Delay, memory, and messaging tradeoffs in distributed service systems

John N. Tsitsiklis

(with D. Gamarnik and M. Zubeldia)

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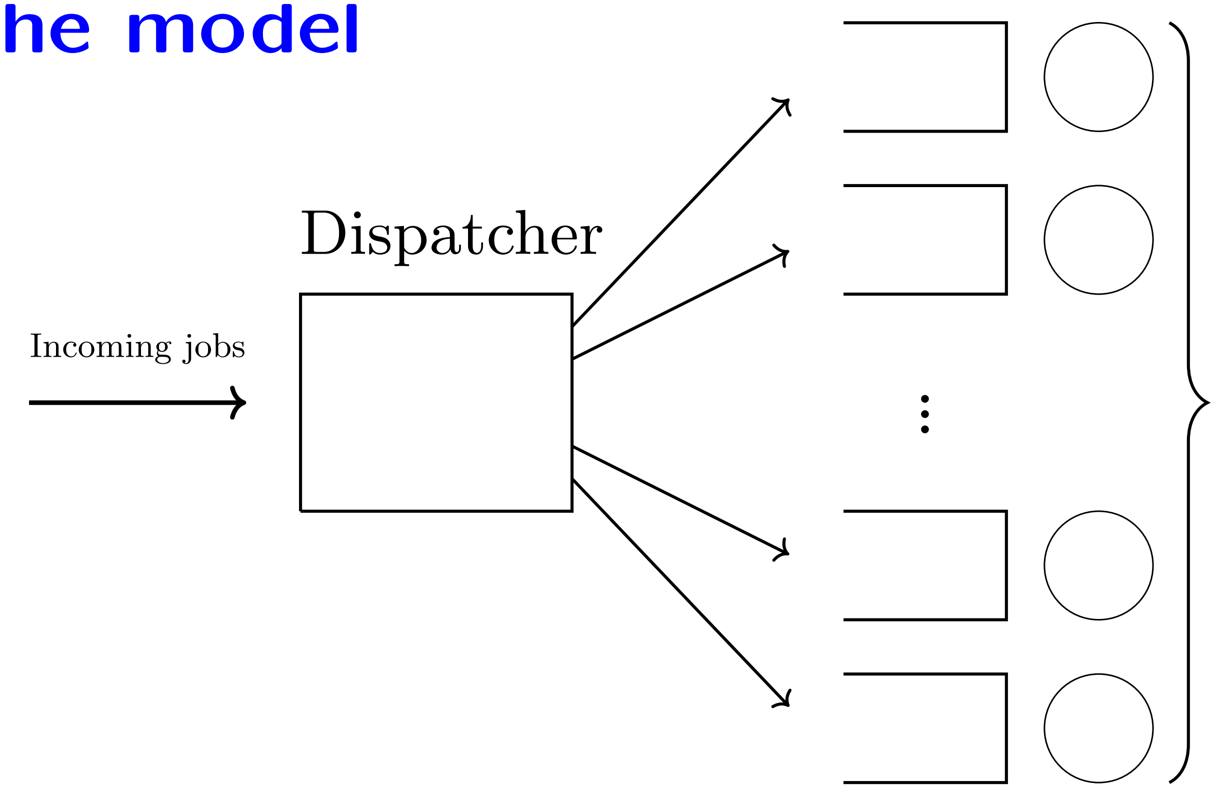
Motivation

- Many modern queueing systems are large scale
- Operating optimally requires large scale resources
- Understand the best performance
  under limited resource availability
- Our context: Dispatching policies
  with limited memory and limited information exchange
  in a many-server queueing system
  (supermarket model)

Outline

- The supermarket model
  - overview and comparison of some policies
- A (somewhat) new policy
  - performance in three regimes
- Lower bound on resources required
- Technical details
- Conclusion

The model



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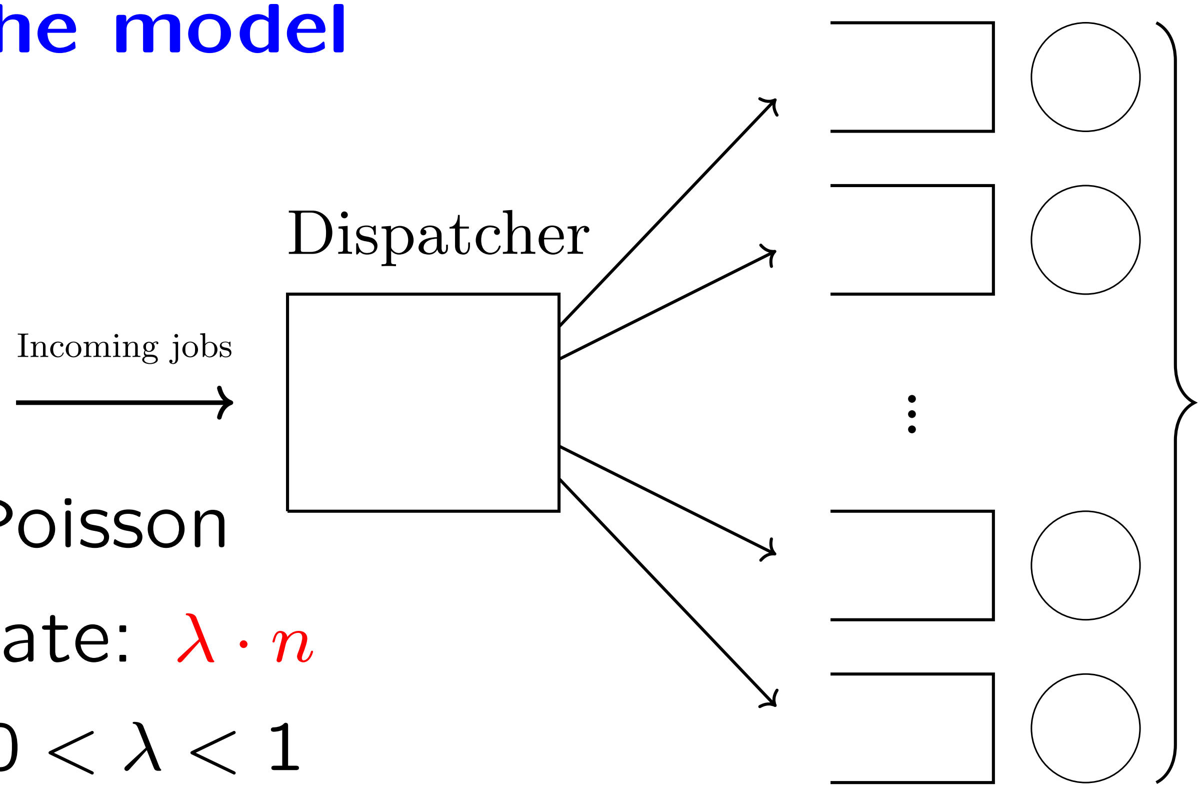

Poisson

rate: $\lambda \cdot n$

$0 < \lambda < 1$

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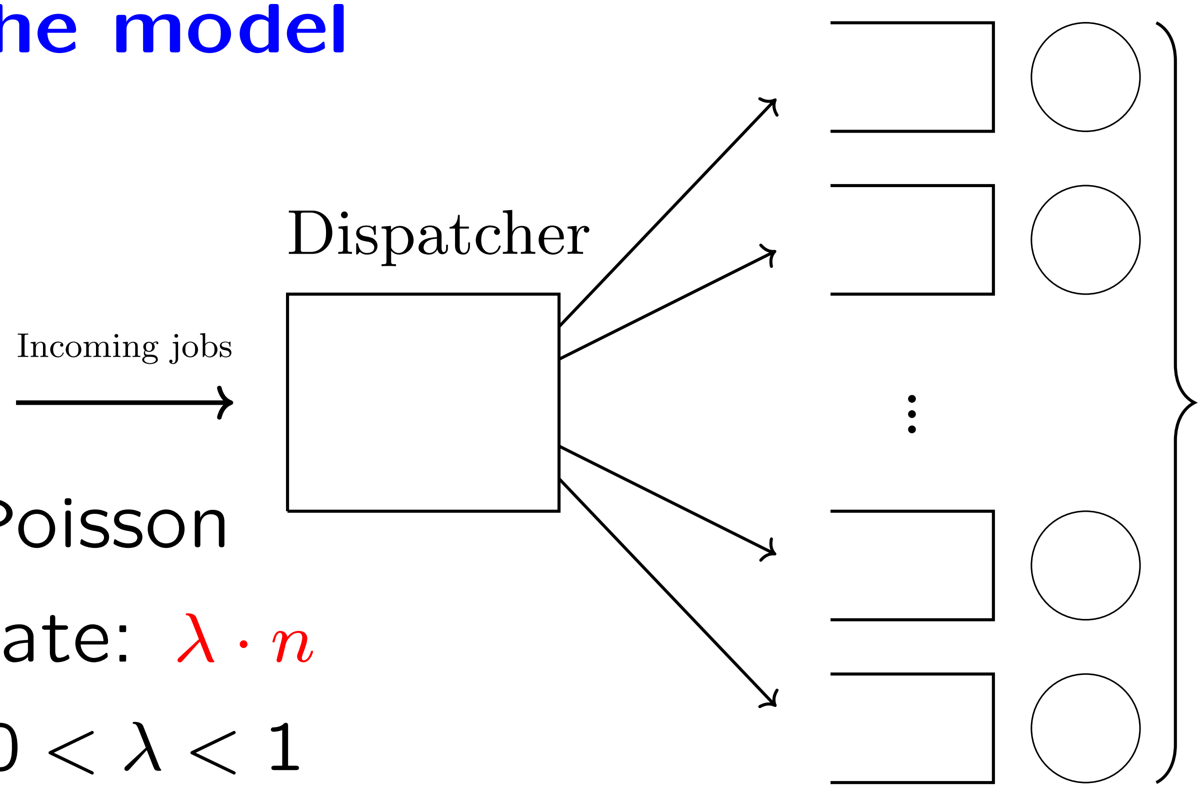

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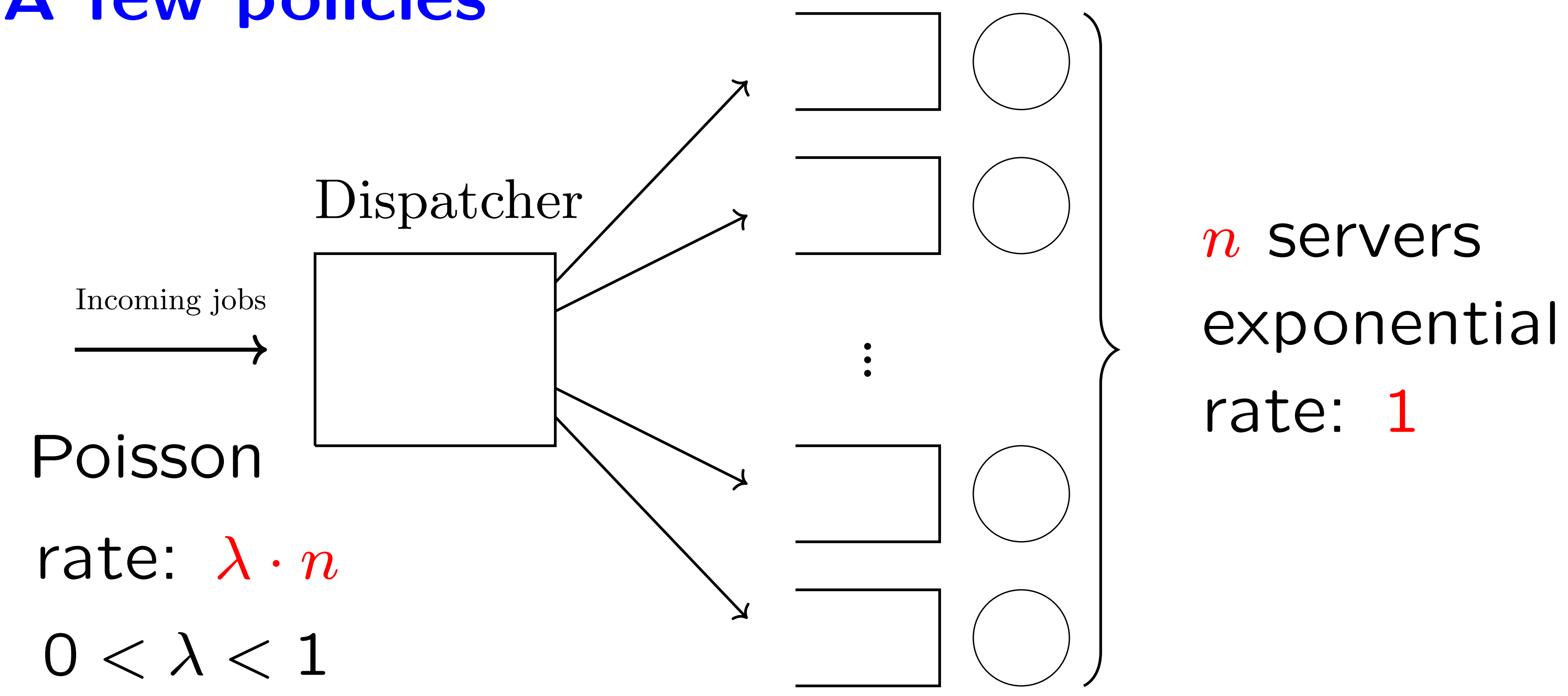
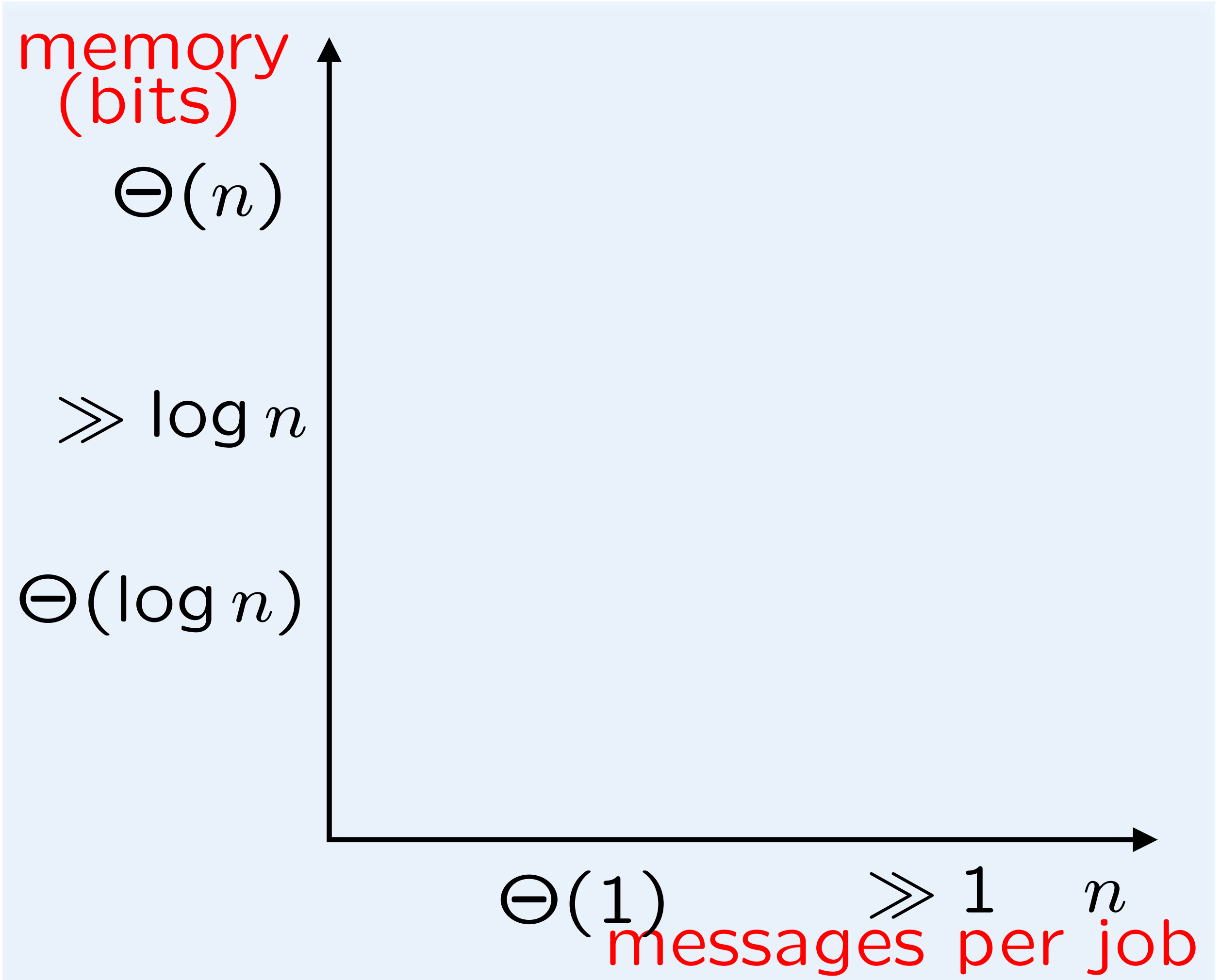
exponential

rate: 1

A few policies





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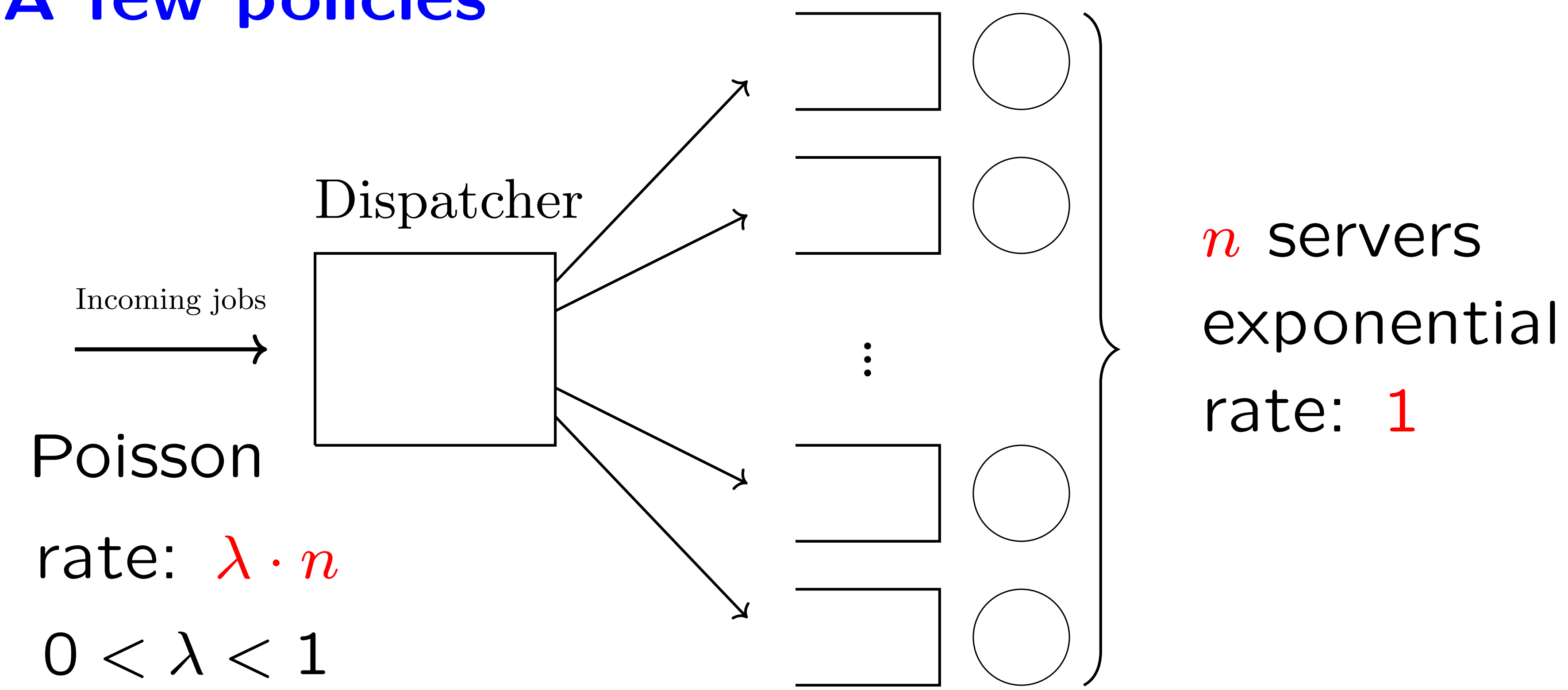

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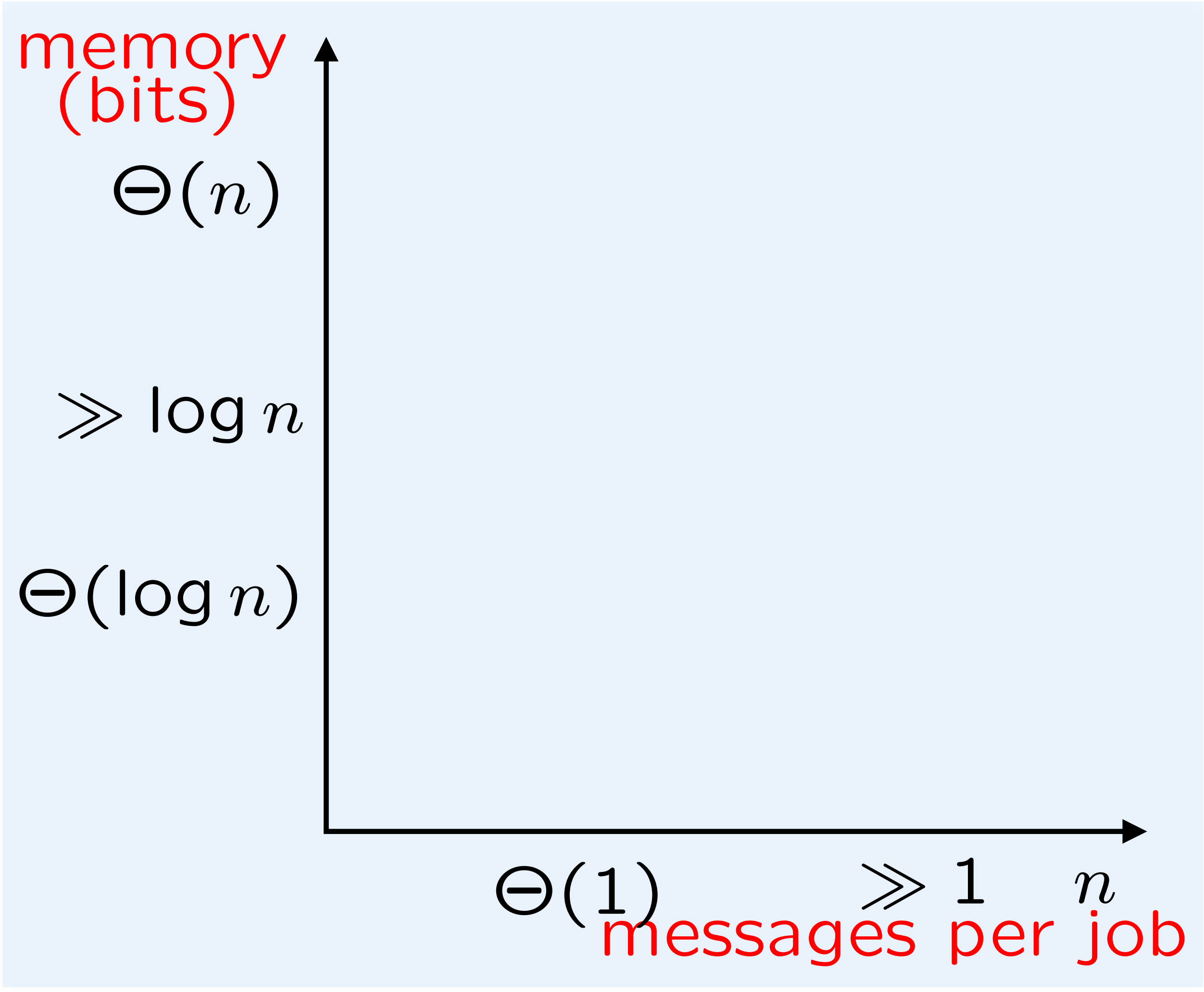

