Self-Adjusting Binary Search Trees: Recent Results

Talk based on papers in WADS 2015, ESA 2015, FOCS 2015, and unpublished work.

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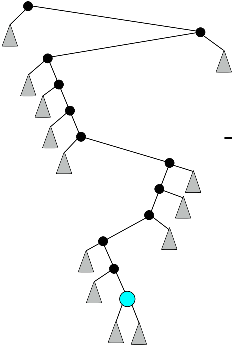
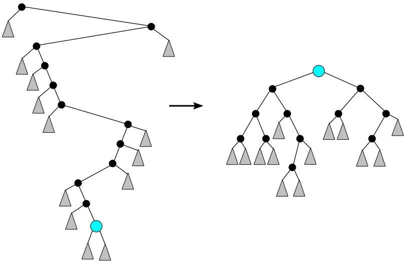
Binary Search Trees (BSTs)



- A search for x in a binary search tree walks down a path. If x is equal to the key stored in the current node, we have found x. If x is smaller than the key stored in the node, we go left. If x is larger than the key stored in the node, we go right.
- Different flavors of BSTs:
  - static BSTs
  - balanced BSTs: AVL-trees, 2-4-trees, red-black trees, ...
  - self-adjusting BSTs: Splay trees.

The Self-Adjusting BST-Model



- After an access, replace the search path by an arbitrary tree (the after-tree) on the same set of nodes rooted at the accessed element.
- Reattach the dangling subtrees (uniquely defined).
- **Cost = length of search path.**

Question: Which re-arrangements lead to an efficient online algorithm?

OPT = cost of the offline optimum.

OPT knows the entire access sequence in advance and can act accordingly. The online algorithm has to rebuild without knowing future accesses.

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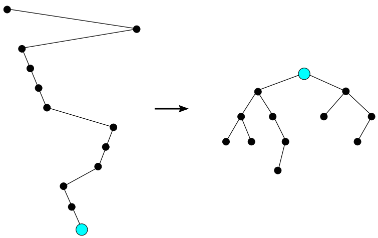

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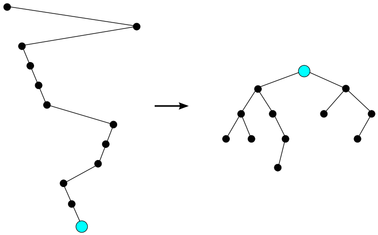

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The Dynamic Optimality Conjecture (Sleator/Tarjan '85)

Splay trees are $O(1)$-competitive, i.e., for every access sequence X, the cost of serving X by splay trees is at most a constant factor larger than serving X optimally.

A path towards proving or disproving the conjecture:

- Understand better which variants of splay trees might also work.
- Show special cases of the dynamic optimality conjecture.
- Exhibit easy sequences, i.e., sequences which OPT serves in time $o(n \log n)$.

ESA-paper addresses the first item.

Self-Adjusting BSTs: What Makes them Tick? (ESA 2015)

Splay trees have many nice properties, e.g.,

- Logarithmic access cost — Static optimality
- Working set property — Static finger property
- Sequential access — Dynamic finger property

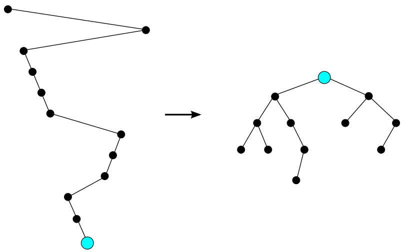
We give sufficient (and necessary) conditions for the first four properties.

Previous work: Sleator, Tarjan, Subramanian, Georgakopoulos, McClurkin prove first four properties for splay-trees and variants thereof.

All of these results are corollaries of the main theorem in the ESA paper. Also prove new results about depth-halving.

Main Result

Characteristic quantities of the search path and the after-tree.



- length of the search path: $|P|$ 12
- number of side changes: z 4
- number of leaves: ℓ 5
- max left-depth of left subtree (max right-depth of right subtree): d 3

Theorem: If accessed element goes to root, $d = O(1)$, and $\ell = \Omega(|P| - z)$, then the BST has the first four properties.

