathbf{b}_{-i})$ is the jth lowest winning bid under $\mathbf{b}_{-i}$.

Then the Bayes-Nash price of anarchy is at most:

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Idea: identify deviation strategy $\mathbf{B}'_i$ for each player i that

- is mixed (i.e., satisfies the theorem *in expectation*)
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Discriminatory Auction: we show that improving our $e/(e-1)$ bound through the use of any of the currently known techniques is impossible

Uniform Price Auction: lower bound of 2 for submodular valuation functions

Further Implications:

- can extend our results to the smoothness framework of [Syrgkanis, Tardos, STOC'13]
- using their framework, we obtain improved bounds for
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Part IV

Concluding Remarks

Final Remarks

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- bounds suggest that the Discriminatory Auction is superior to the Uniform Price Auction

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THANK YOU!