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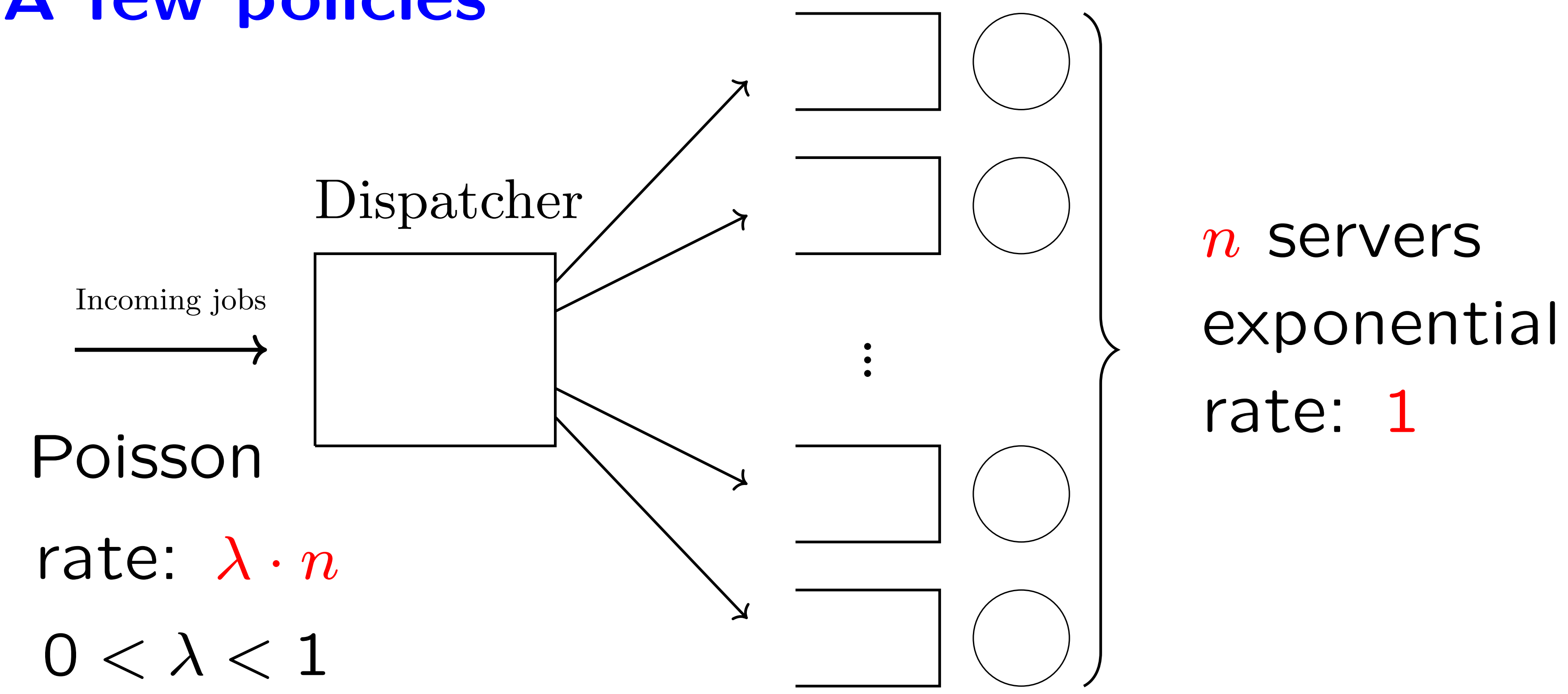

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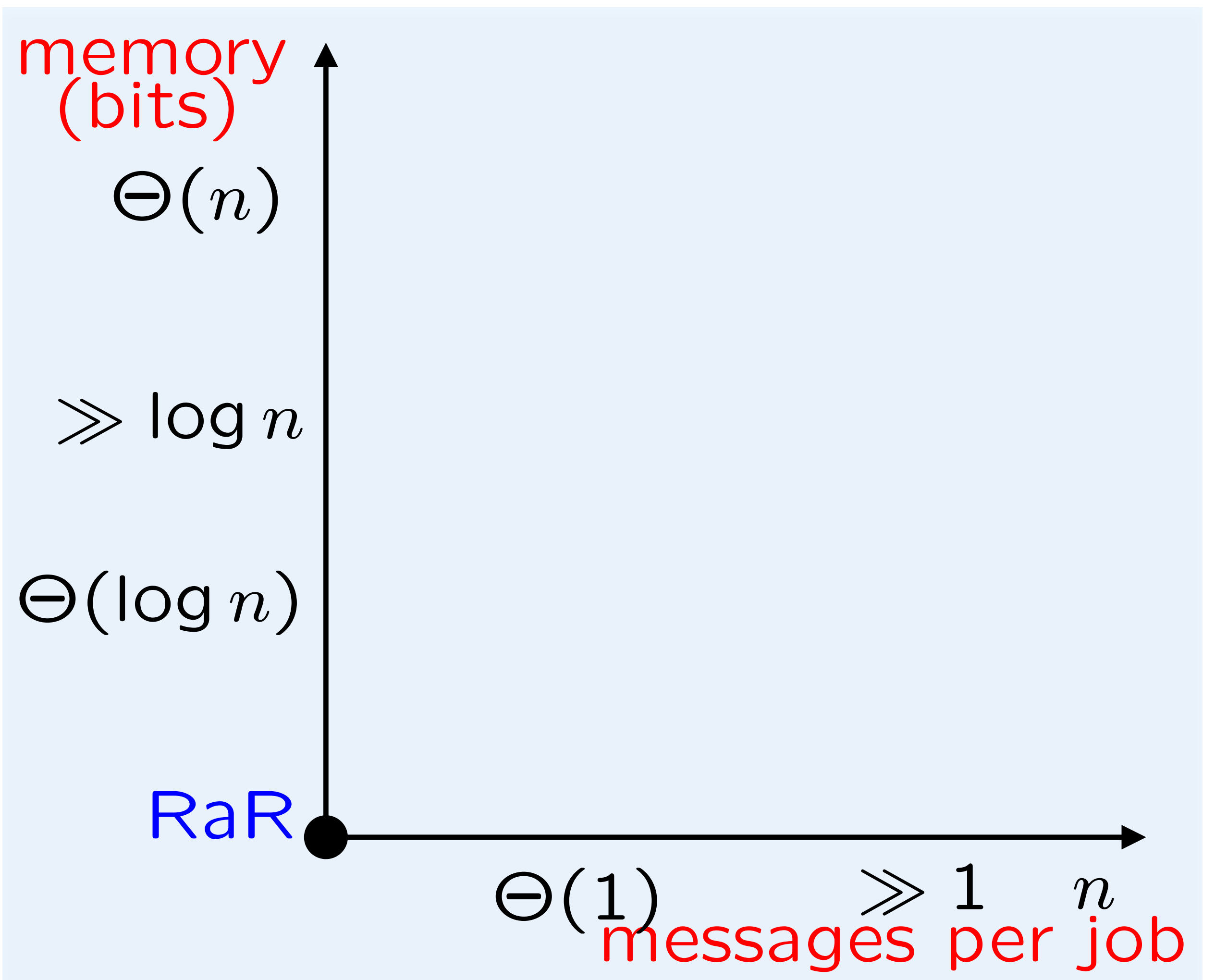

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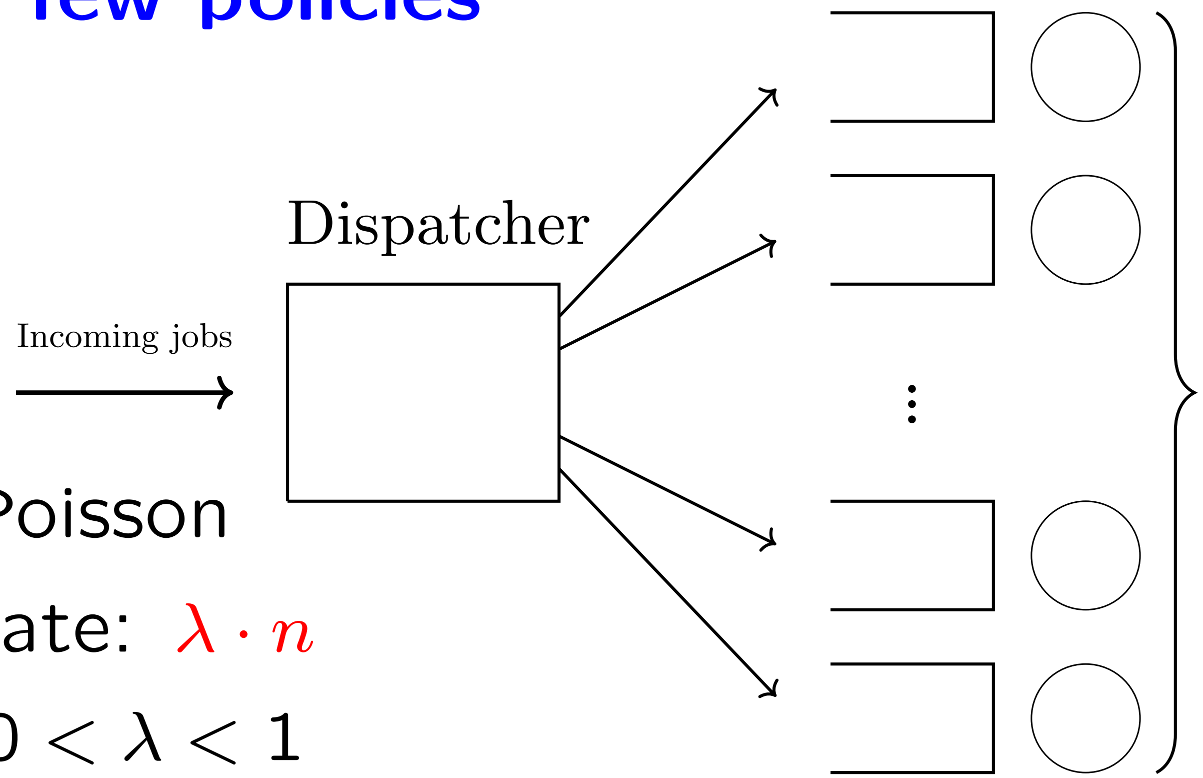

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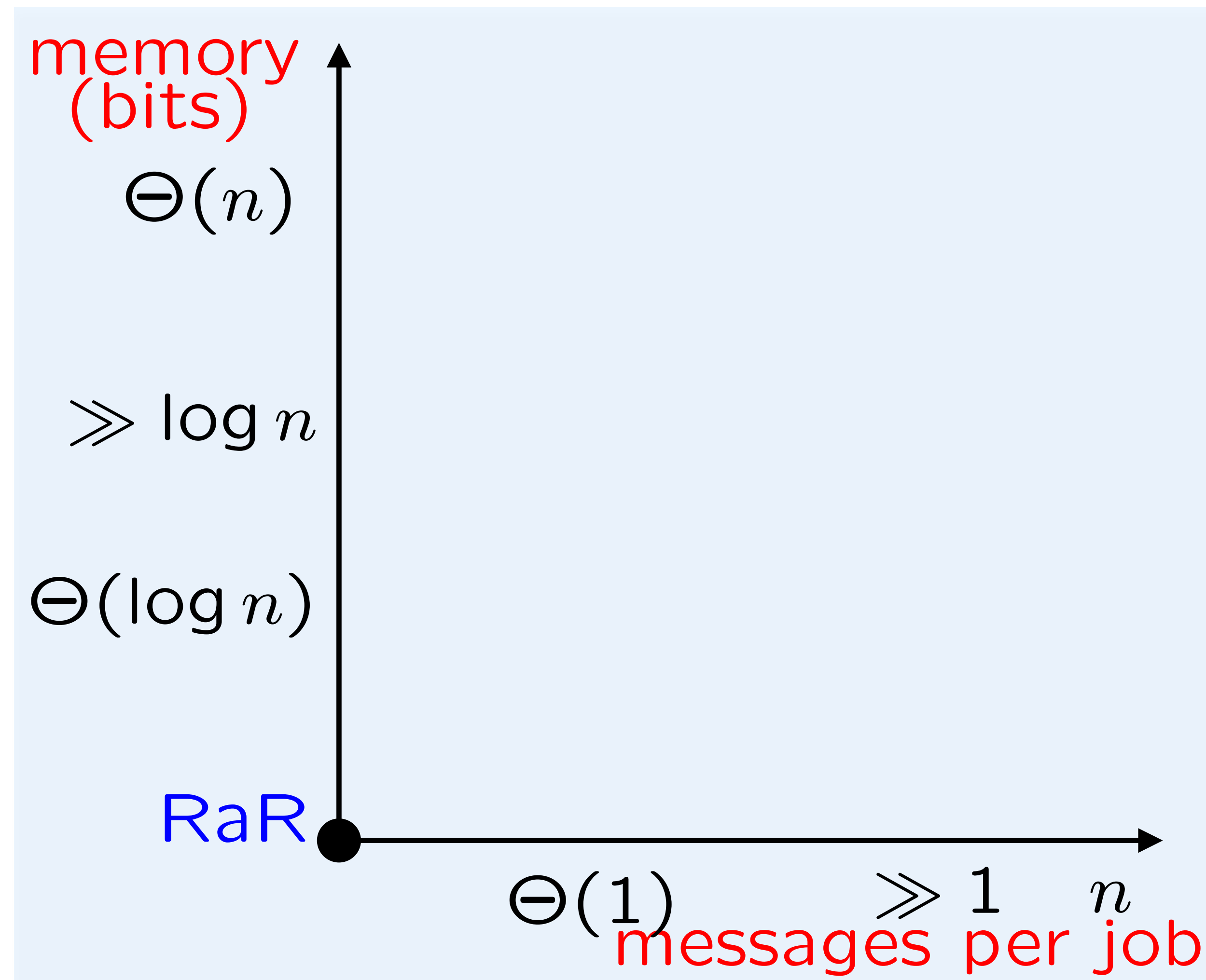

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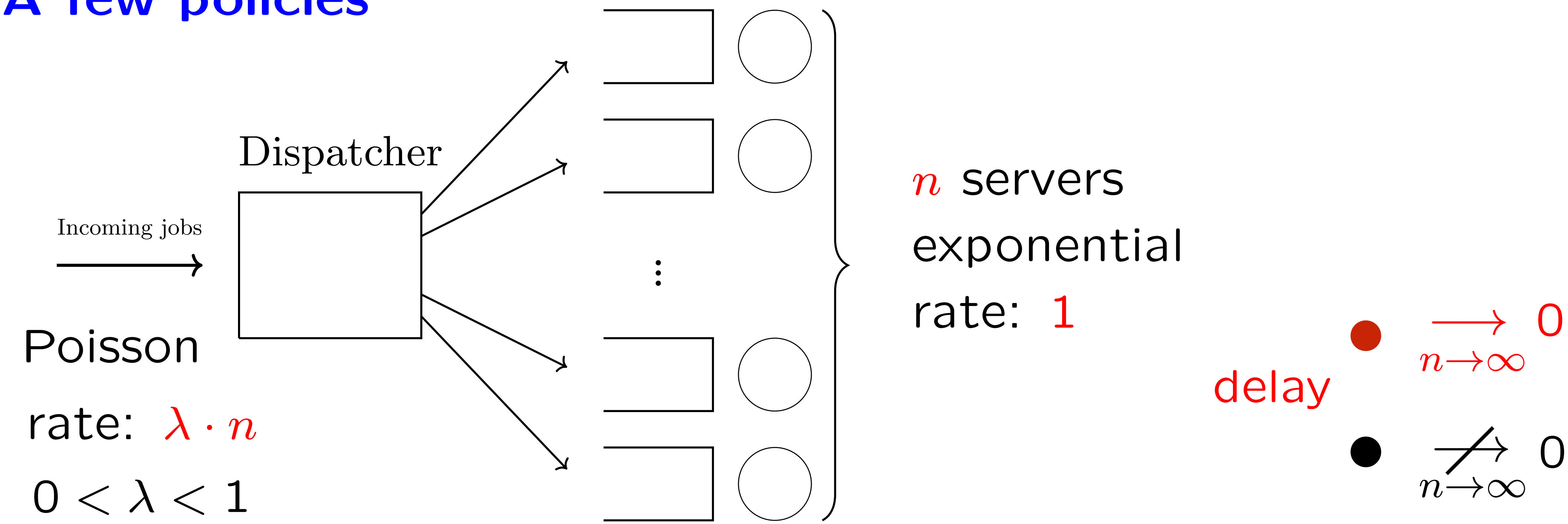

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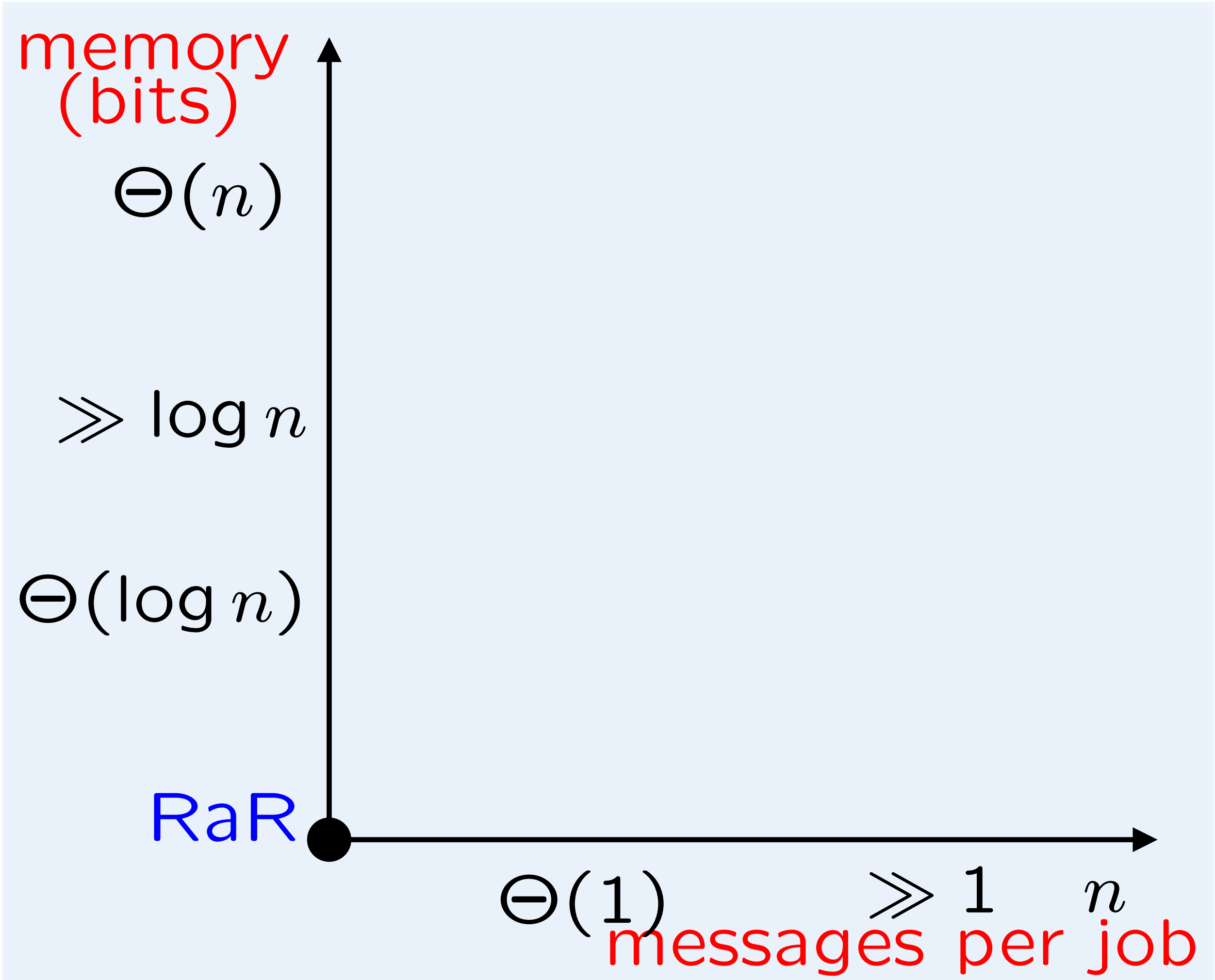

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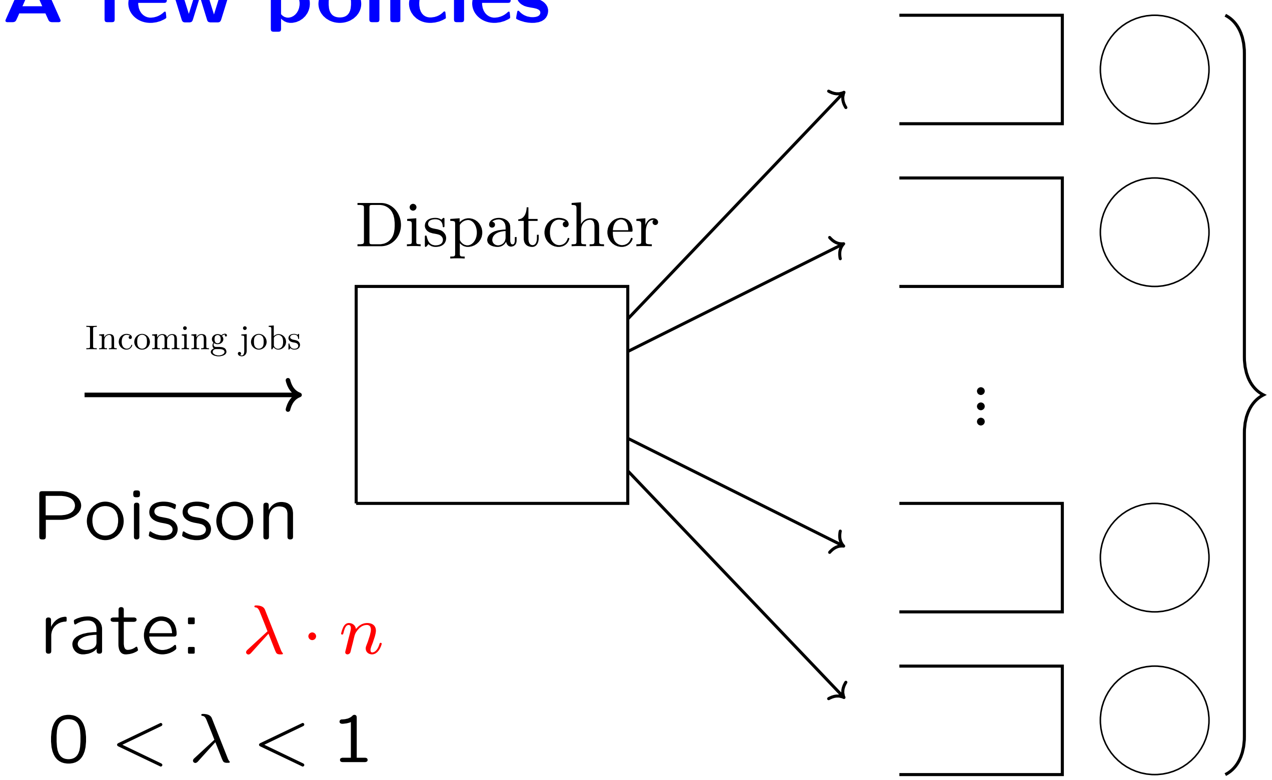