We also have a partial converse (more later).

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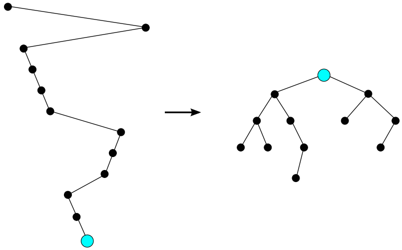

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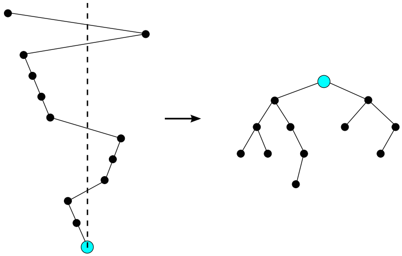

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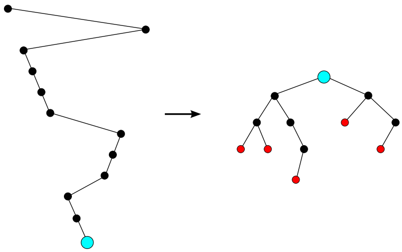

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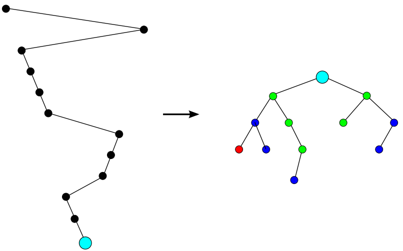

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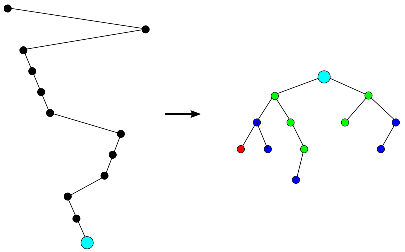

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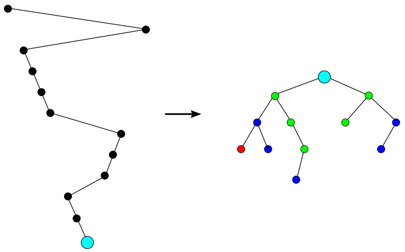

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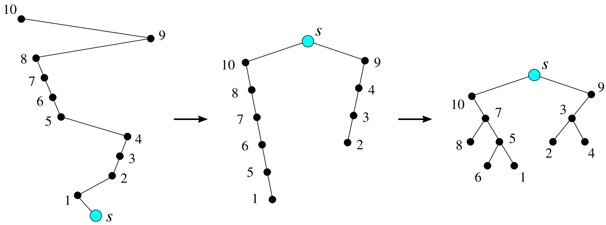

- Split the search path at *s* and swap adjacent odd-even pairs.
- **This is a global view on splay trees; seems to be new.**

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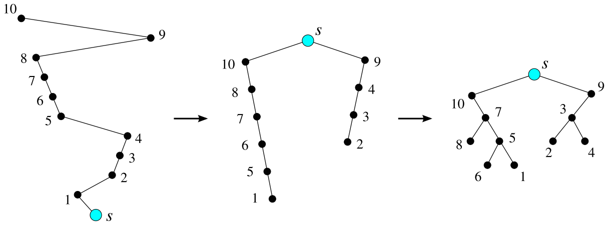

- accessed element becomes root
- max right-(left) depth is $d = 2$
- $z + \ell \geq |P|/2 - 1$

  Proof: There are $|P|/2 - 1$ odd-even pairs. Each side change can move the elements of one pair to different sides. Each odd-even pair on the same side creates a leaf. Thus

$$\text{\# of leaves} \geq |P|/2 - 1 - \text{\# of side changes}$$

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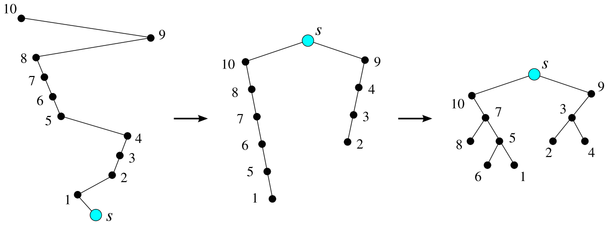

