Modern Benders (in a nutshell)

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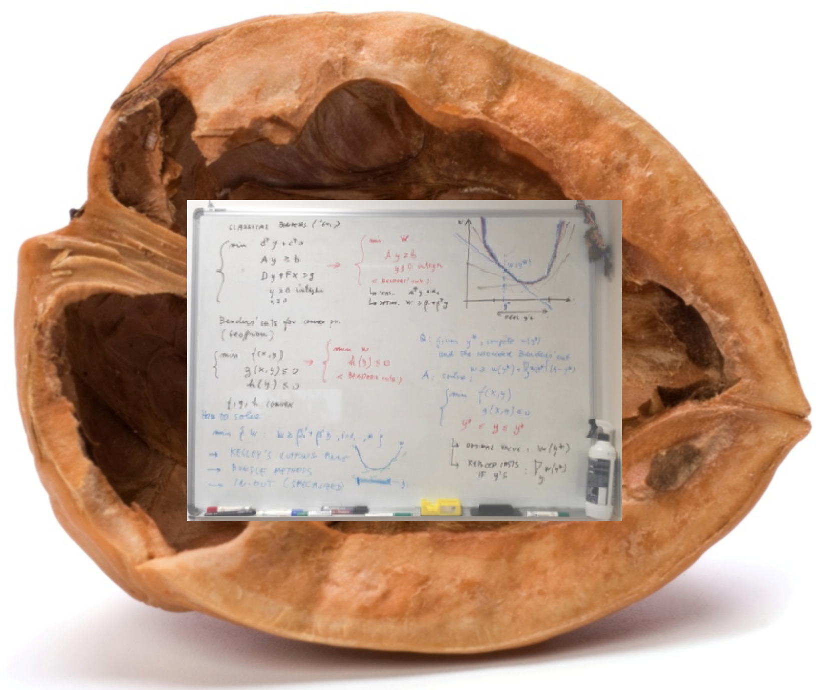
(based on joint work with Ivana Ljubic and Markus Sinnl)

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Discrete Optimization

Benders decomposition without separability: A computational study for capacitated facility location problems

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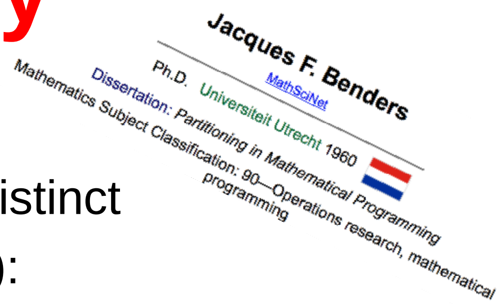
What do you actually mean by “Benders decomposition”?



- The original Benders decomposition from the ‘60s uses **two** distinct ingredients for solving a Mixed-Integer Linear Program (MILP):
  1) A **search strategy** where a relaxed **(NP-hard)** MILP on a variable **subspace** is solved exactly (i.e., to **integrality**) by a black-box solver, and then is iteratively tightened by means of additional **“Benders” linear cuts**
  2) The **technicality** of how to actually compute those cuts (Farkas’ projection)
  – Papers proposing “a new Benders-like scheme” typically refer to 1)
  – Students scared by “Benders implementations” typically refer to 2)

  Later developments in the ‘70s:
  – Folklore (Miliotios for TSP?): generate Benders cuts within a **single B&B tree** to cut any infeasible integer solution that is going to update the incumbent
  – McDaniel & Devine (1977): use Benders cuts to cut **fractional sol.s** as well (root node only)

- Everything fits very naturally within a modern **Branch-and-Cut** (B&C) framework.

B&C for Mixed-Integer Programming

- We will focus on the MIP

$$\min f(x,y)$$
$$g(x,y) \leq 0$$
$$Ay \leq b$$
$$y \text{ integer}$$

  where *f* and *g* are **convex functions**

- Non-convexity only comes from integrality requirement on *y*, so it can be handled by a **branch-and-bound** scheme (possibly using on-the-fly cutting planes) → **Branch and Cut (B&C) solution scheme**