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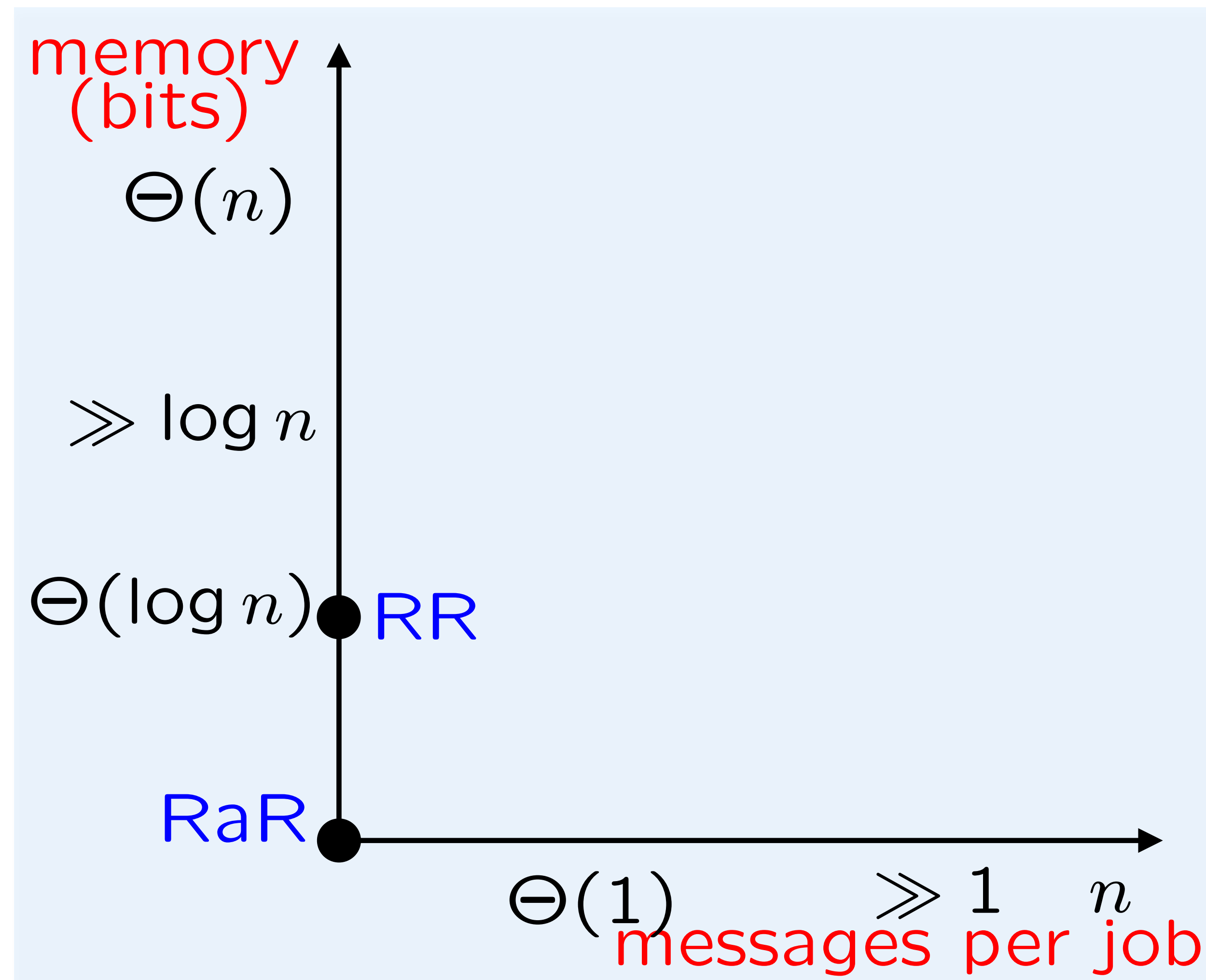

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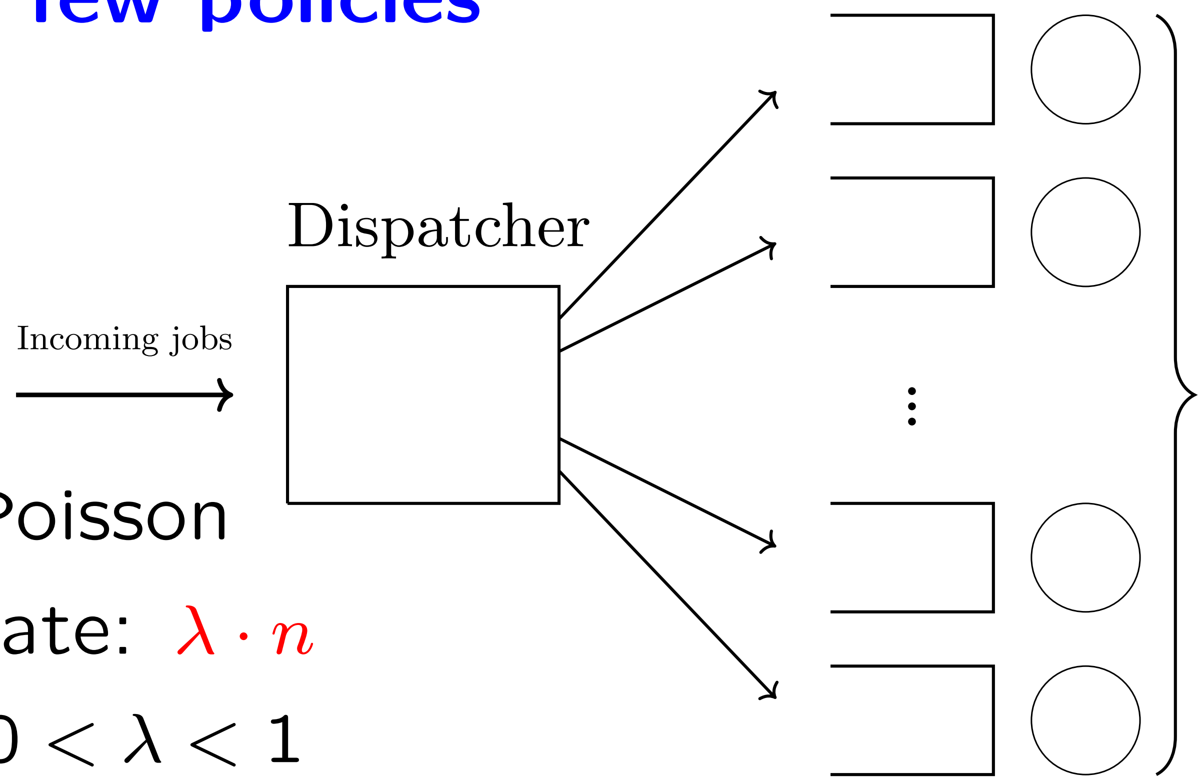

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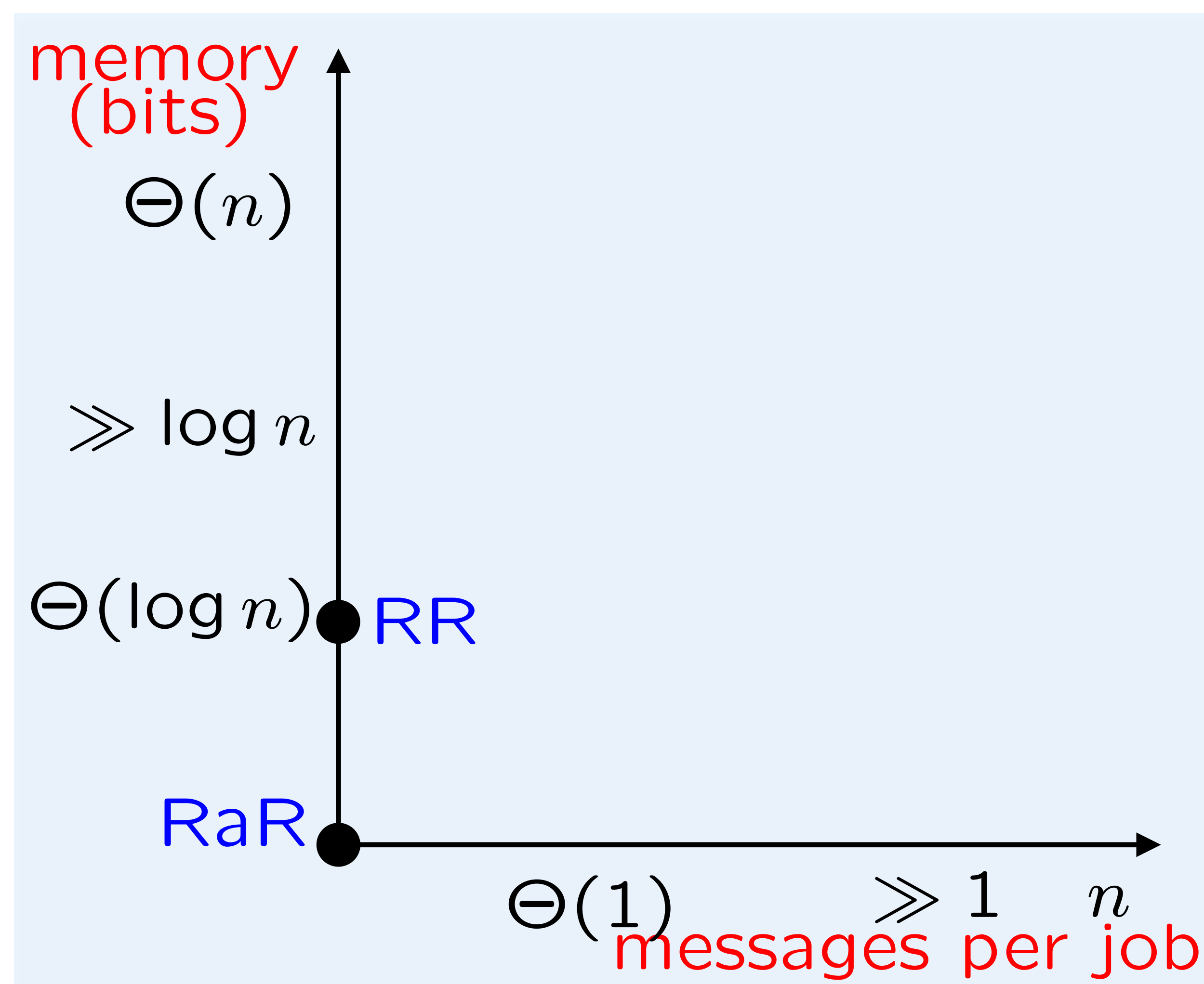

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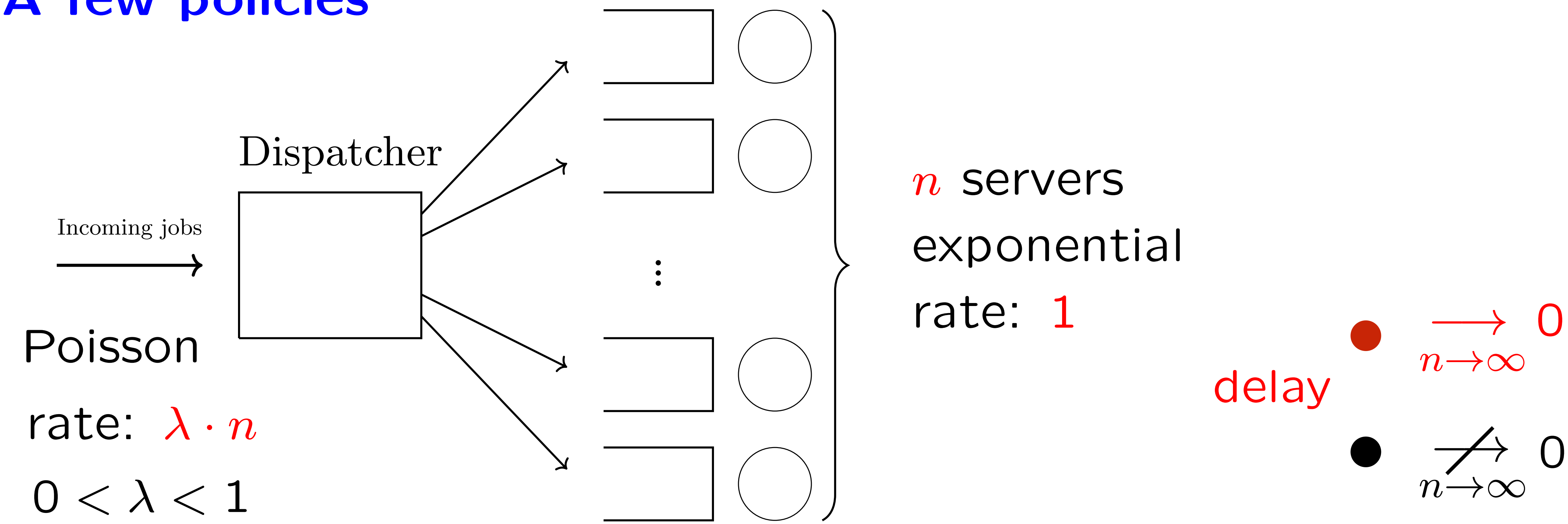

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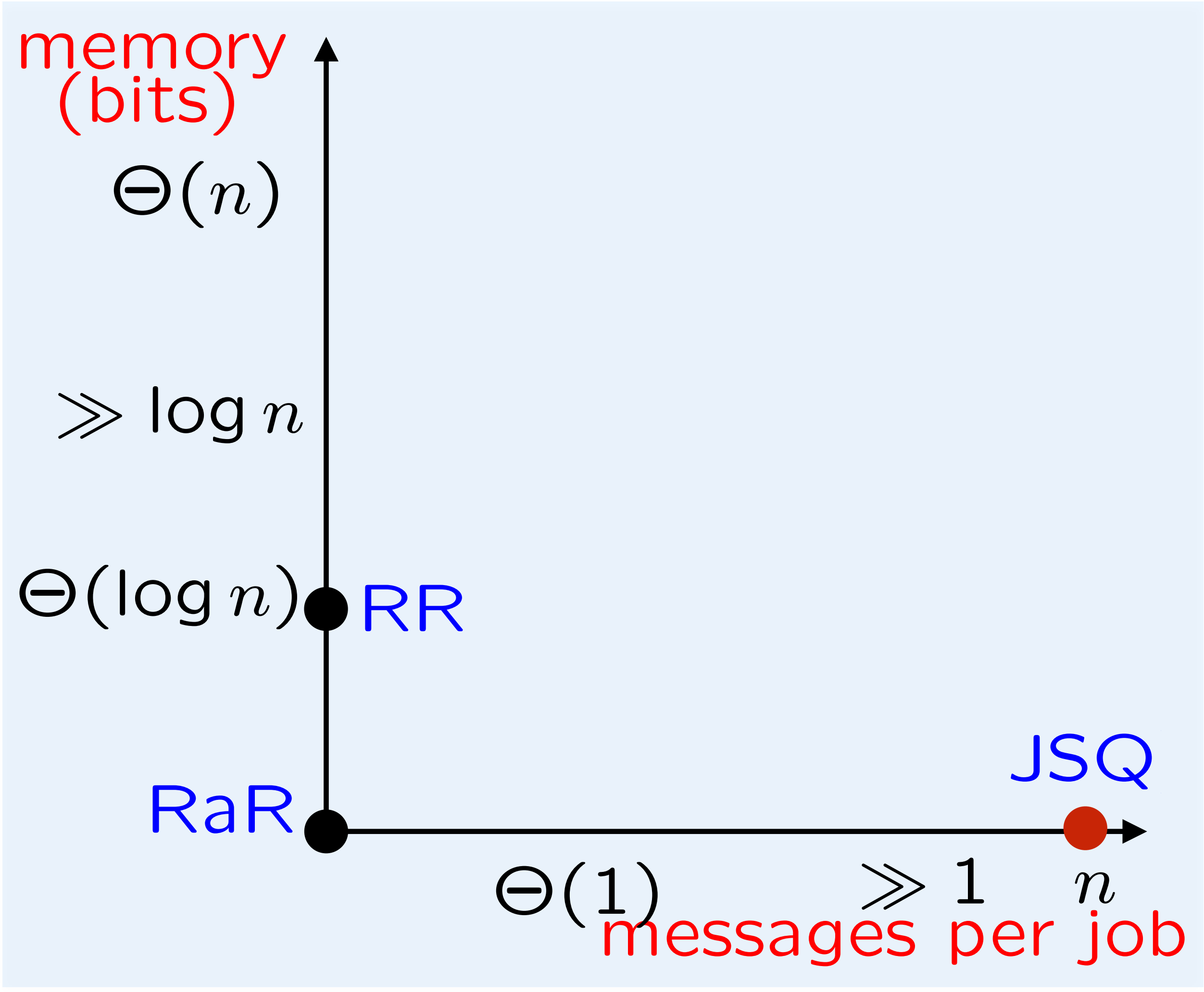

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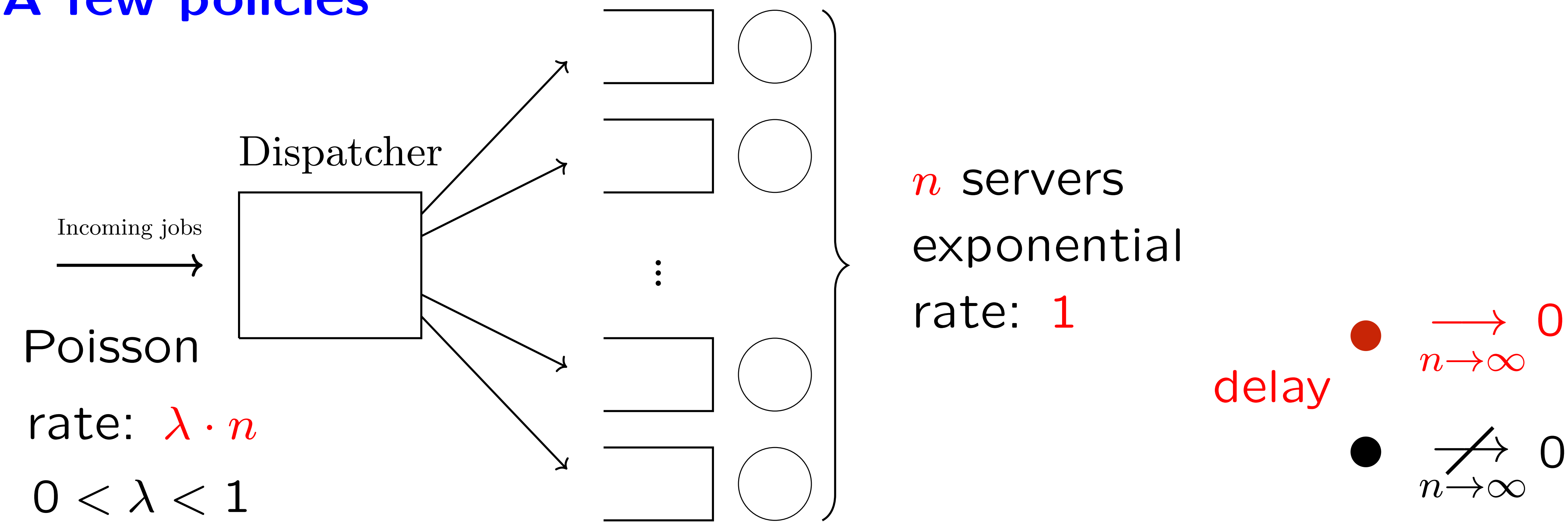

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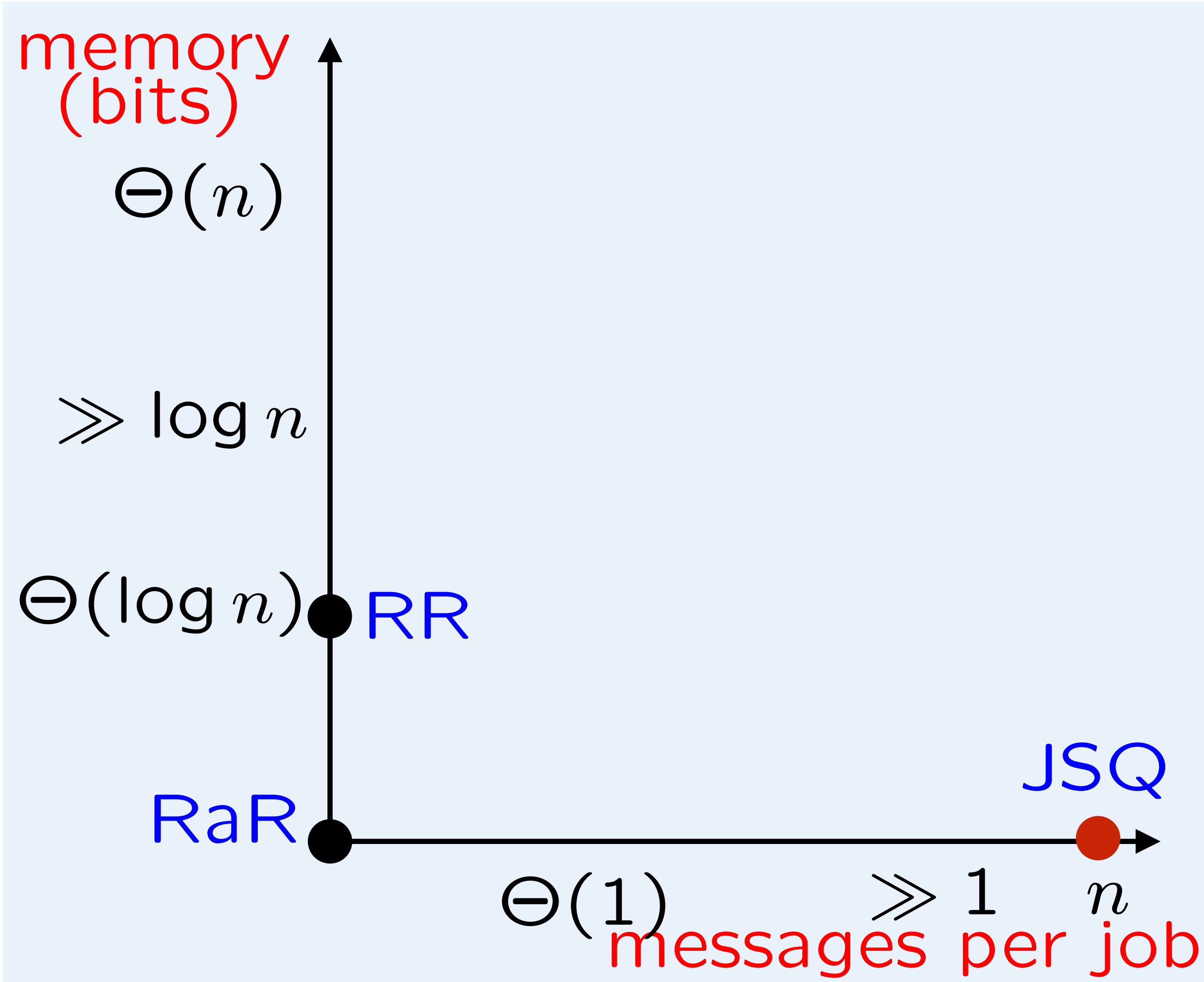