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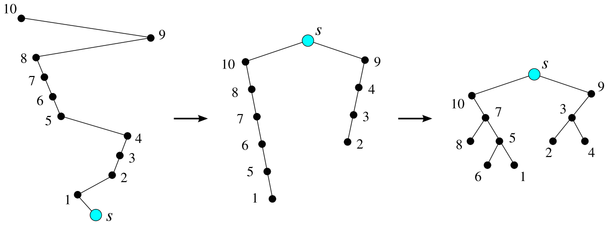

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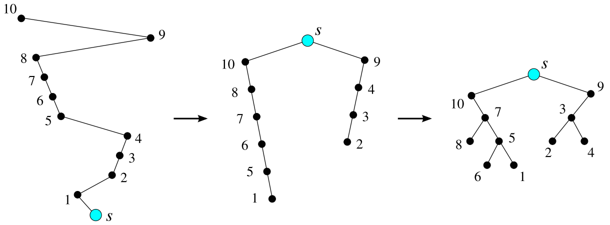

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Depth Halving

In splay every node on the search path roughly halves its depth.

Sleator: is this property sufficient?

We don't know, but strict depth-halving is sufficient: the accessed element becomes the root and every node x on the search path loses at least $(1/2 + \epsilon)d(x) - O(1)$ ancestors and gains at most $O(1)$ new descendants.

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Partial Converses

If the after-tree may have non-constant left-depth or right-depth, then the good properties (logarithmic access, static optimality, ...) cannot be shown with the sum-of-logs potential function.

If the number of leaves of the after-tree is allowed to be $o(|P| - \text{number of side changes})$, then the traversal conjecture does not hold.

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- Exhibit additional easy sequences, i.e., sequences which OPT serves in time $o(n \log n)$.

FOCS-paper addresses items 2 and 3.

Traversal conjecture: Let X be the preorder traversal of a tree T. Process X starting with a tree T'. OPT = $O(n)$.

Only shown for $T' = T$ or $X = 1, 2, \ldots, n$.

Pattern Avoiding Accesses (FOCS 2015)

An access sequence X avoids a pattern P if there is no subsequence of X that is order-isomorphic to P.

- $X = 1, 2, \ldots, n$ avoids $2, 1$.
- Preorder traversal of a tree avoids $2, 3, 1$.
- Special cases of the optimality conjecture.
  - GREEDY serves any sequence that avoids a permutation pattern of size k with cost $O(2^{\alpha(n)^{O(k^2)}} \cdot n)$.
  - GREEDY with chosen initial tree serves any such sequence with cost $O(2^{O(k^2)} \cdot n)$.
  - Traversal conjecture: $k = 3$.
- New easy sequences.
  - OPT serves any k-decomposable sequence with cost $O(n \log k)$.

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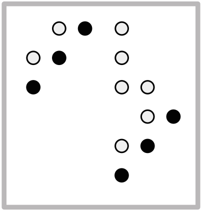
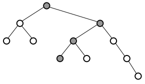
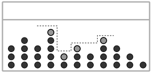
Satisfied Point Sets



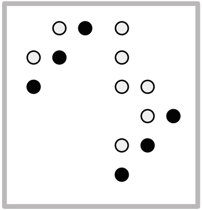




- M = a $\{0,1\}$-matrix (a point set).
- Ignore the colors for the moment.
- $\Box_{pq}$ = closed rectangle with corners p and q.
- M is satisfied if for any two points $p, q \in M$ with distinct x and y coordinates there is another point from M in the rectangle.
- Access sequence $X \to$ matrix X.

  Point $(x, t) \in X$ iff the element x is accessed at time t .
- A tree T gives rise to a matrix T.