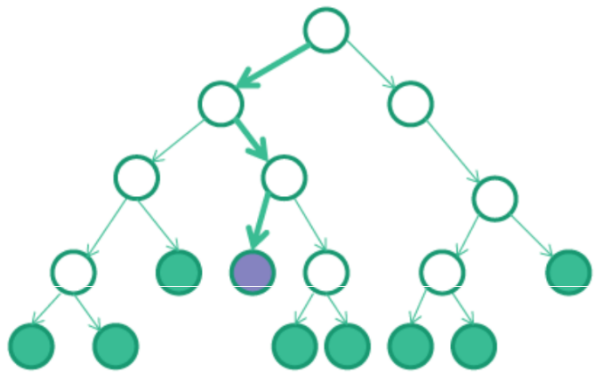
- B&C was proposed by Padberg and Rinaldi in the '90s (i.e., well after Benders seminal work) and is nowadays the method of choice for solving MIP

- **This talk:** rephrase Benders in "modern slang" **#BendersIsEasy**

Modern B&C implementation

- Modern commercial B&C solvers such as IBM ILOG Cplex, Gurobi etc. can be fully **customized** by using ***callback functions***

- Callback functions are just **entry points** in the B&C code where an advanced user (you!) can add his/her customizations



- Most-used callbacks (using Cplex's jargon)
  - **Lazy constraint**: add "lazy constr.s" that should be part of the original model
  - **User cut**: add additional contr.s that hopefully help enforcing feasibility/integrality
  - Heuristic: try to improve the incumbent (primal solution) as soon as possible
  - Branch: modify the branching strategy
  - …

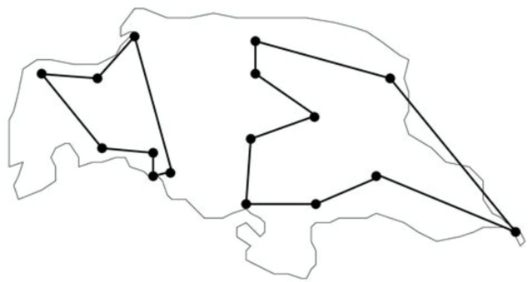
Lazy constraint callback

- Automatically invoked when a solution is going to update the **incumbent** (meaning it is **integer** and **feasible** w.r.t. current model)

- This is the **last checkpoint** where we can discard a solution for whatever reason (e.g., because it violates a constraint that is not part of the current model)

- To avoid be bothered by this solution again and again, we can/should return a **violated constraint (cut)** that is added (globally or locally) to the current model

- Cut generation is often **simplified** by the fact that the solution to be cut is known to be **integer** (e.g., SECs for TSP)



User cut callback

- Automatically invoked at every B&B node when the current solution is **not integer** (say: just before branching)
- A **violated cut** can possibly be returned, to be added (locally or globally) to the current model → often leads to an improved convergence to integer solutions
- If no cut is returned, **branching** occurs as usual

- Cut generation **can be hard** as the point is not integer (heuristic approaches can be used)
- User cuts are **not mandatory** for B&C correctness → being too clever on them can actually **slow-down** the solver because of the overhead in generating and using them (larger/denser LPs etc.)

Modern Benders

- Consider again the convex MINLP in the *(x,y)* space

$$\min f(x,y)$$
$$g(x,y) \le 0$$
$$Ay \le b$$
$$y \text{ integer}$$

and assume for the sake of simplicity that $S := \{y : Ay \le b\}$ is nonempty and bounded, and that

$$X(y) := \{x : g(x,y) \le 0\}$$

is **nonempty**, closed and bounded for all y ∈ S

→ the **convex function** $\Phi(y) := \min_{x \in X(y)} f(x,y)$ is well defined for all y ∈ S

→ no "feasibility cuts" needed (this kind of cuts will be discussed later on)

Working on the y-space (projection)

(1)

$$\min_y \min_x f(x,y)$$
$$g(x,y) \le 0$$
$$Ay \le b$$
$$y \text{ integer}$$

(2)