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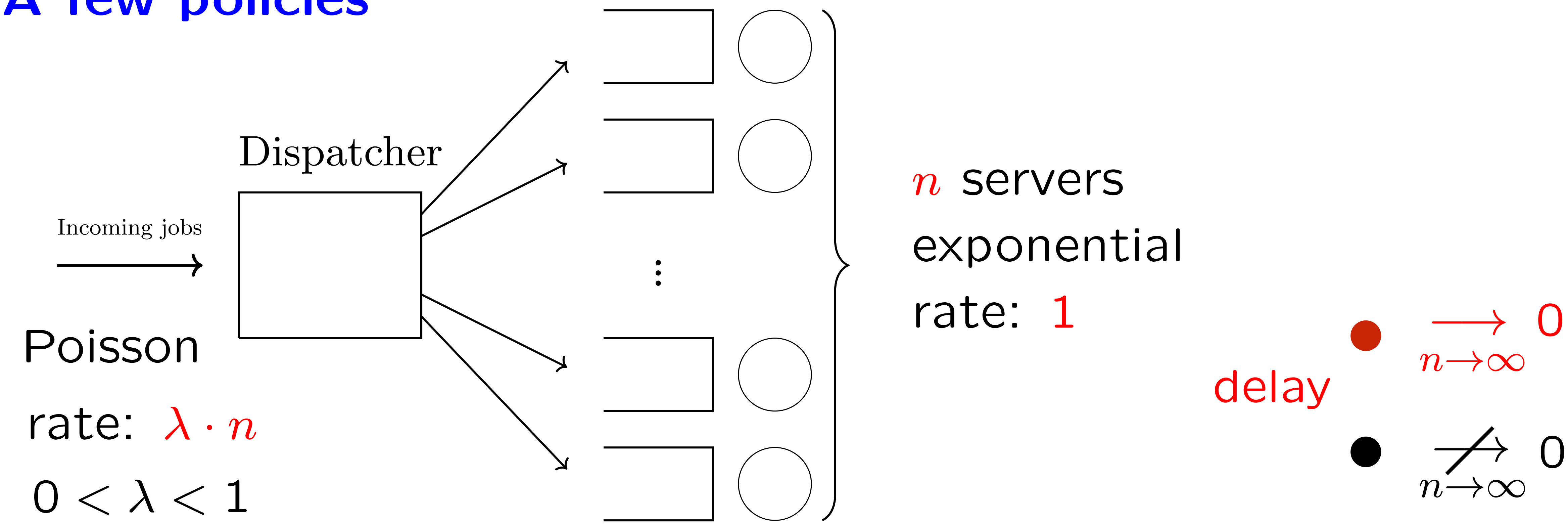

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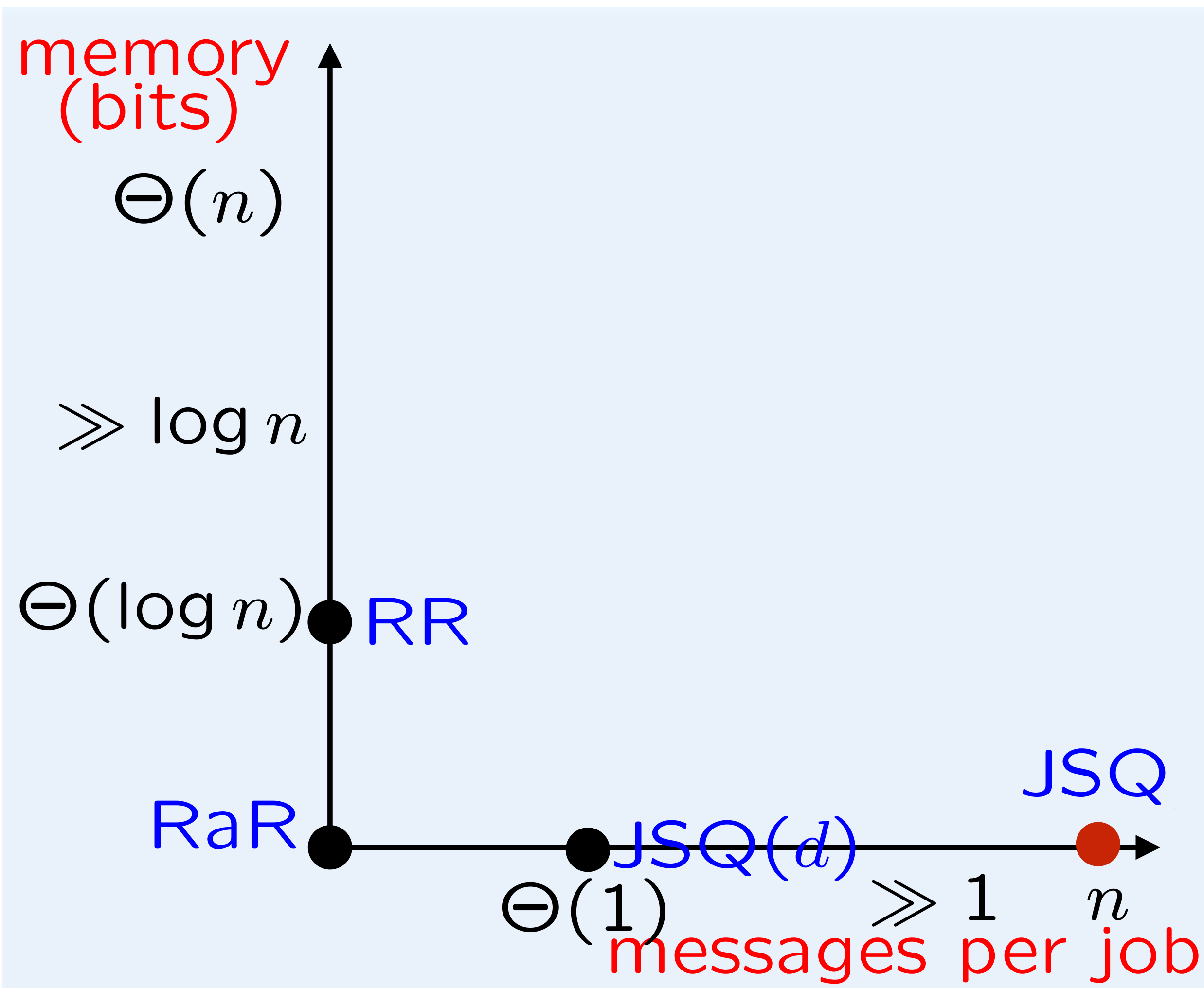

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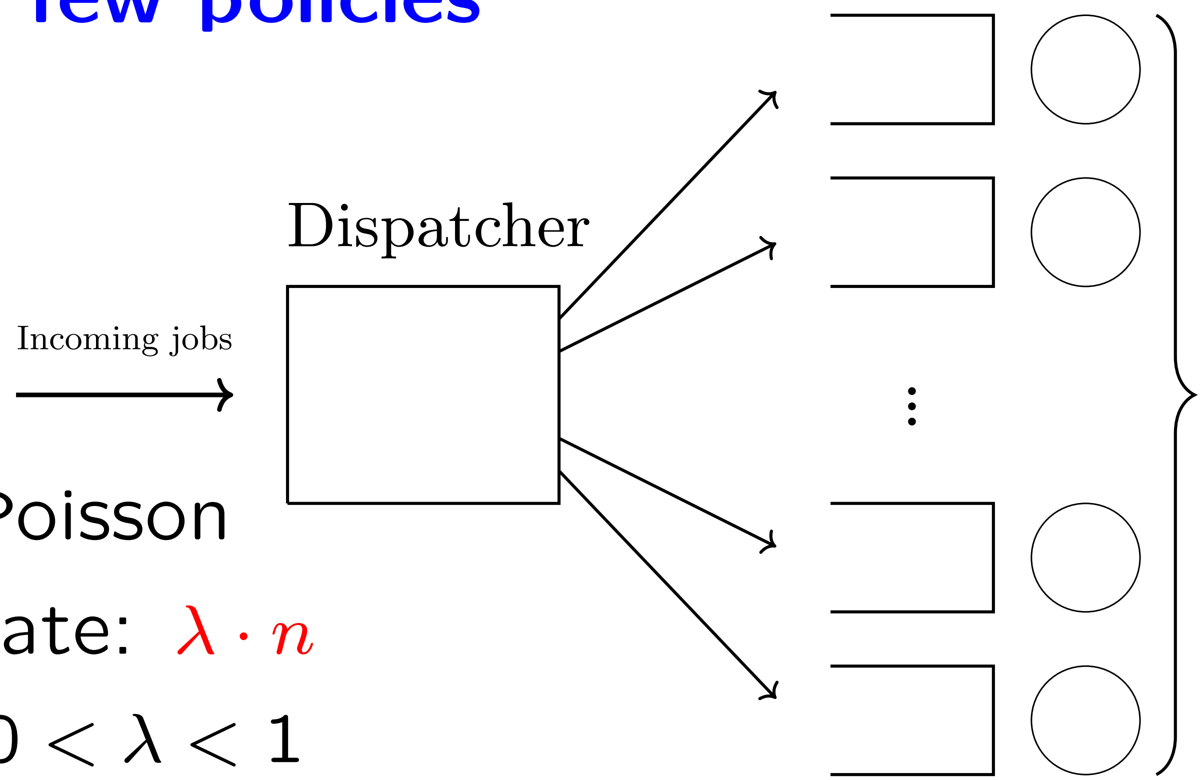

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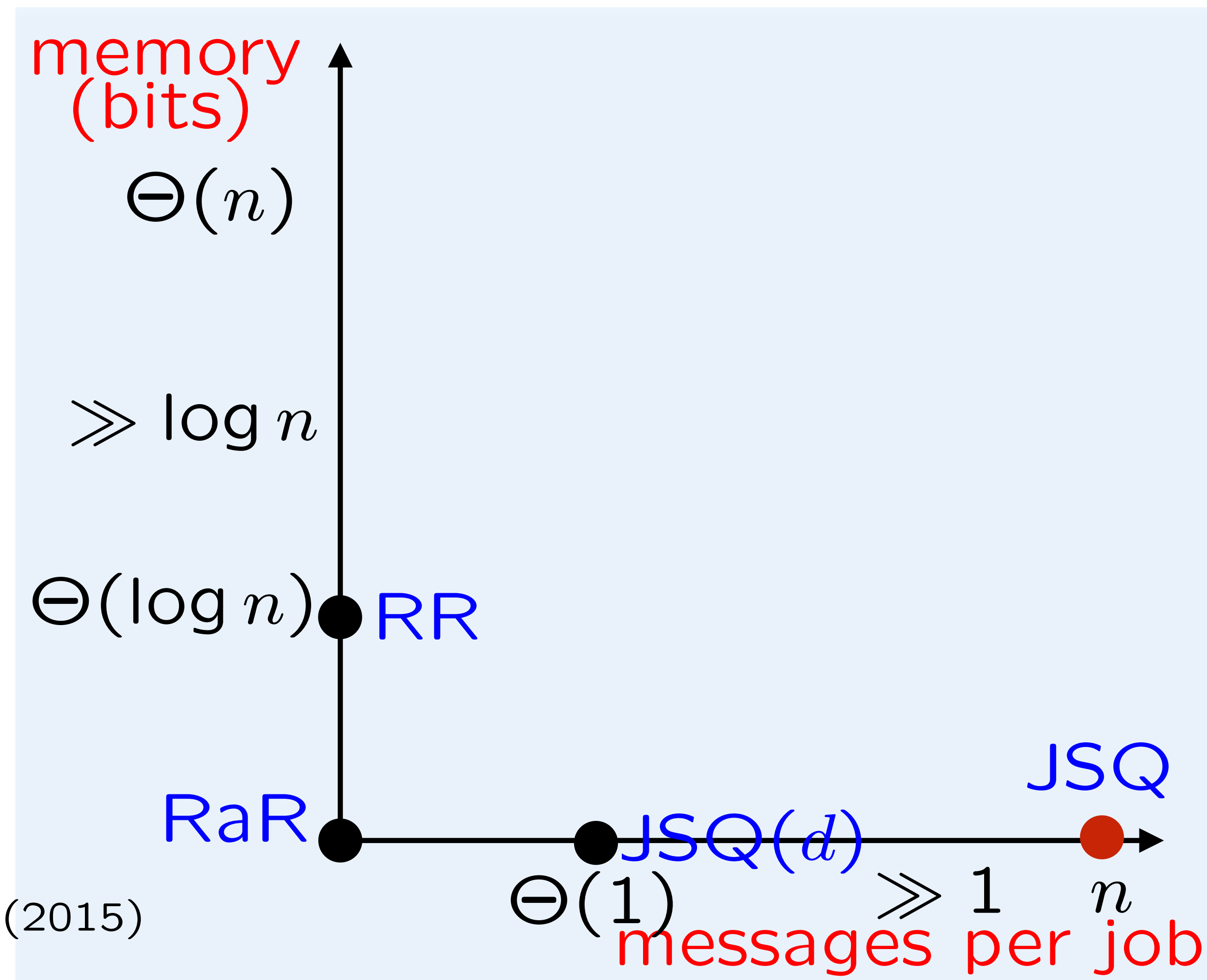
- Idle servers pull jobs (Pull)

Badonnel & Burgess (2008), Y. Lu et al. (2011), Stolyar (2015)



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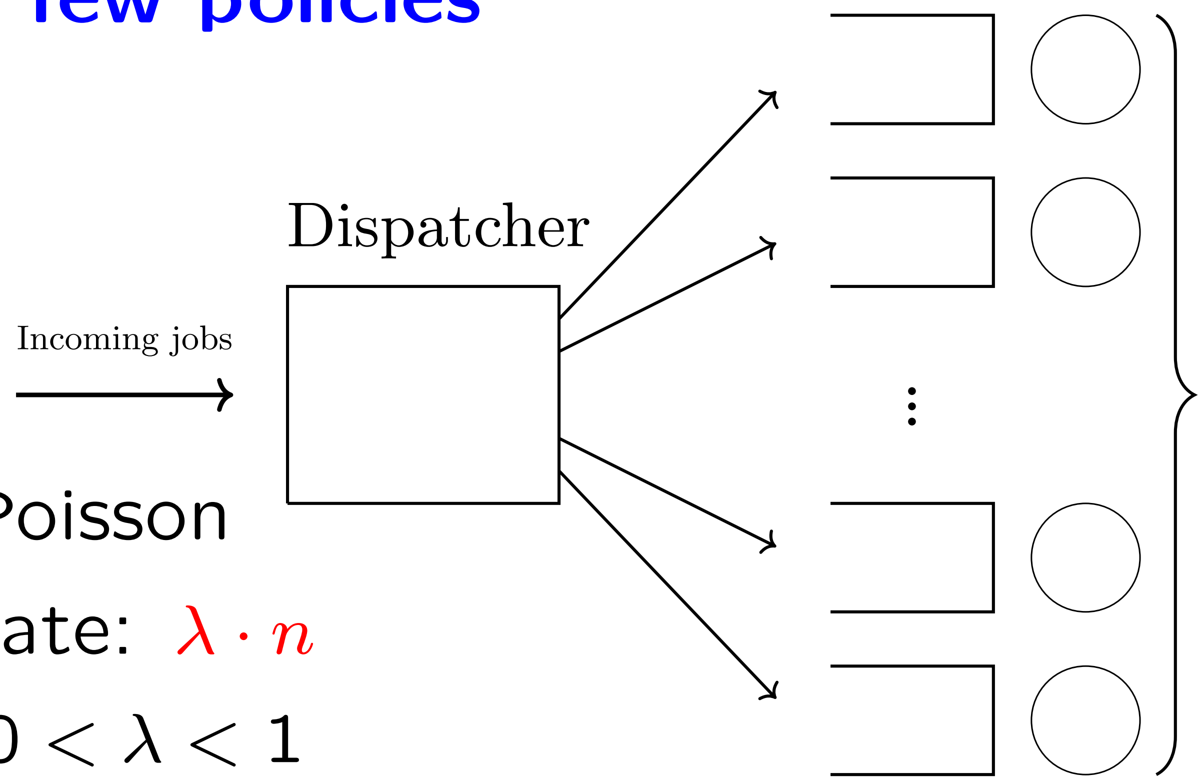

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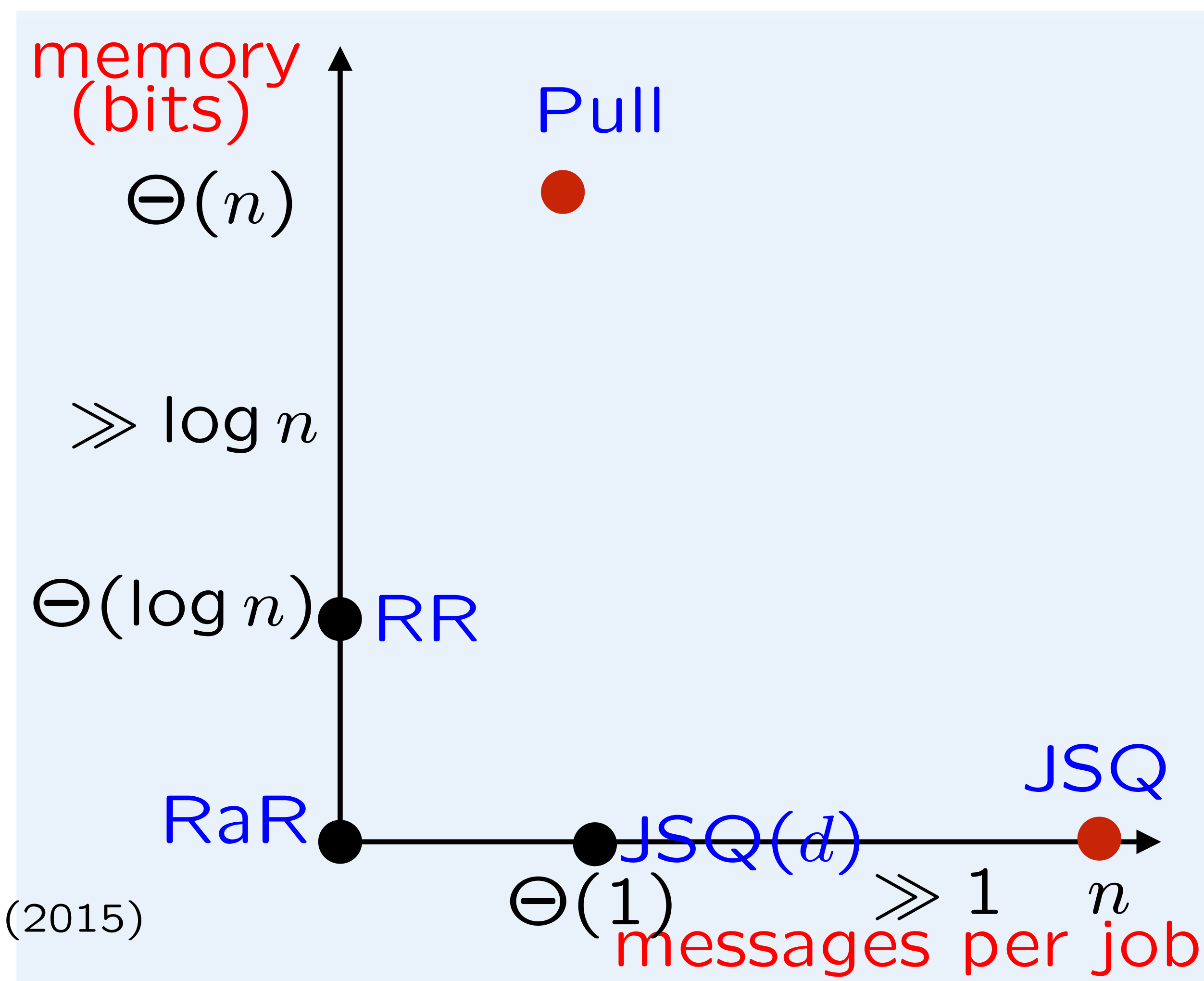
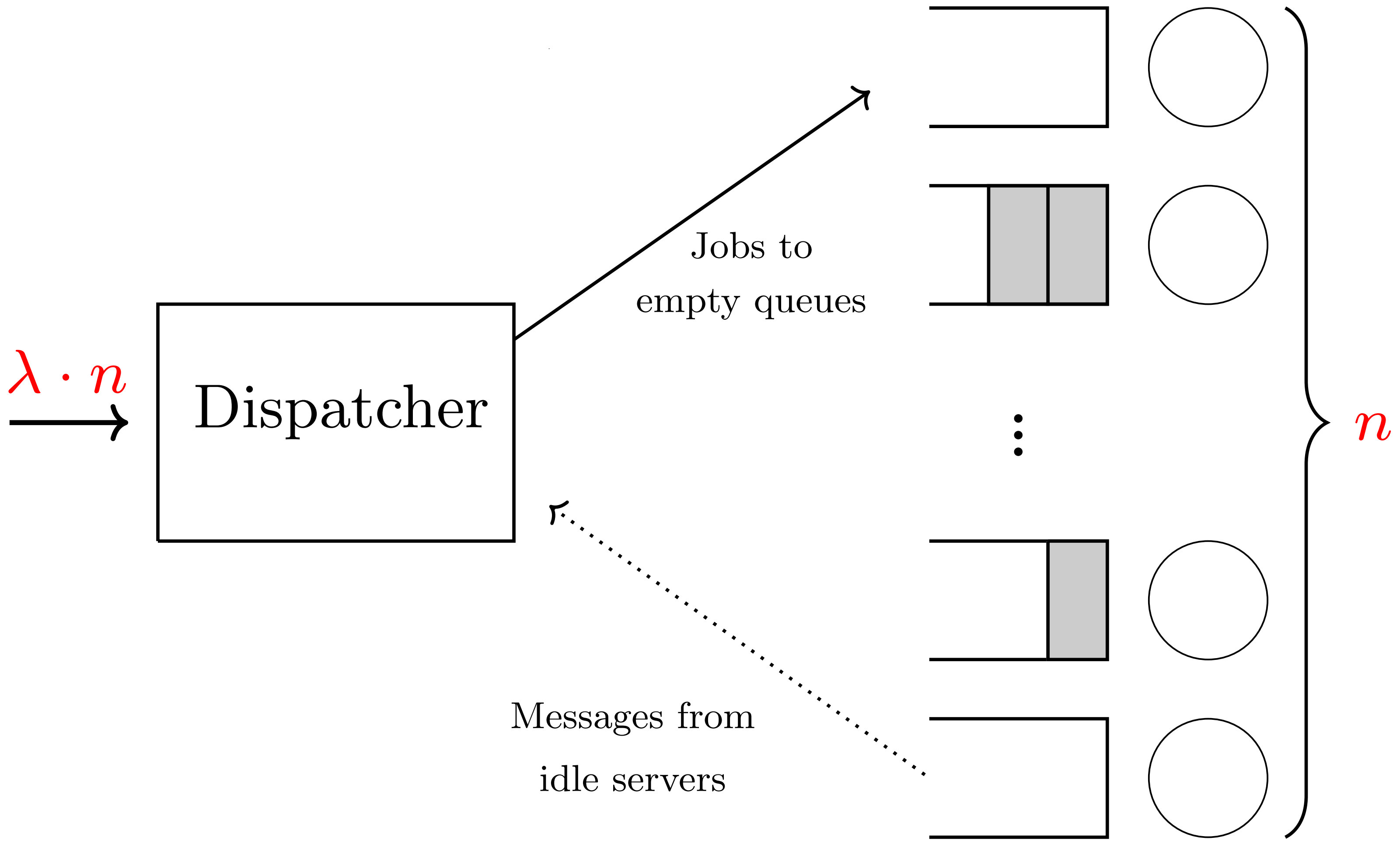
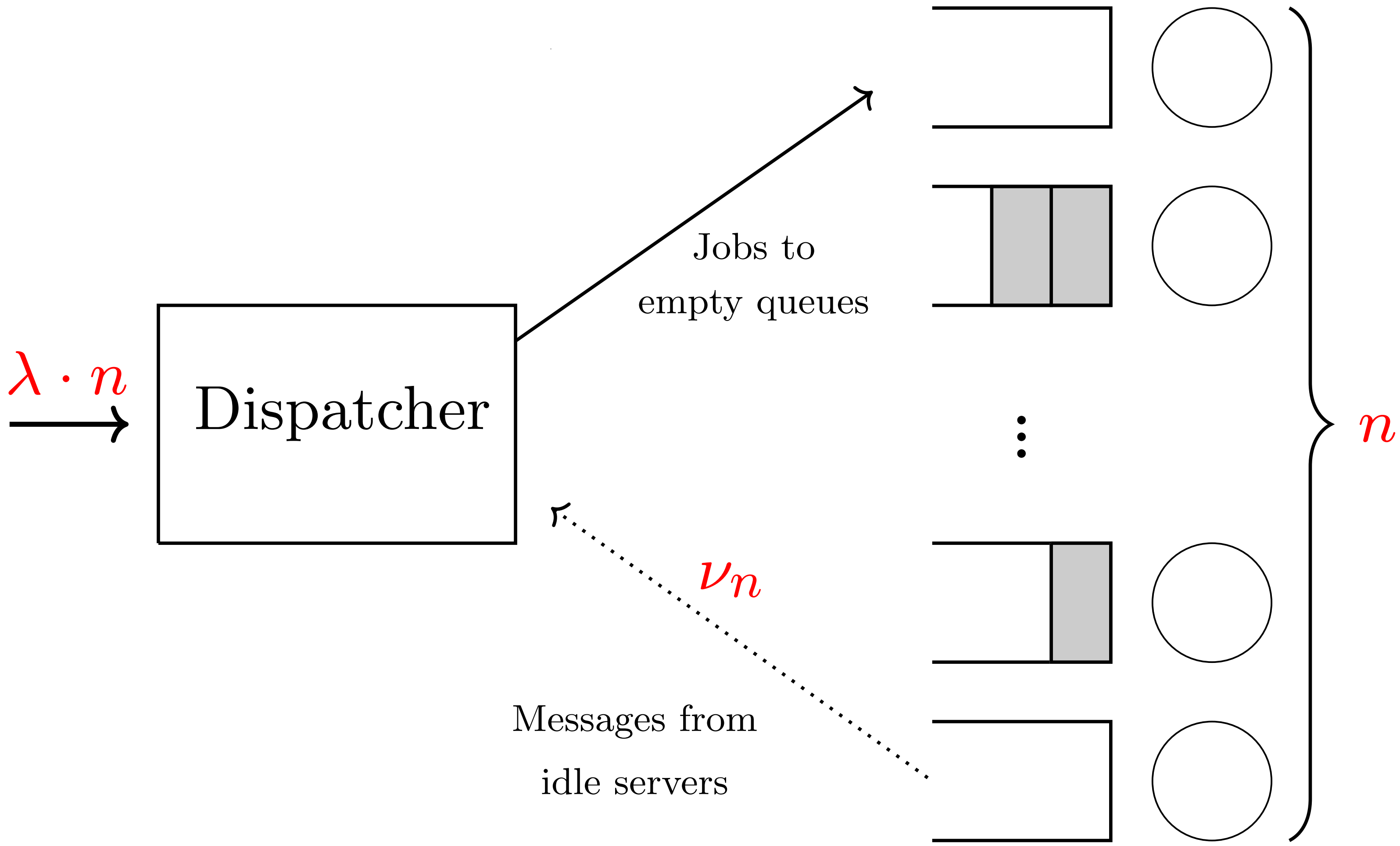