Geometric BSTs (Demaine, Harmon, Iacono, Kane, Patrascu (SODA '09))



Cost = 8

Geometric BST $\mathcal{A}$ on input $\left[\begin{smallmatrix} X \\ T \end{smallmatrix}\right]$ outputs a satisfied matrix $\left[\begin{smallmatrix} \mathcal{A}_T(X) \\ T \end{smallmatrix}\right]$, where $\mathcal{A}_T(X) \supseteq X$.

Chosen initial tree: On input X outputs $\mathcal{A}(X)$.

Cost = number of points (ones) in $\mathcal{A}_T(X)$.

Offline versus online

Theorem(DHIKP): (online) geometric-BSTs = (online) BSTs.

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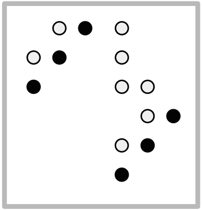

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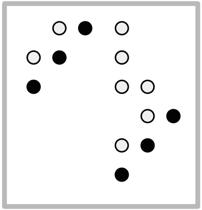

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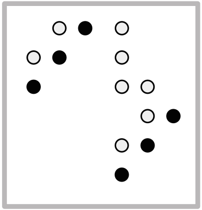

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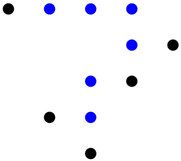

Accessed points in black.

Points added by Greedy in blue.

- After access to x at time t add exactly the (y, t) that are needed for satisfaction.
- Greedy is not optimal.
- Conjecture (Lucas, Munro, DHIKP): Greedy is $O(1)$-competitive.
- Theorem: Greedy almost satisfies traversal conjecture (cost $n \cdot 2^{\alpha(n)^{O(1)}}$).

  Greedy with chosen initial tree satisfies traversal conjecture.

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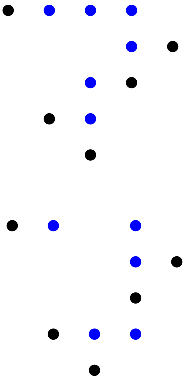

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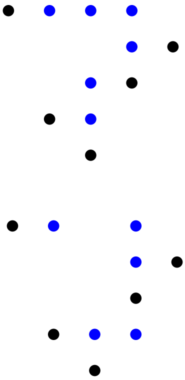

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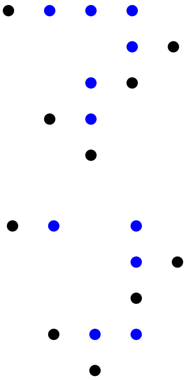

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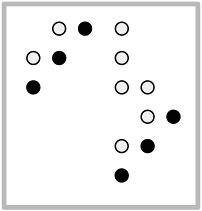
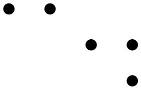
Forbidden Matrix Theory



contains



Let M and P be $n \times n$ and $k \times k$ matrices s.t. M avoids P.

- (Marcus, Tardos, Fox): If P is a permutation matrix, the number of ones in M is at most $n2^{O(k)}$.
- (Klasar, Keszegh): If P is light (only one 1 per column), the number of ones in M is at most $n2^{\alpha(n)^{O(k^2)}}$.

S. Pettie pioneered the use of forbidden matrix theory for the study of data structures.

Greedy and Forbidden Matrices I

Theorem

If the access sequence X avoids a pattern P, then for any initial tree T, $\textsc{Greedy}(X)$ avoids $P \otimes \mathtt{Cap}$, where $\mathtt{Cap} = \begin{pmatrix} & \bullet & \\ \bullet & & \bullet \end{pmatrix}$.

For $P = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix}$, $P \otimes \mathtt{Cap} = \begin{pmatrix} & \bullet & & & & & & & \\ \bullet & & \bullet & & & & & & \\ & & & & & & & \bullet & \\ & & & & & & \bullet & & \bullet \\ & & & & \bullet & & & & \\ & & & \bullet & & \bullet & & & \end{pmatrix}$.