"isolate the inner minimization over *x*"

$$\Phi(y) := \min_x f(x,y)$$
$$g(x,y) \le 0$$

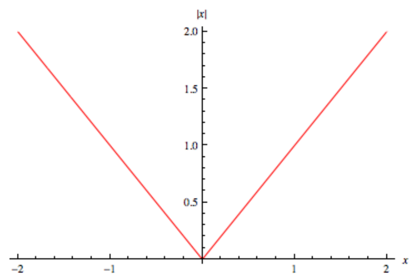
(3)

$$\min \Phi(y)$$
$$Ay \le b$$
$$y \text{ integer}$$

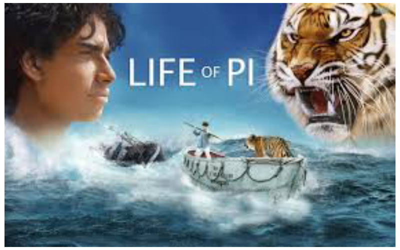
Original MINLP in the (x,y) space → Projected "**master"** problem in the *y* space

Warning: projection changes the objective function shape!

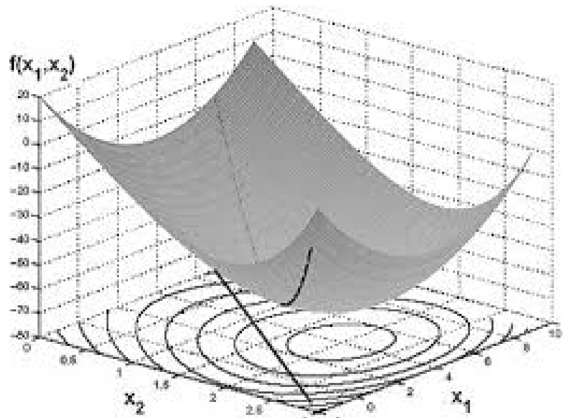
$$\min x$$
$$x \ge y$$
$$x \ge -y$$
$$y \in [-1,1]$$



$$\min \Phi(y) = |y|$$
$$y \in [-1,1]$$

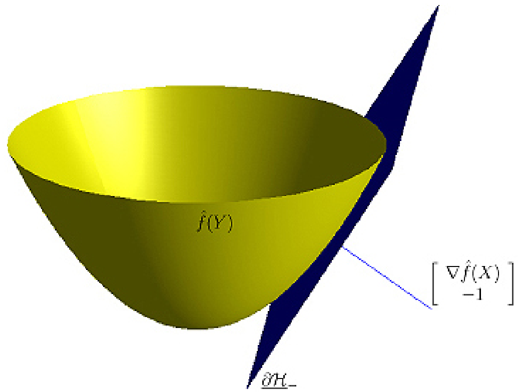


Life of P(H)I



- Solving Benders' master problem calls for the minimization of a **nonlinear** convex function **(even if you start from a linear problem!)**

- Branch-and-cut MINLP solvers generate a sequence of **linear cuts** to approximate this function from below (**outer-approximation**)



$$\begin{aligned} \min\ & w \\ \text{s.t.}\ & w \geq \Phi(y) \\ & Ay \leq b \\ & y \text{ integer} \end{aligned}$$

$$w \geq \Phi(y) \geq \Phi(y^*) + \xi(y^*)^T(y - y^*)$$

Benders cut computation

- **Benders** (for linear) and **Geoffrion** (general convex) told us how to compute a **(sub)gradient** to be used in the cut derivation, by using the optimal primal-dual solution *(x*,u*)* available after computing $\Phi(y^*)$

$$\xi(y^*) = \nabla_y f(x^*, y^*) + u^* \nabla_y g(x^*, y^*)$$

- The above formula is **problem-specific** and perhaps **#scaring**
- By rewriting

$$\Phi(y^*) = \min\{f(x,\mathbf{q}) \mid g(x,\mathbf{q}) \le 0,\ y^* \le \mathbf{q} \le y^*\}$$

we obtain a much **simpler recipe** to derive the same Benders cut:

1) solve the original convex problem with new var. bounds $y^* \le y \le y^*$

2) take opt_val and reduced costs r_j's

3) write $w \ge opt\_val + \sum_j r_j (y_j - y_j^*)$

Benders feasibility cuts

- For some important applications, the set

$$X(y) := \{x : g(x,y) \le 0\}$$

can be empty for some "**infeasible**" $y \in S$

$\rightarrow \quad \Phi(y) := \min_{x \in X(y)} f(x,y)$ undefined

- This situation can be handled by considering the "phase-1" feasibility condition

$$0 \ge \Psi(y) := \min\{1^T s \,|\, g(x,y) \le s,\ s \ge 0\}$$

where the function $\Psi(y)$ is **convex**
$\rightarrow$ it can be approximated by the usual (sub)gradient **"feasibility cut"**

$$0 \ge \Psi(y) \ge \Psi(y^*) + \xi(y^*)^T (y - y^*)$$

to be computed by the same machinery as the usual **"optimality cut"**

$$w \ge \Phi(y) \ge \Phi(y^*) + \xi(y^*)^T (y - y^*)$$

Successful Benders applications

- Benders decomposition works well when fixing *y=y** for computing $\Phi(y^*)$ makes the problem **much simpler to solve**.