Our policy



$\lambda \cdot n$
Dispatcher
Jobs to
empty queues
Messages from
idle servers
n

Our policy



- idle processors send messages at rate ν_n

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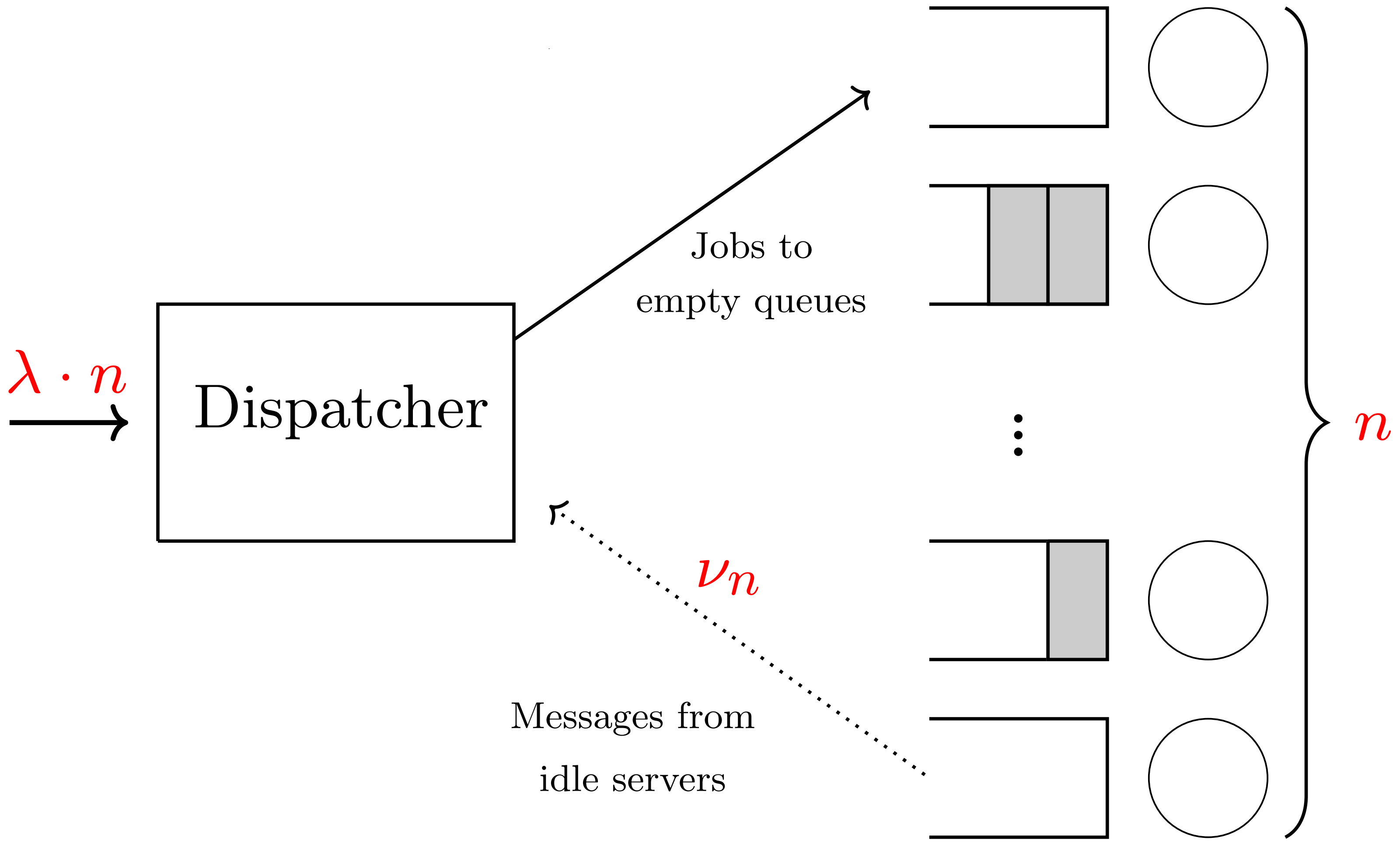

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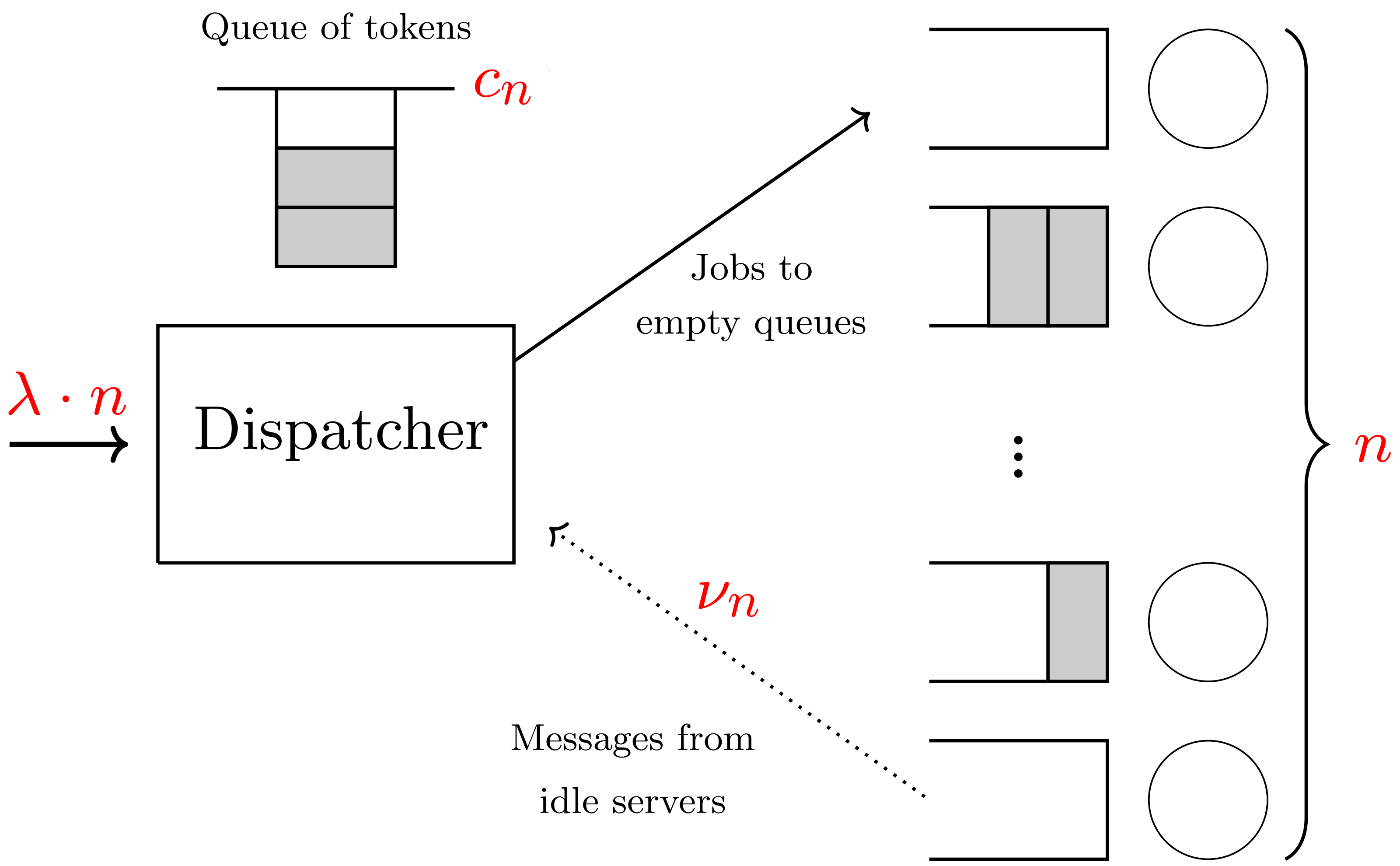

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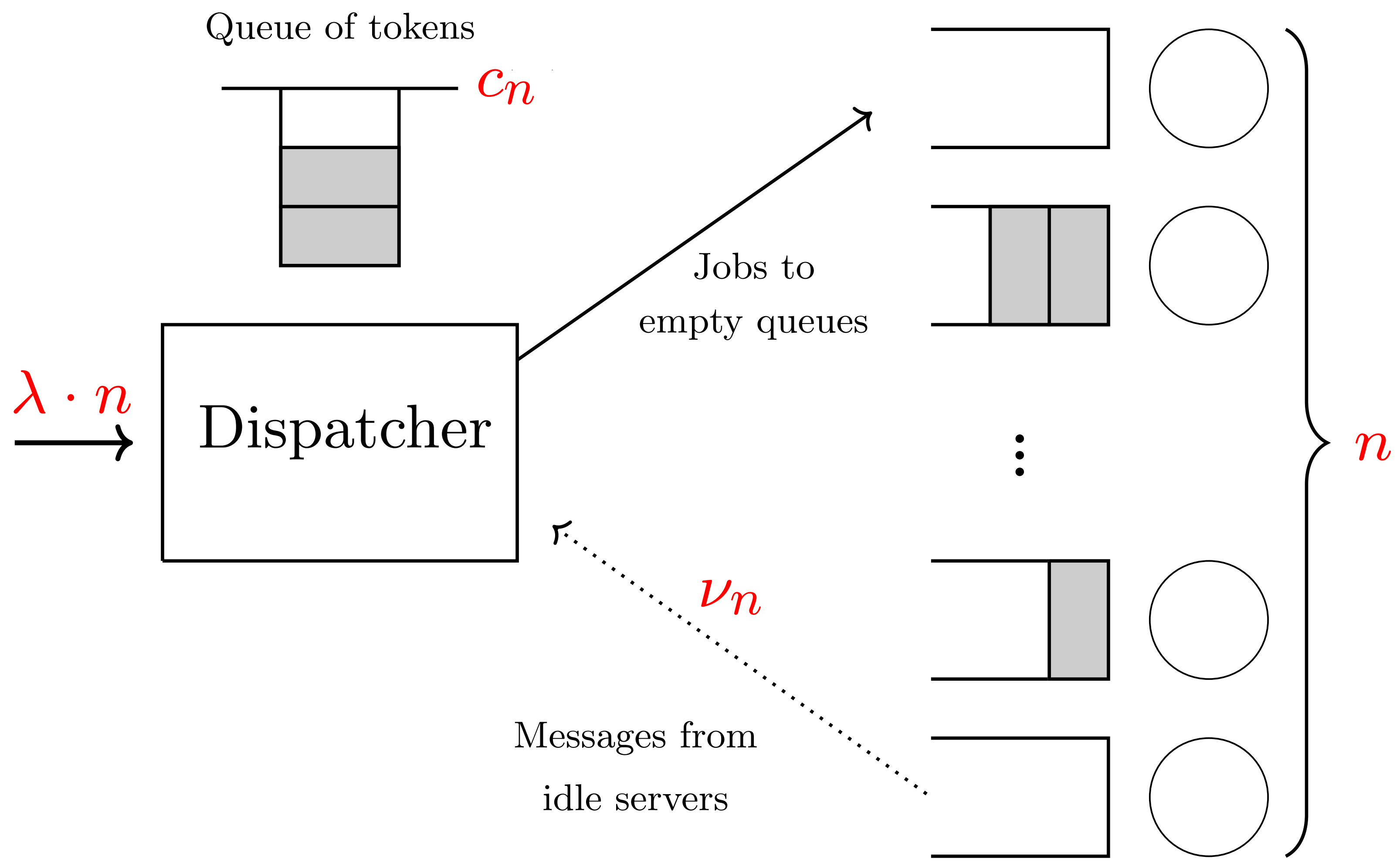

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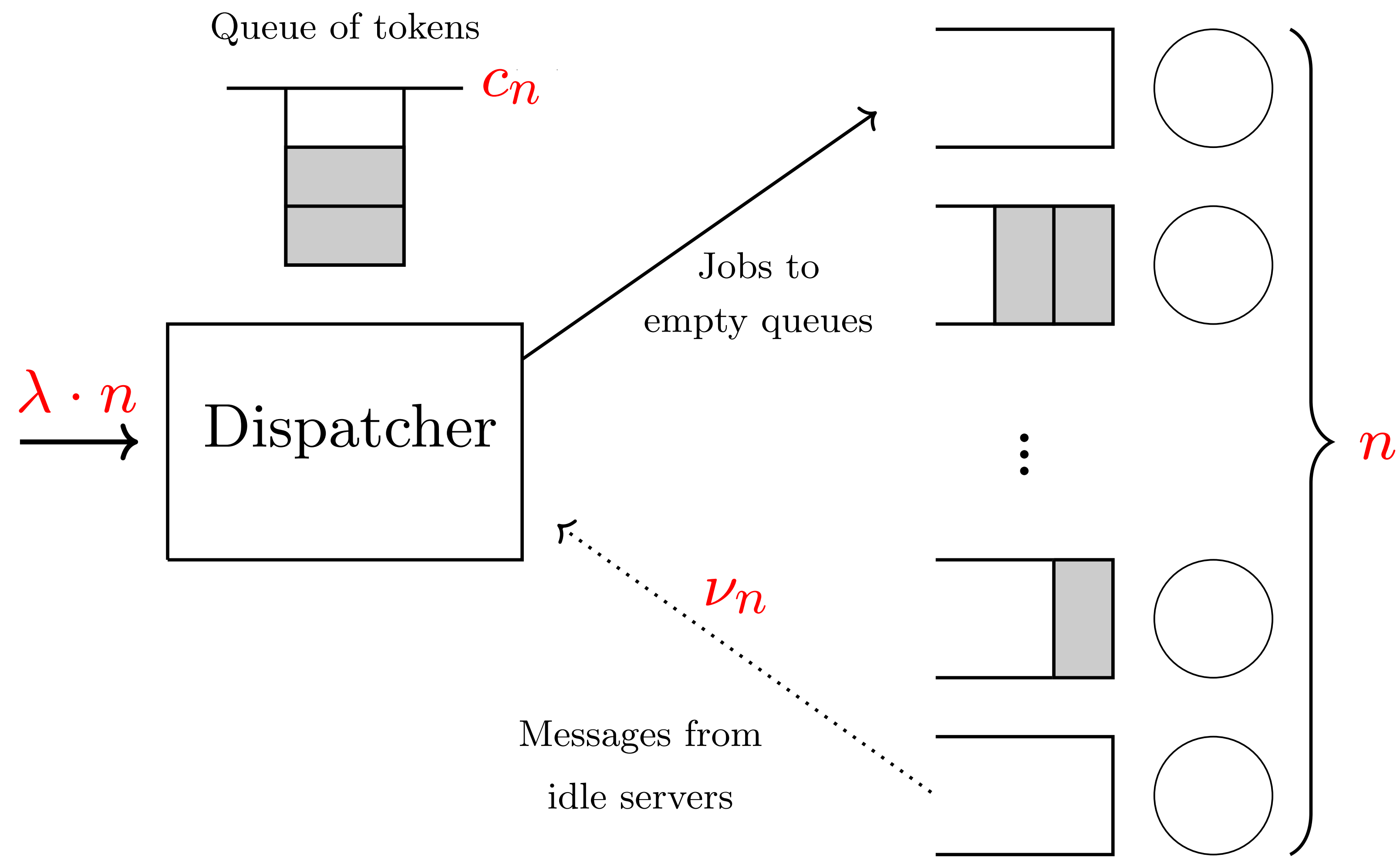

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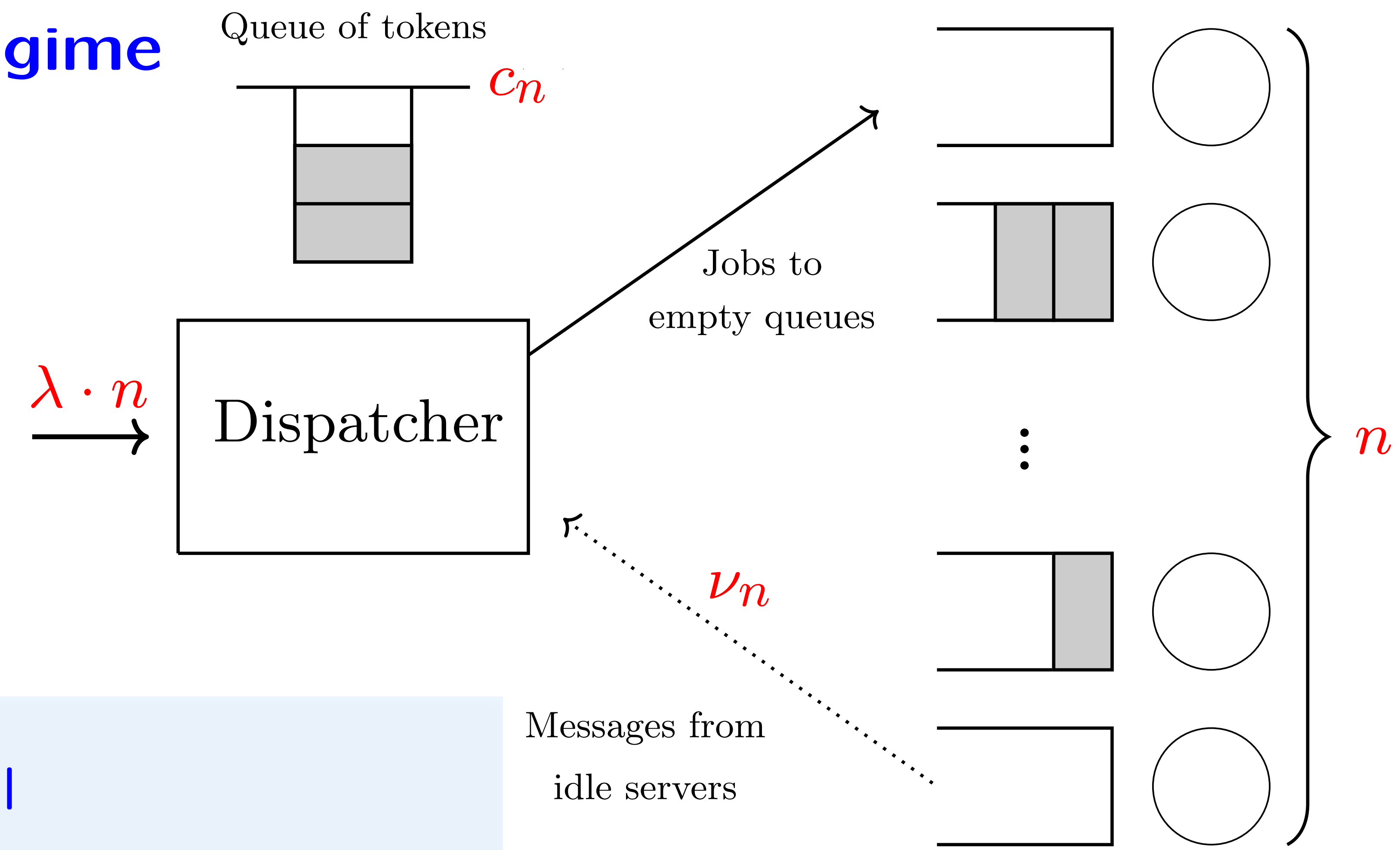
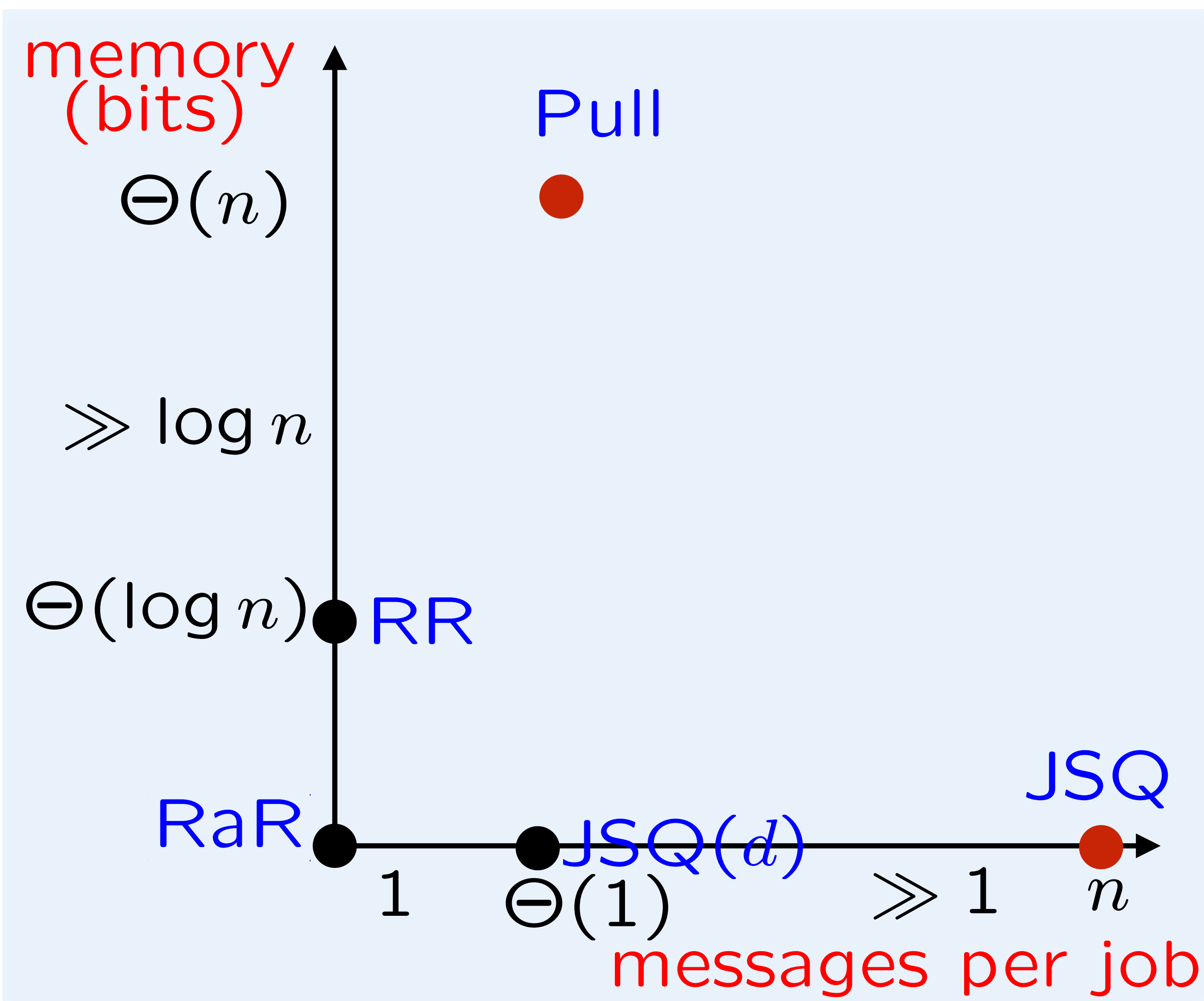
- when job arrives:
  - send to server in memory
  - if empty memory, send to random server