Proof: Assume that at time t columns a and b are touched. If all accesses after time t are to columns $\leq a$ or $\geq b$, then columns $a+1$ to $b-1$ will not be touched after time t.

Thus, if $\textsc{Greedy}(X)$ contains a point in $[a+1, \ldots, b-1] \times [t+1, \ldots]$, X must contain a point in this set.

In particular, if $\textsc{Greedy}(X)$ contains $P \otimes$ *capture* then X contains P.

Greedy and Forbidden Matrices I

Theorem

If the access sequence X avoids a pattern P, then for any initial tree T, $\textsc{Greedy}(X)$ avoids $P \otimes \mathtt{Cap}$, where $\mathtt{Cap} = \begin{pmatrix} & \bullet & \\ \bullet & & \bullet \end{pmatrix}$.

For $P = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix}$, $P \otimes \mathtt{Cap} = \begin{pmatrix} & \bullet & & & & & & & \\ \bullet & & \bullet & & & & & & \\ & & & & & & & \bullet & \\ & & & & & & \bullet & & \bullet \\ & & & & \bullet & & & & \\ & & & \bullet & & \bullet & & & \end{pmatrix}$.

For preorder sequence X, $\textsc{Greedy}(X)$ avoids a light 6×9 matrix.

Theorem

$\textsc{Greedy}$ *with arbitrary initial tree serves preorder sequences with cost $n \cdot 2^{\alpha(n)^{O(1)}}$.*

Greedy and Forbidden Matrices II

Theorem

If access sequence X avoids a pattern P, then $\textsc{Greedy}(X)$ with chosen initial tree avoids $P \otimes P'$, where P' is a particular permutation matrix (of the same size as P).

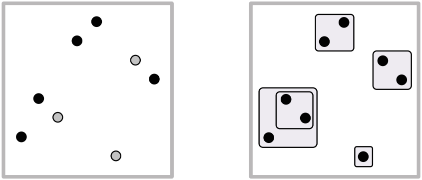
This proof is more involved.

For $P = \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix}$, $P \otimes P'$ is 9×9.

Theorem

Greedy with chosen initial tree satisfies traversal conjecture.

Decomposable Permutations



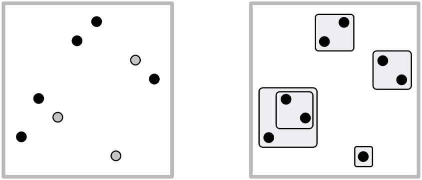
A 4-decomposable permutation.

- k-decomposable = avoid all non-decomposable permutations of size $k + 1$ or more.
- OPT serves k-decomposable permutations with cost $O(n \log k)$.

A new challenge!!!

Proof Technique: We introduce an offline variant of GREEDY and analyse its behavior on k-decomposable permutations.

Decomposable Permutations



A 4-decomposable permutation.

- k-decomposable = avoid all non-decomposable permutations of size $k+1$ or more.
- OPT serves k-decomposable permutations with cost $O(n \log k)$. A new challenge!!!

Theorem

- GREEDY *serves k-decomposable sequences with cost $O(n2^{\alpha(n)^{O(k^2)}})$.*
- GREEDY *with chosen initial tree matches performance of OPT.*

Iacono-Langermann (SODA 2016): Dynamic Finger Property

GREEDY has dynamic finger property, i.e., total search cost is $O(\sum_i \log |x_i - x_{i-1}|)$.

Cole has previously proven dynamic finger property for splay trees (80 page paper, complex proof).

Proof for GREEDY is 10 pages and easy to check.

Summary

- A wide class of BSTs with logarithmic access cost, static optimality, working set and static finger property.
- GREEDY does well on inputs that avoid patterns. In particular,
  - Traversal conjecture almost holds for GREEDY.
  - Traversal conjecture holds for GREEDY with chosen initial tree.
- New challenges for self-adjusting BSTs: OPT serves any sequence that can be decomposed into k monotone sequences with cost $O(n \log k)$.
- Next steps:
  - Show that GREEDY does well on k-monotone sequences
  - Traversal conjecture for arbitrary initial tree.

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