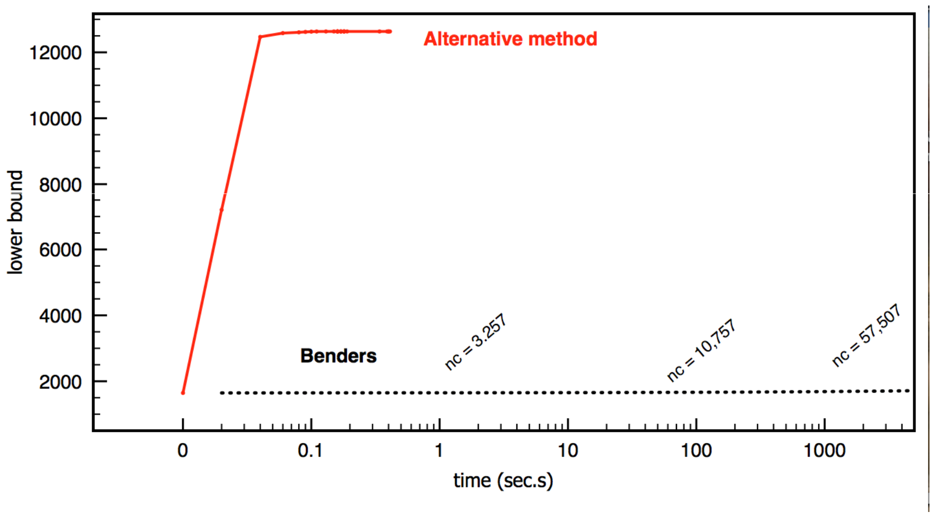
- This usually happens when
  - The problem for *y=y** decomposes into a number of **independent subproblems**
    - Stochastic Programming
    - Uncapacitated Facility Location
    - etc.

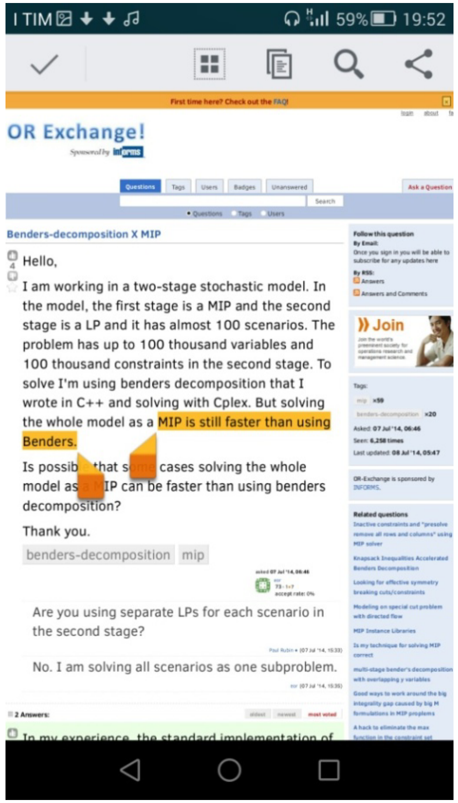
$$
\begin{aligned}
\min \sum_{i\in I} f_i y_i + \sum_{i\in I}\sum_{j\in J} c_{ij} x_{ij} & \\
\text{s.t. } \sum_{i\in I} x_{ij} = 1 & \quad \forall j \in J \\
x_{ij} \le y_i & \quad \forall i \in I, j \in J \\
x_{ij} \ge 0 & \quad \forall i \in I, j \in J \\
y_i \in \{0,1\} & \quad \forall i \in I
\end{aligned}
$$

  - Fixing *y=y** **changes the nature** of some constraints:
    - in Capacitated Facility Location, tons of contr.s of the form $x_{ij} \le y_j$ become just variable bounds
    - Second Order Constraints $x_{ij}^2 \le z_{ij}\, y_i$ become quadratic contr.s
    - etc.