High message rate regime





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- $\nu_n \to \infty$
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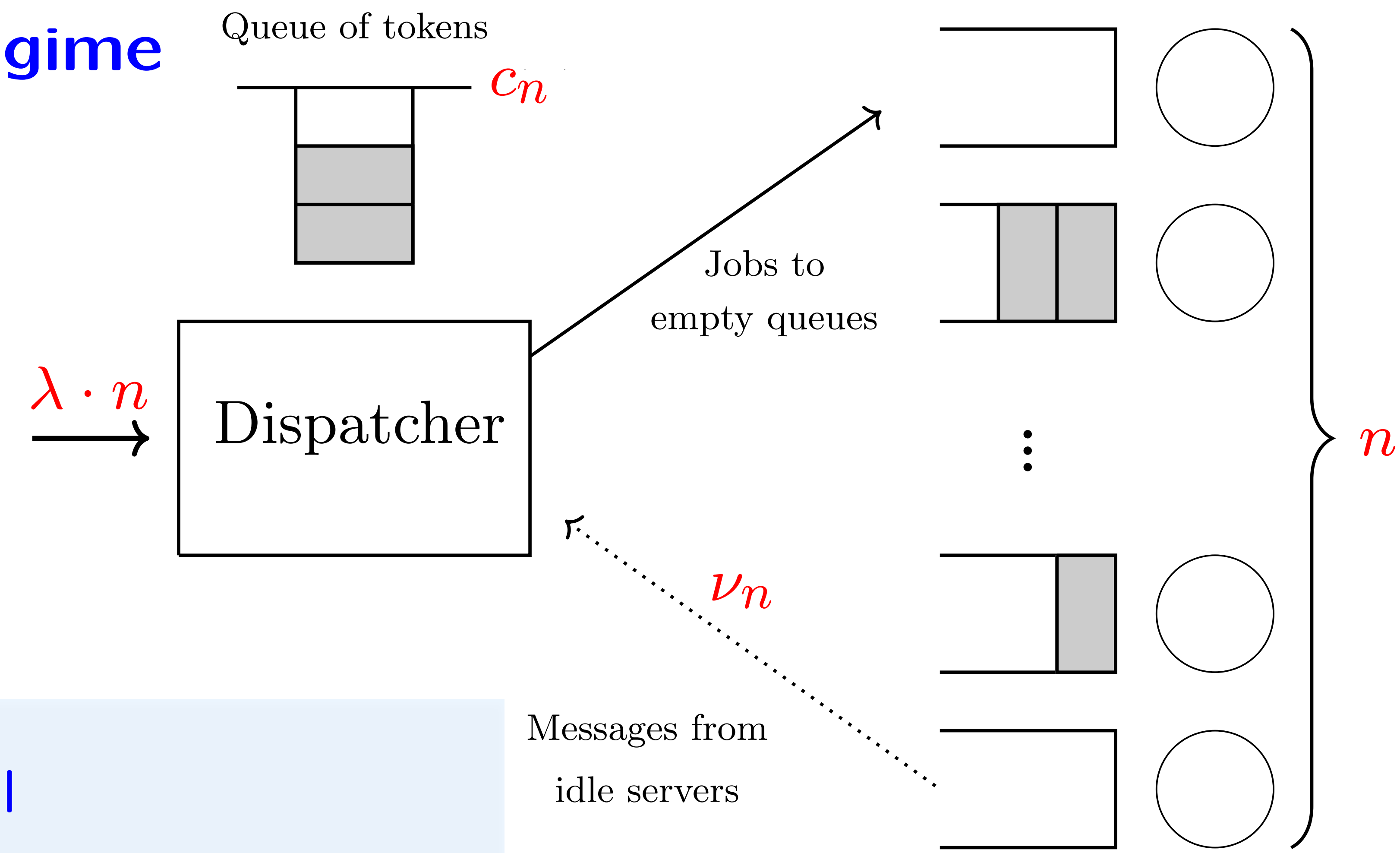

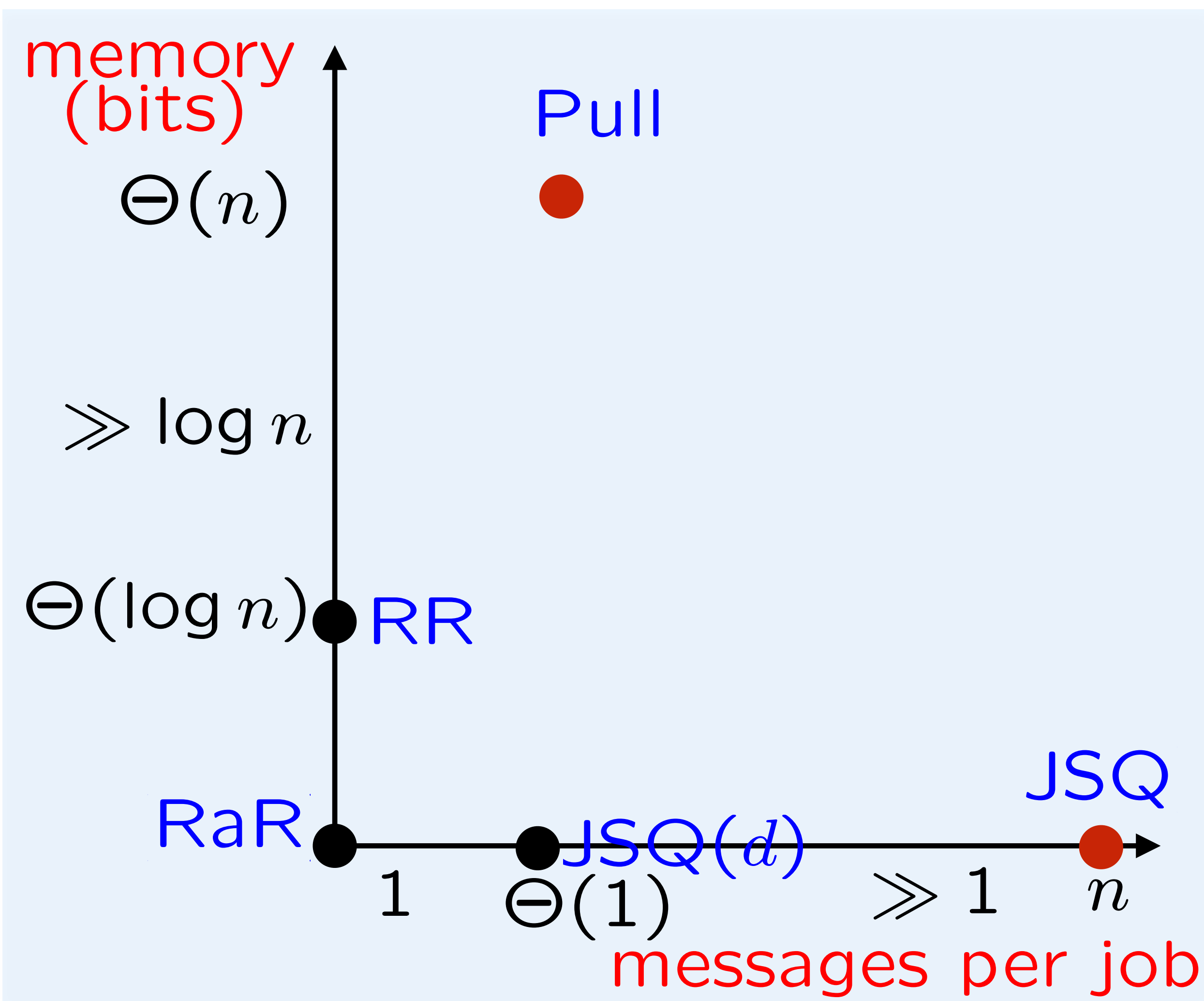

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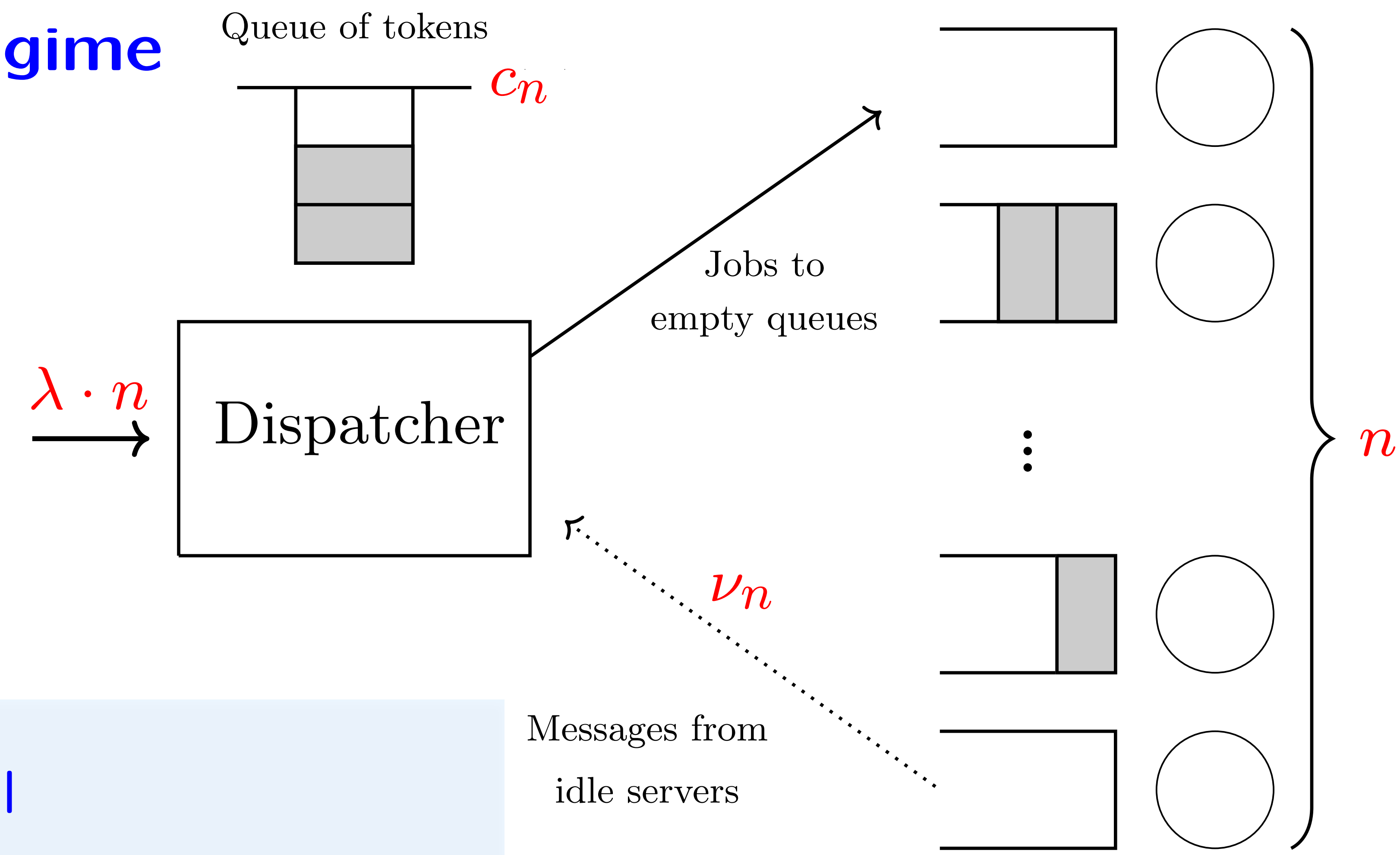

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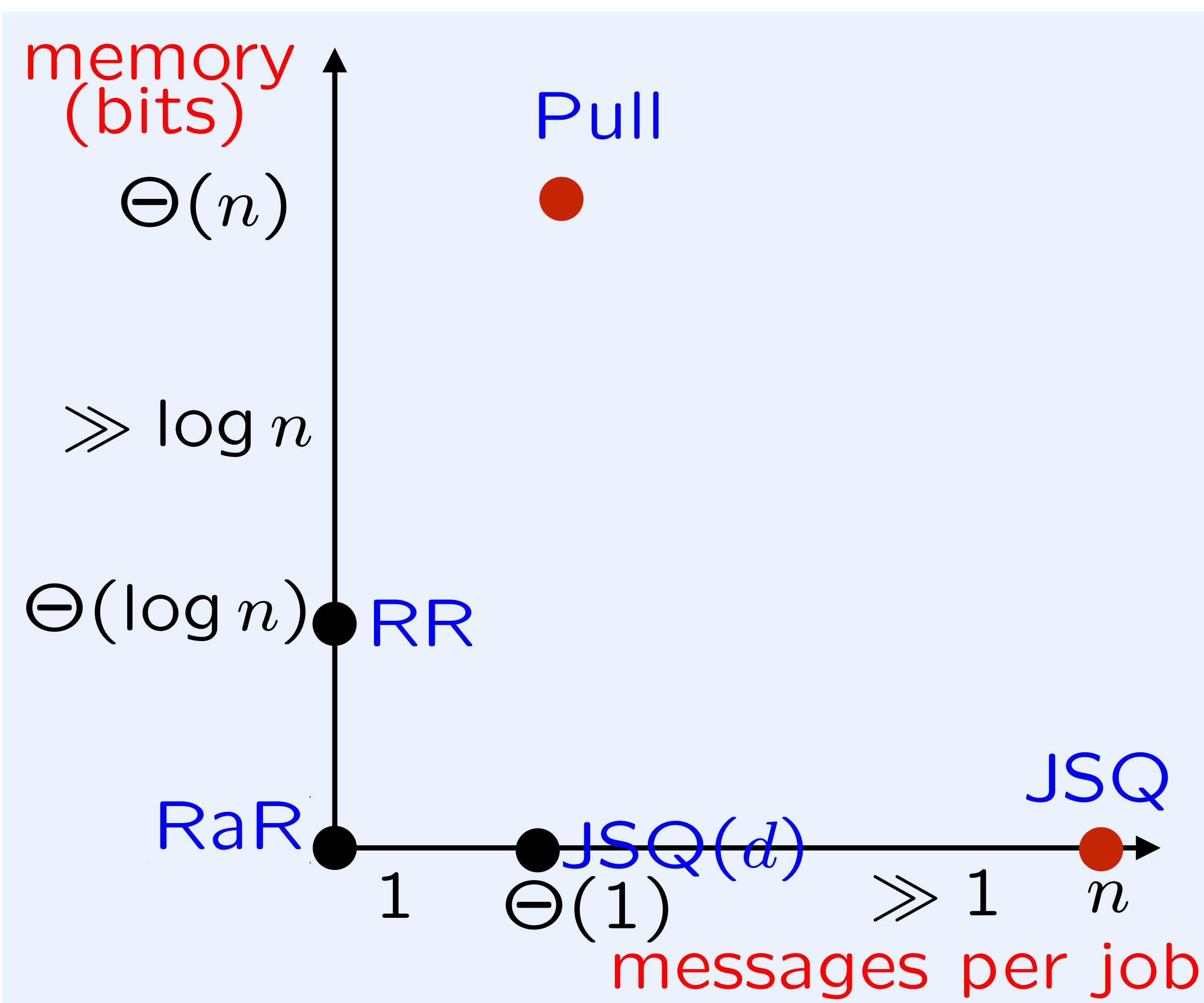