That's it … or not?

- In practice, Benders decomposition can work quite well, but sometimes it is **desperately slow**
  … as the root node bound does not improve even after the addition of tons of Benders cuts





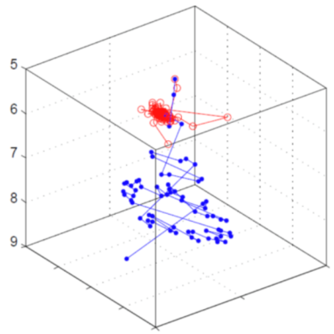
- Slow convergence is generally attributed to the **poor quality** of Benders cuts, to be cured by a more clever **selection policy** (Pareto optimality of Magnanti and Wong, 1981, etc.) but **there is more**…

Role of the cut loop

- B&C codes generate cuts, on the fly, in a **sequential** fashion
- Consider e.g. the **root B&C node** (arguably, the most critical one)
- A classical **cut-loop scheme** (described here for MILPs)

  J. E. Kelley. The cutting plane method for solving convex programs, Journal of the SIAM, 8:703-712, 1960.

  – Find an optimal **vertex** x^* of the current LP relaxation
  – Invoke a separation function on x^*, add the returned violated cut (if any) to the current LP, and repeat

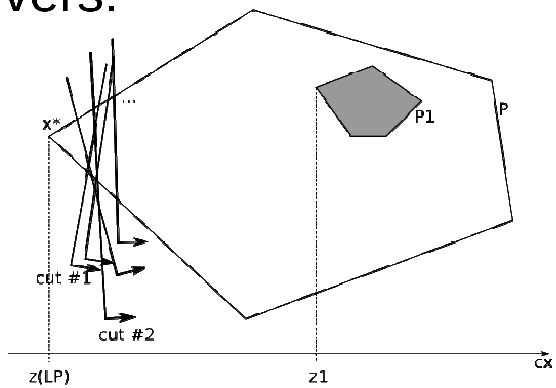


- Can be very **ineffective** in the **first iterations** when few constraints are specified, and x^* moves along an **unstable zig-zag trajectory**

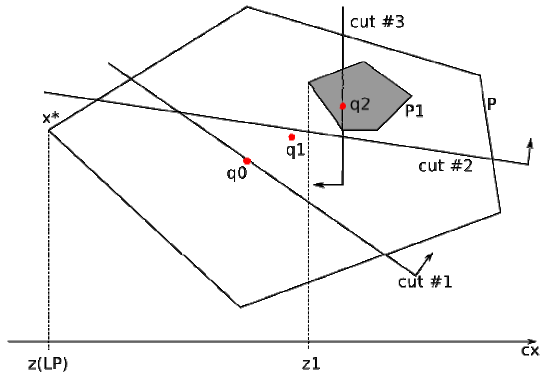
  ... which is precisely what often happens with Benders cuts

But… alternative cut loops do exist!

- **Kelley's** cut loop implemented in standard MI(L)P solvers:
  - PROS: natural, efficient reopt., often works well
  - CONS: can be VERY ineffective, e.g., in column generation or in some under-constrained cutting plane methods



- **Ellipsoid & Analytic Center** cut loops:
  kind of **binary search** in the multi-dimensional space: at each iteration, a **core point** *q* "well inside" the current relaxation is computed and separated
  - CONS: q can be difficult to find and to separate
  - PROS: overall convergence does not depend on the quality of the cut (facets not required here!)



- Cheaper alternatives often preferred: **bundle** (Lemaréchal) or **in-out** (Ben-Ameur and Neto) methods

Stabilizing Benders can be easy!

- To summarize:

  - Benders cut machinery is easy to implement …

  … but the root node cut loop can be **very critical**
  → many implementations sank here!