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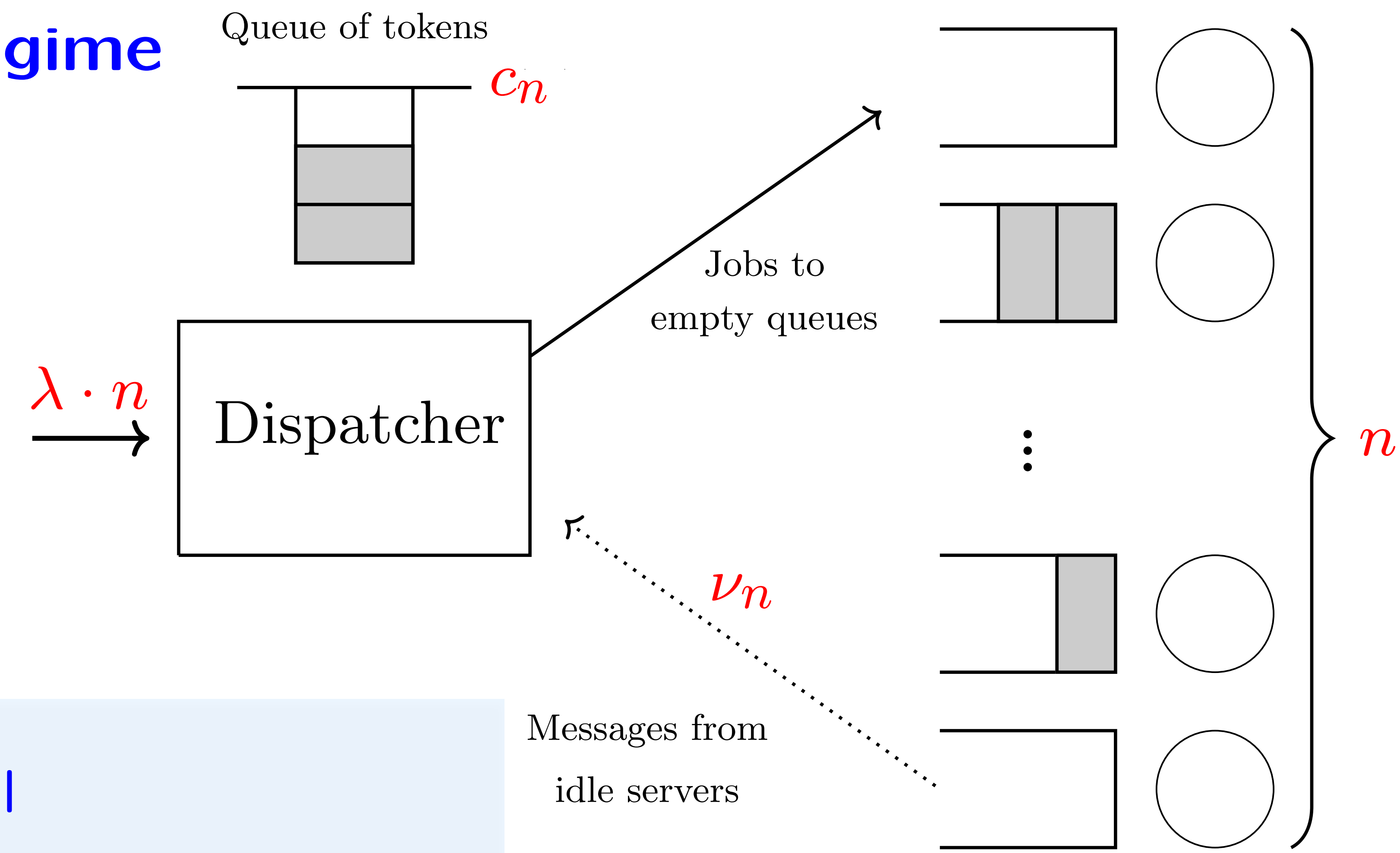
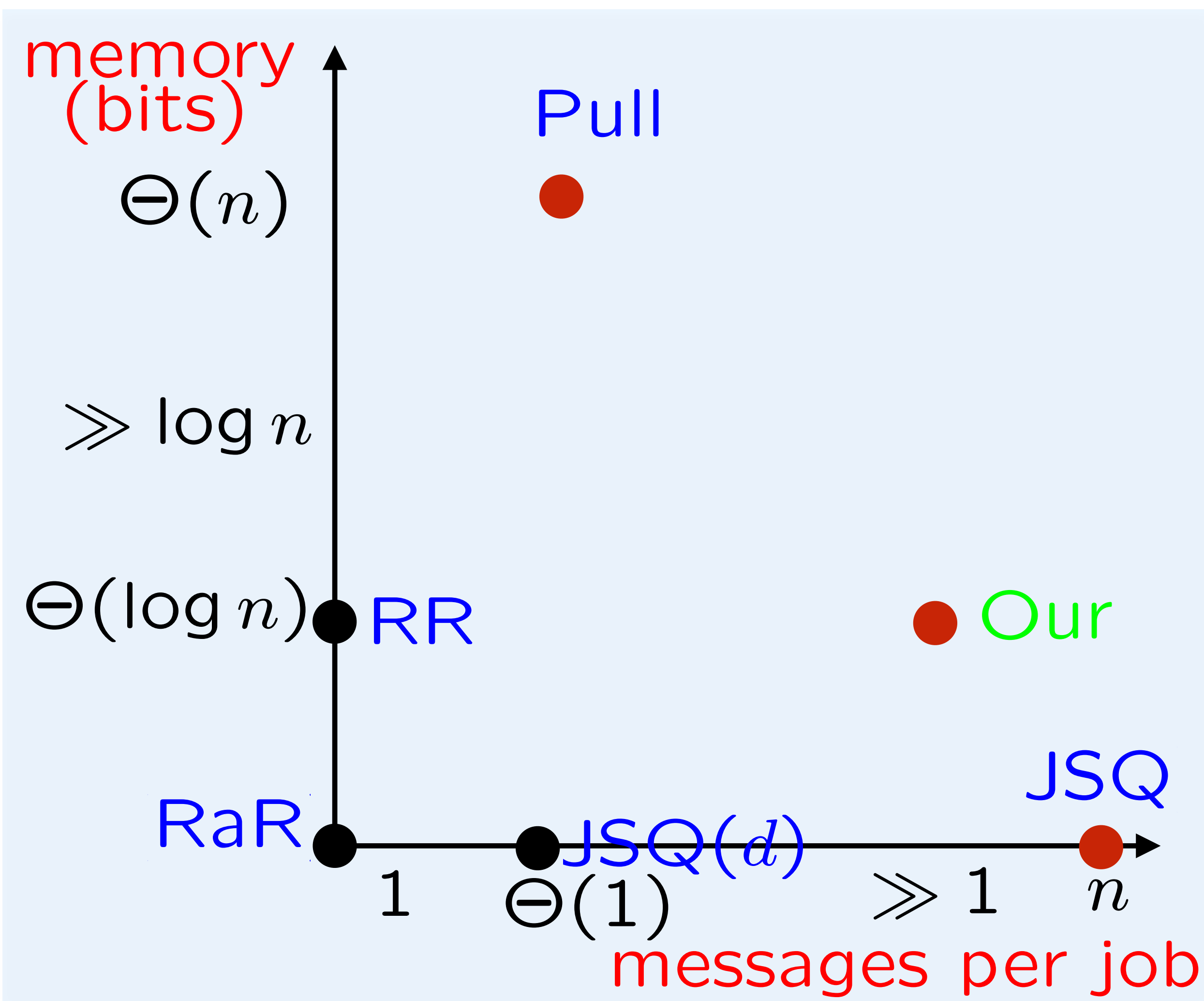
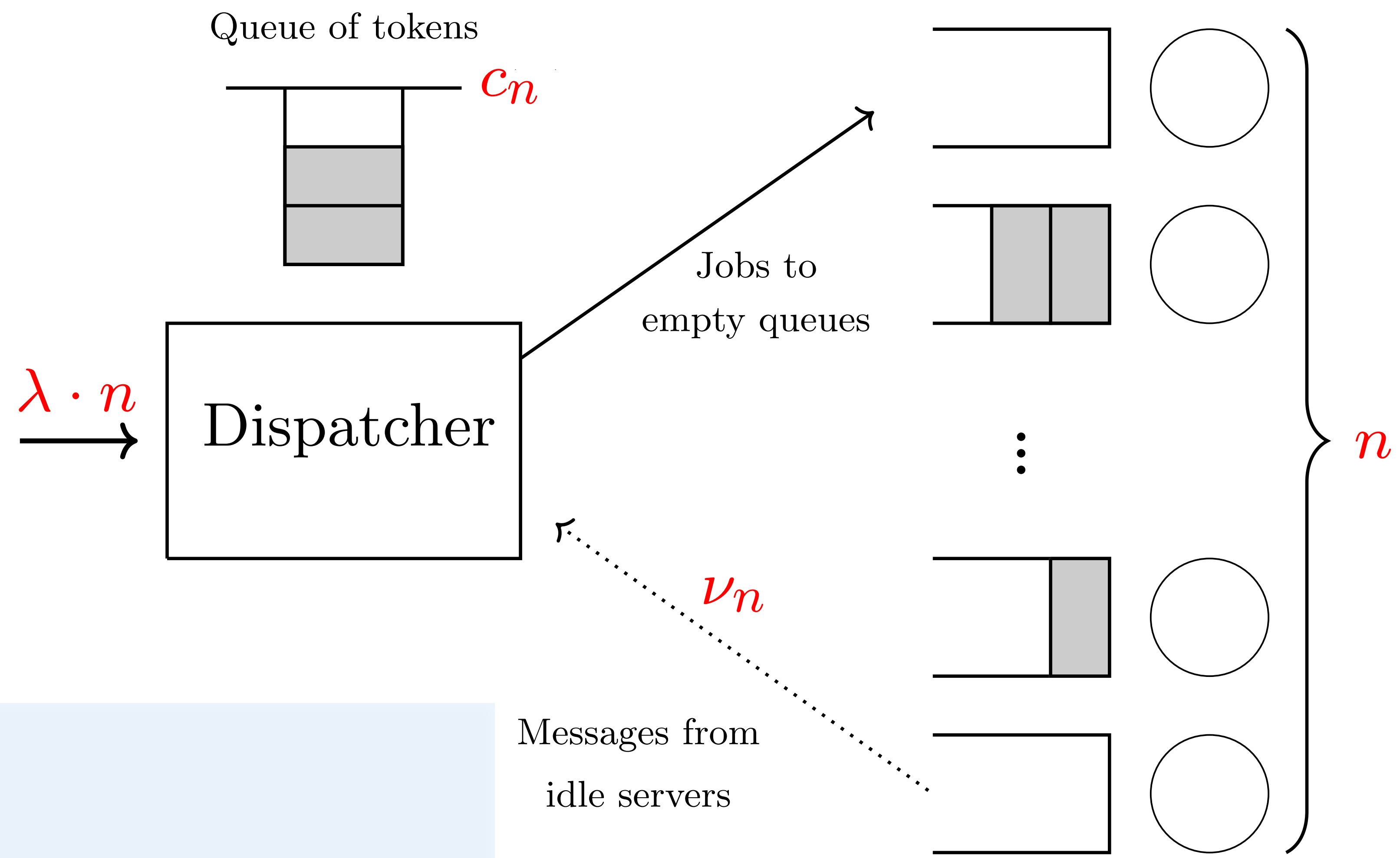
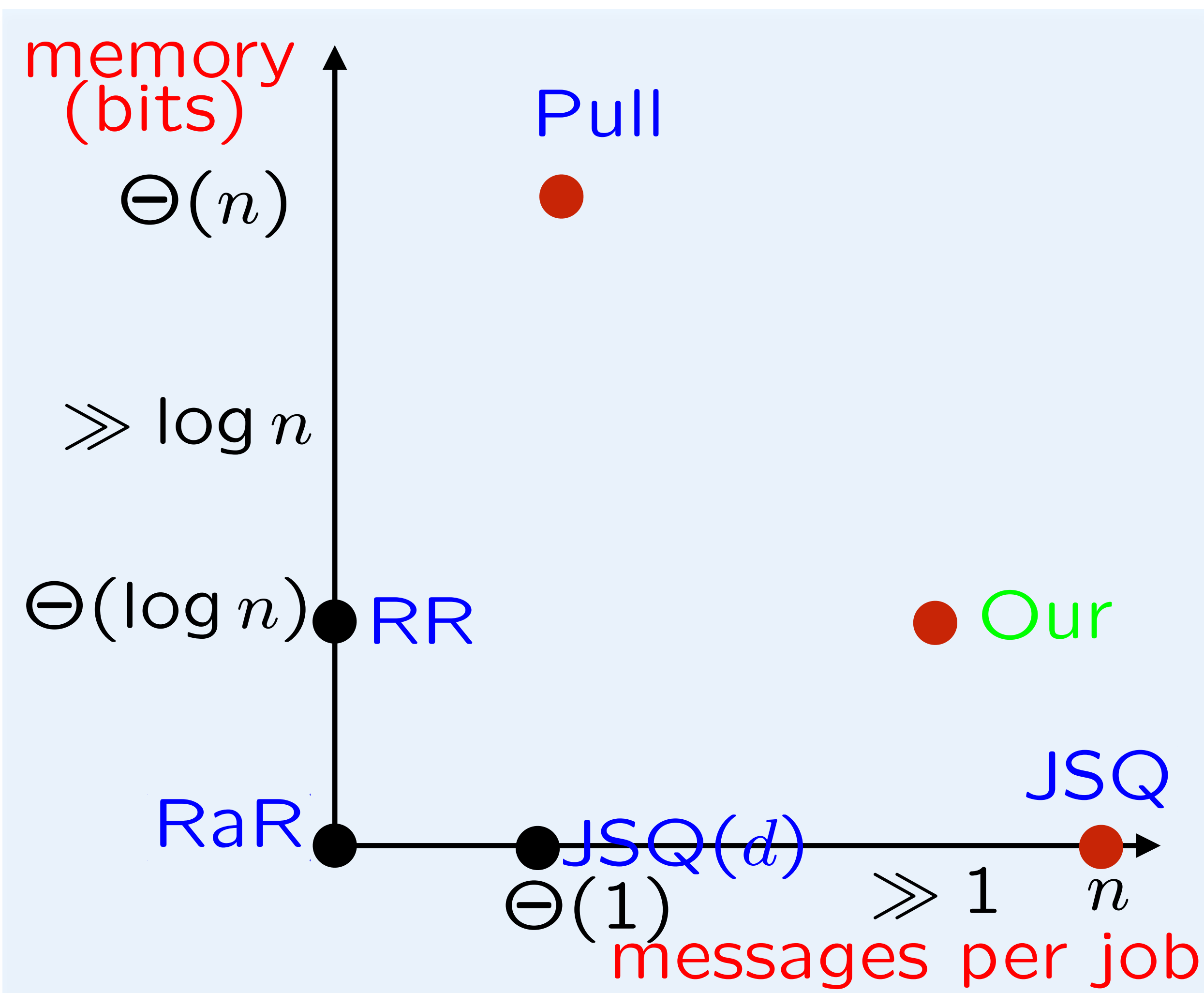




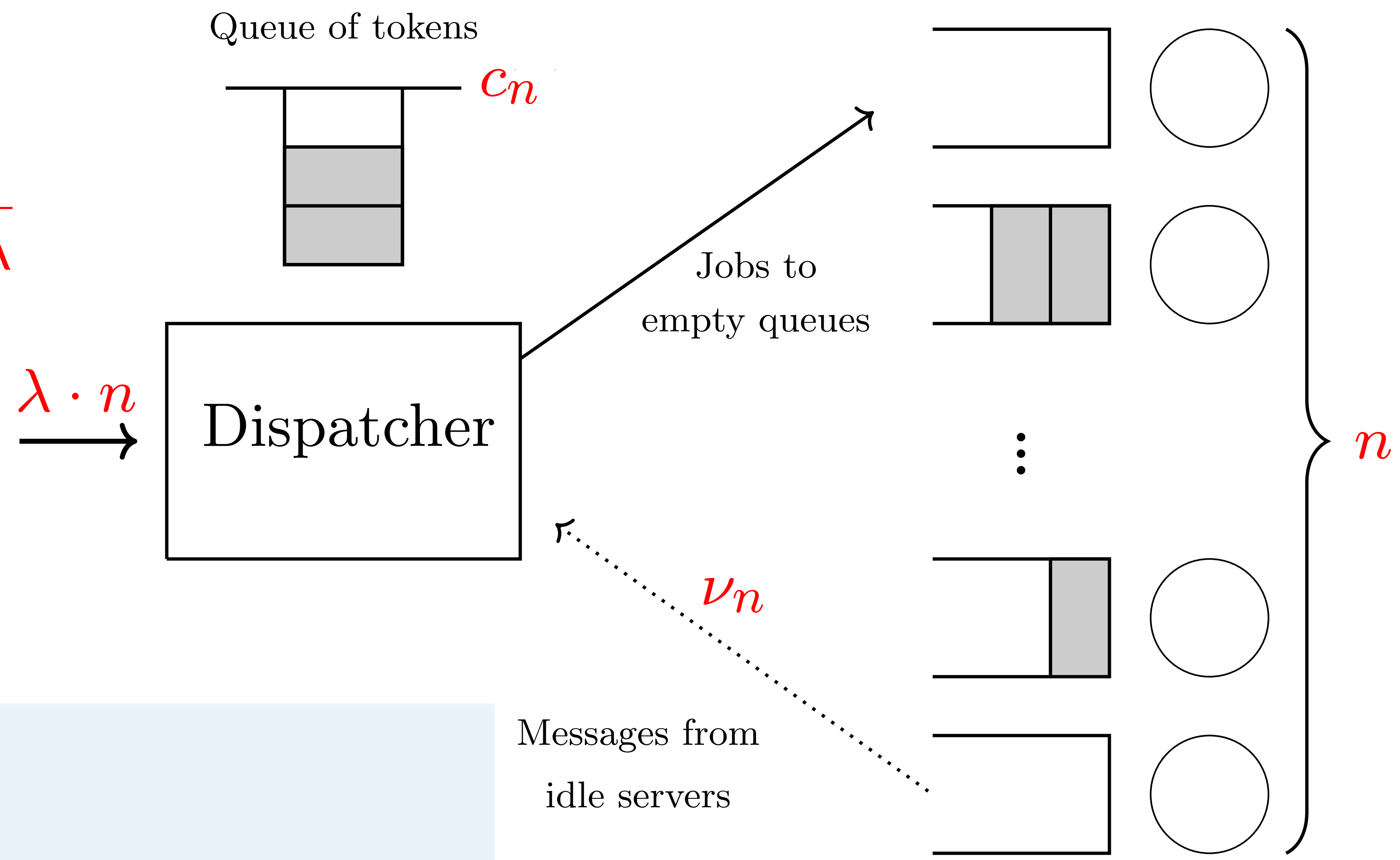
High memory regime

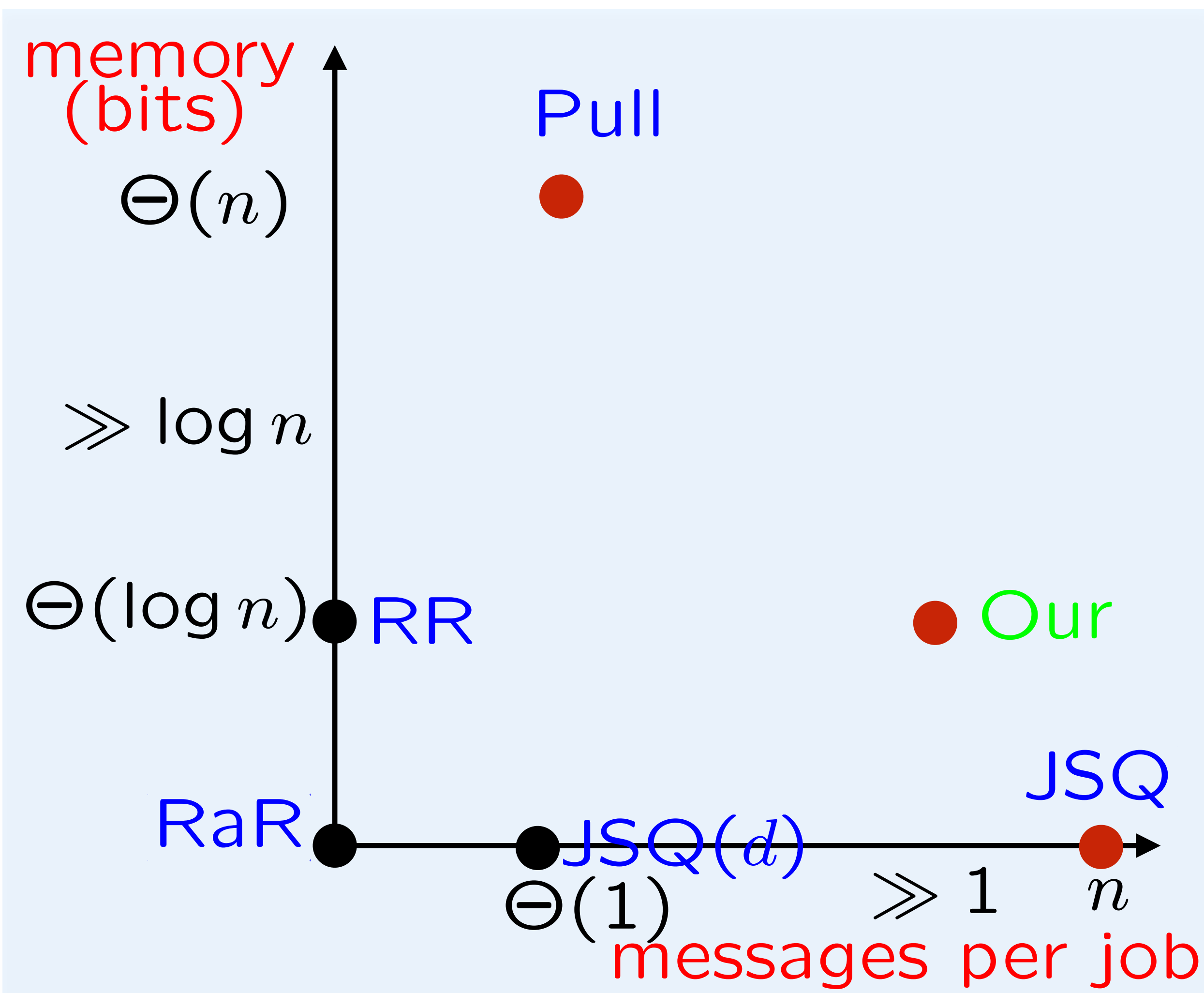




High memory regime

- $\nu_n = \text{constant} \geq \dfrac{\lambda}{1-\lambda}$





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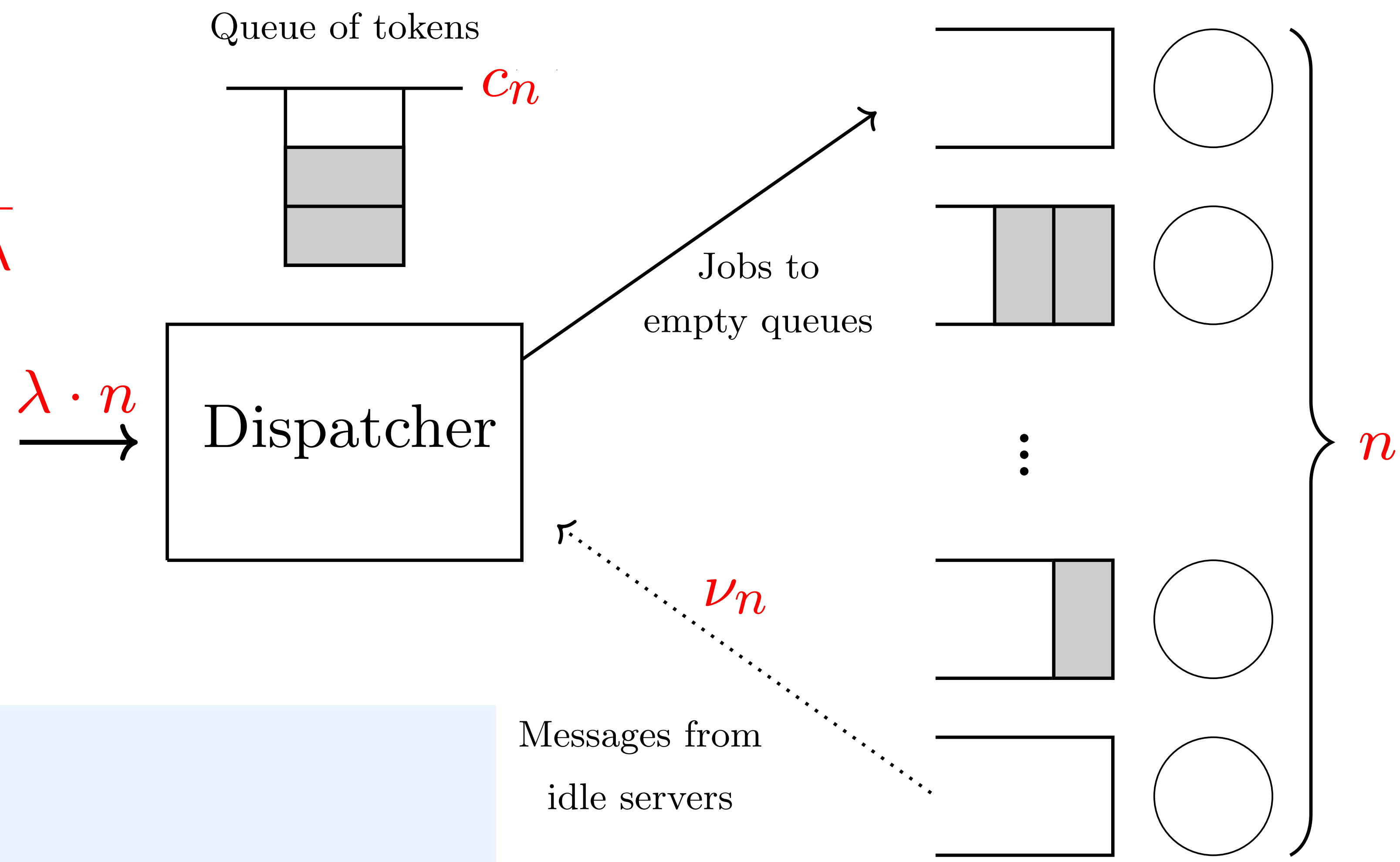