  - Kelley's cut loop can be **desperately slow**

  - Stabilization using "interior points" is a **must**
    → this is well-known in subgradient optimization and Dantzig-Wolfe decomposition (column generation), but holds for Benders as well

- E.g., for facility location problems, we implemented a very simple **"chase the carrot"** heuristic to determine a stabilized path towards the optimal *y*

- Akin to **Nesterov's Accelerated Gradient** descent method

Our #ChaseTheCarrot heuristic

- We (the donkey) start with ***y* = *(1,1,…,1)*** and optimize the master LP as in Kelley, to get optimal ***y**** (the carrot on the stick).

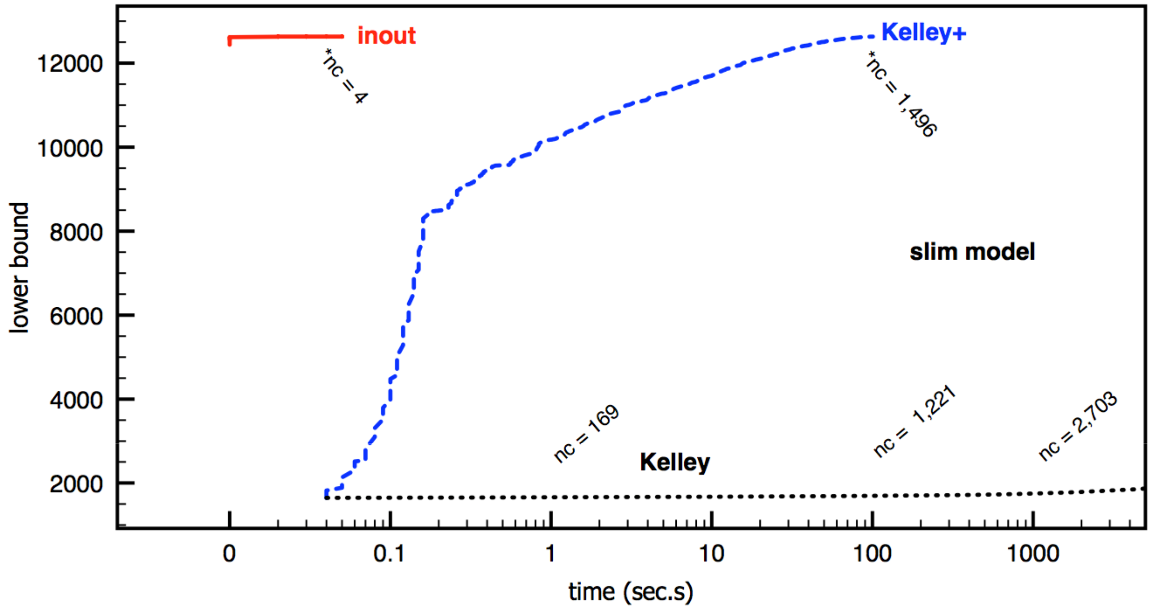
- We move ***y*** just **half-way** towards ***y*.*** We then separate a point *y'* in the segment **[*y, y**]** close to the new ***y***.

- The generated Benders cut is added to the master LP, which is reoptimized to get the new optimal ***y**** (carrot moves).

- Repeat until bound improves, then switch to Kelley for final bound refinement (kind of cross-over)

- **Warning: adaptations needed if feasibility Benders cuts can be generated…**

Effect of the improved cut loop



- Comparing **Kelley** cut loop at the root node with **Kelley+** (add epsilon to y*) and with our chase-the-carrot method (**inout**)
- Koerkel-Ghosh **qUFL** instance gs250a-1 (250x250, quadratic costs)
- ***nc** = n. of Benders cuts generated at the end of the root node
- times in **logarithmic scale**

Conclusions

To summarize:

- Benders cuts are **easy** to implement within modern B&C (just use a callback where you solve the problem for $y=y^*$ and compute reduced costs)

- Kelley's cut loop can be **desperately slow** hence stabilization is a **must**

- Implementations in **general** MIP solvers expected soon (already in Cplex 12.7)

Slides available at http://www.dei.unipd.it/~fisch/papers/slides/

Reference papers:

M. Fischetti, I. Ljubic, M. Sinnl, "Benders decomposition without separability: a computational study for capacitated facility location problems", European Journal of Operational Research, 253, 557-569, 2016.

M. Fischetti, I. Ljubic, M. Sinnl, "Redesigning Benders Decomposition for Large Scale Facility Location", to appear in Management Science, 2016.