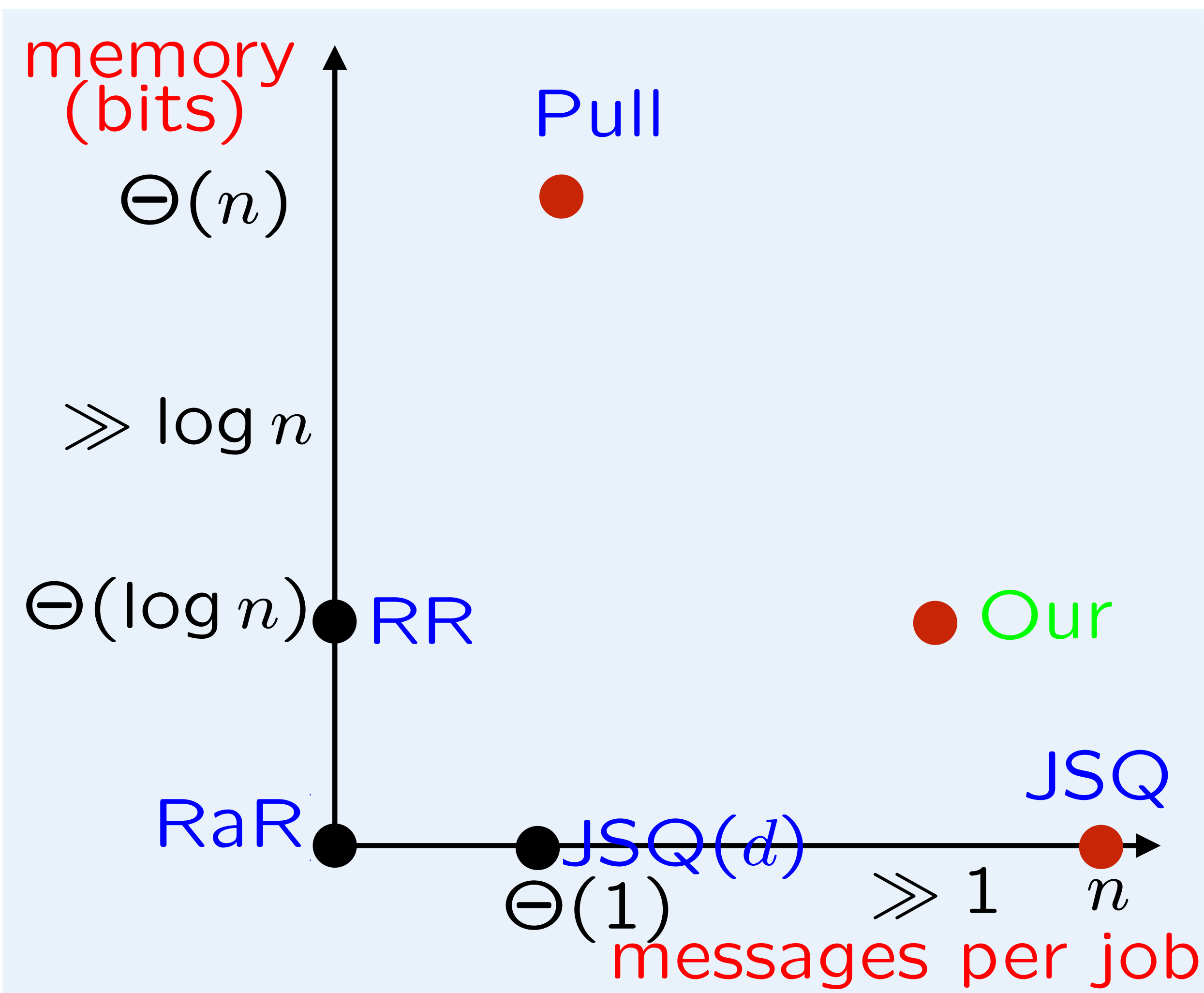

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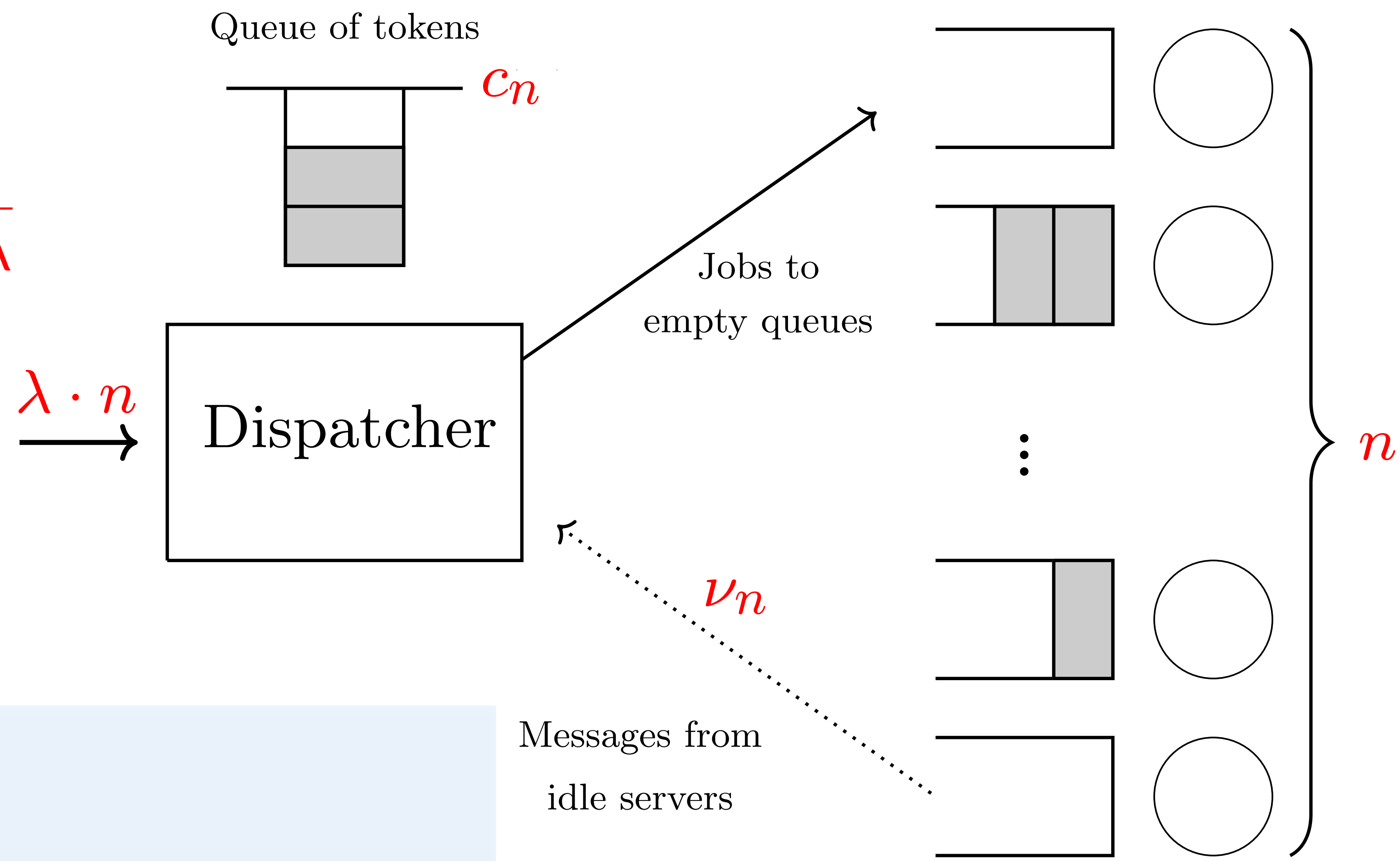

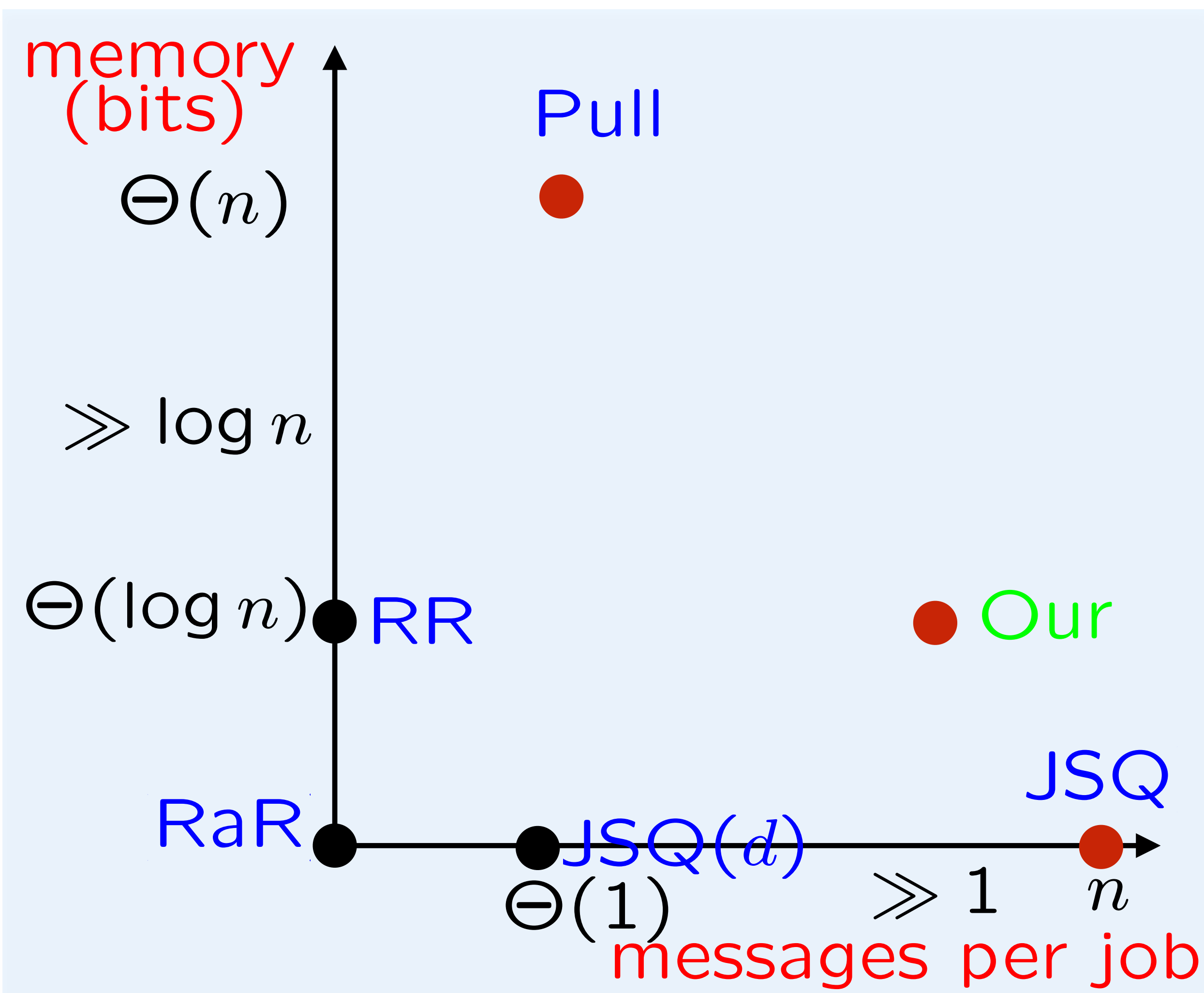

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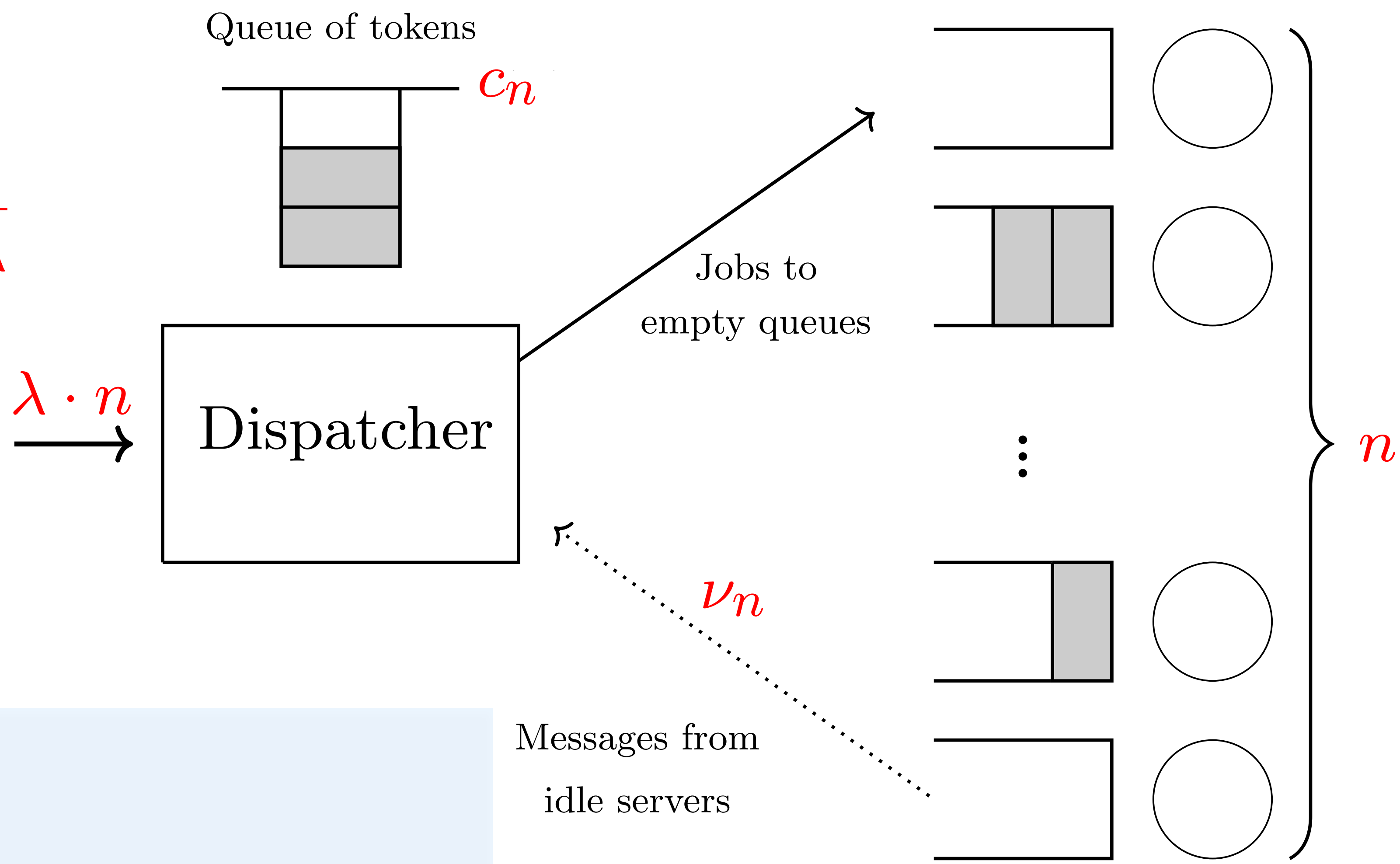
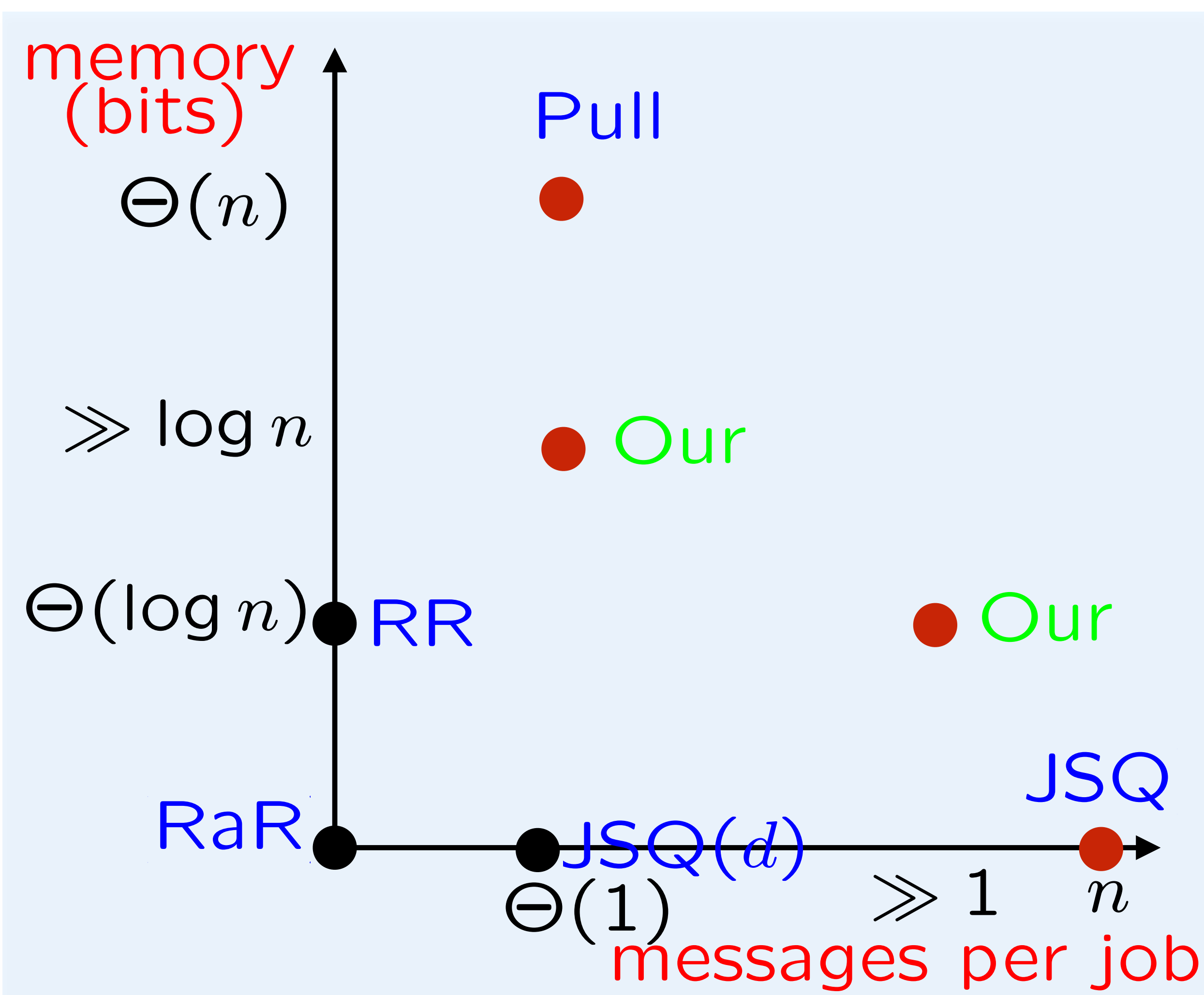
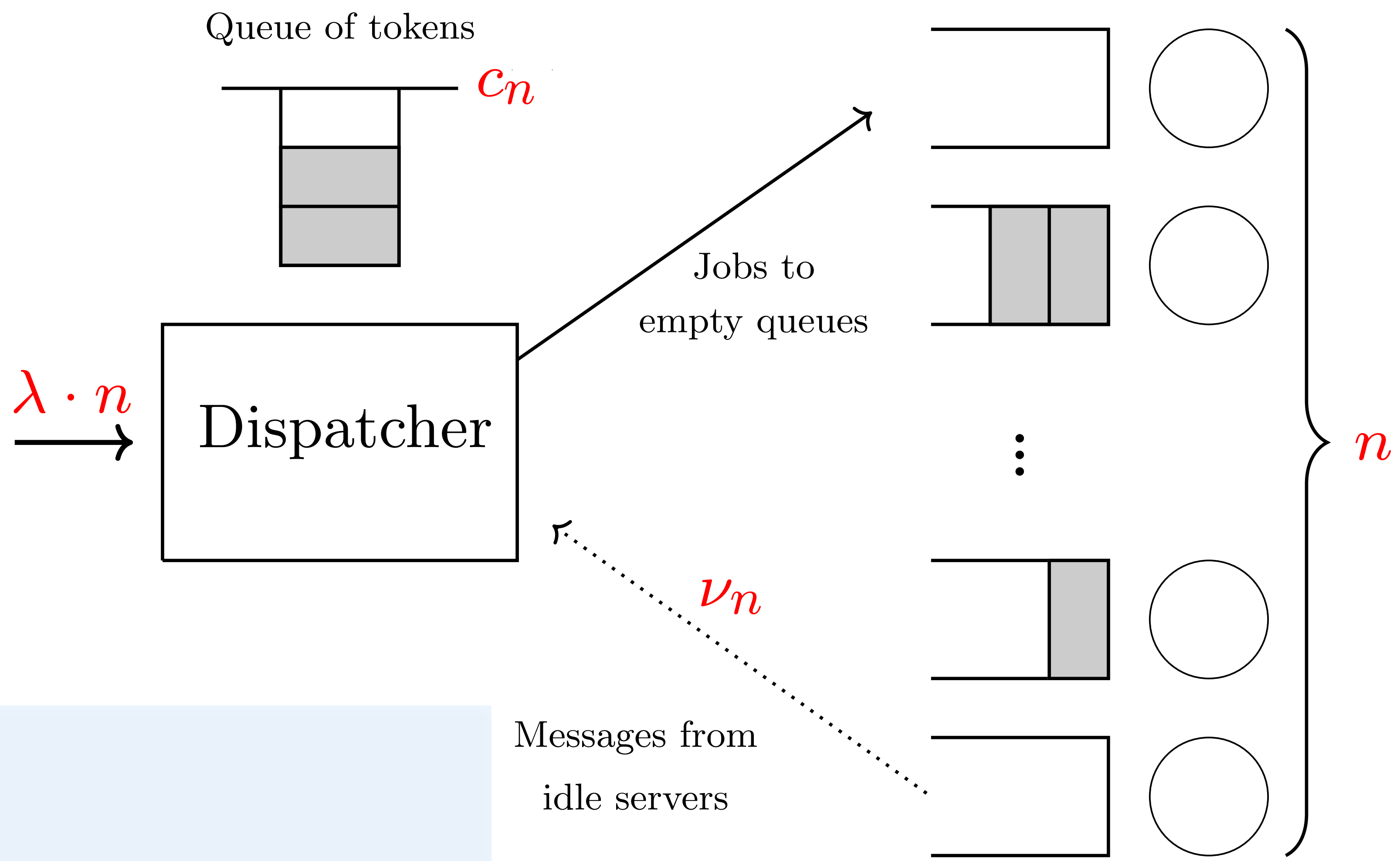
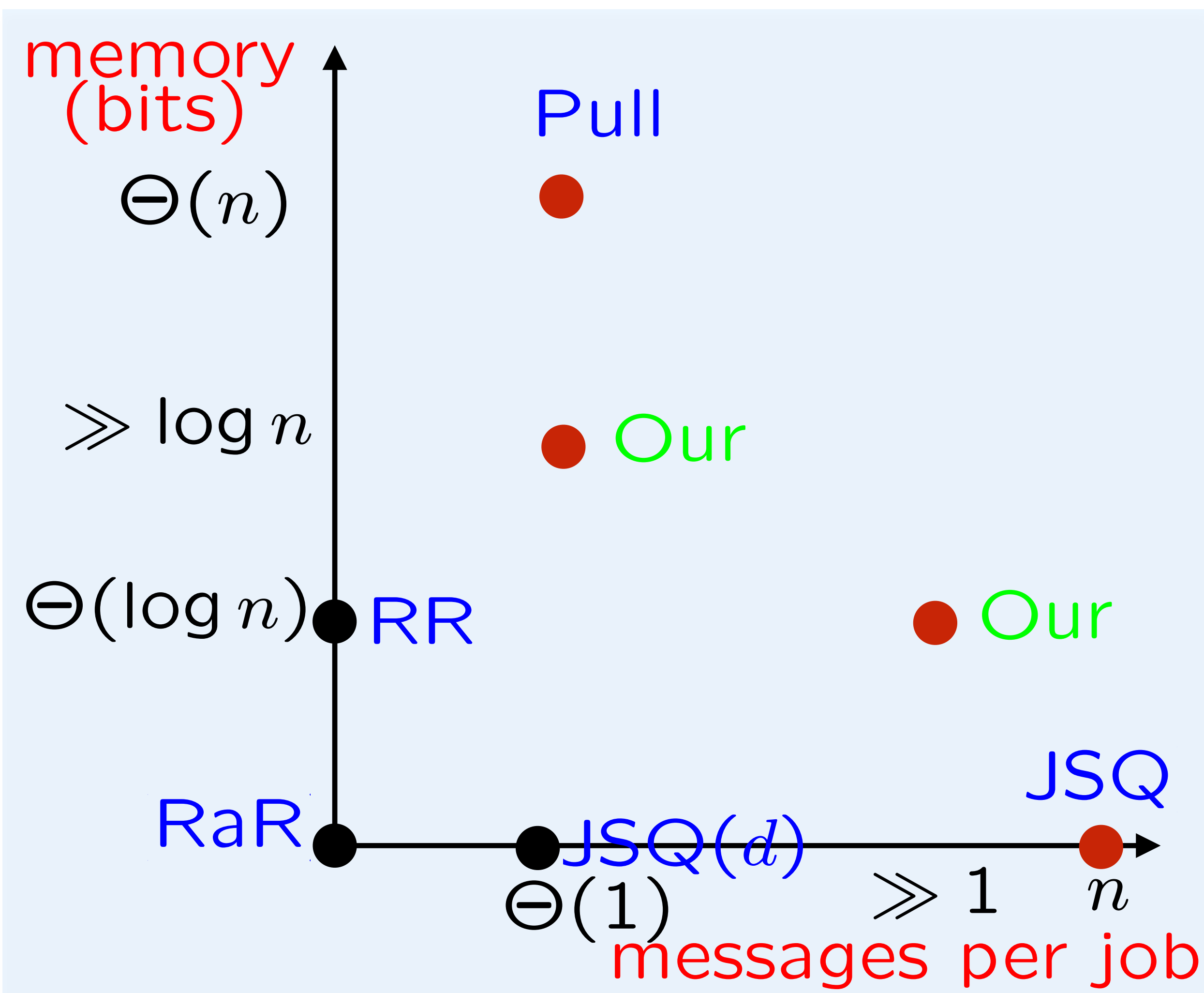




Parsimonious regime





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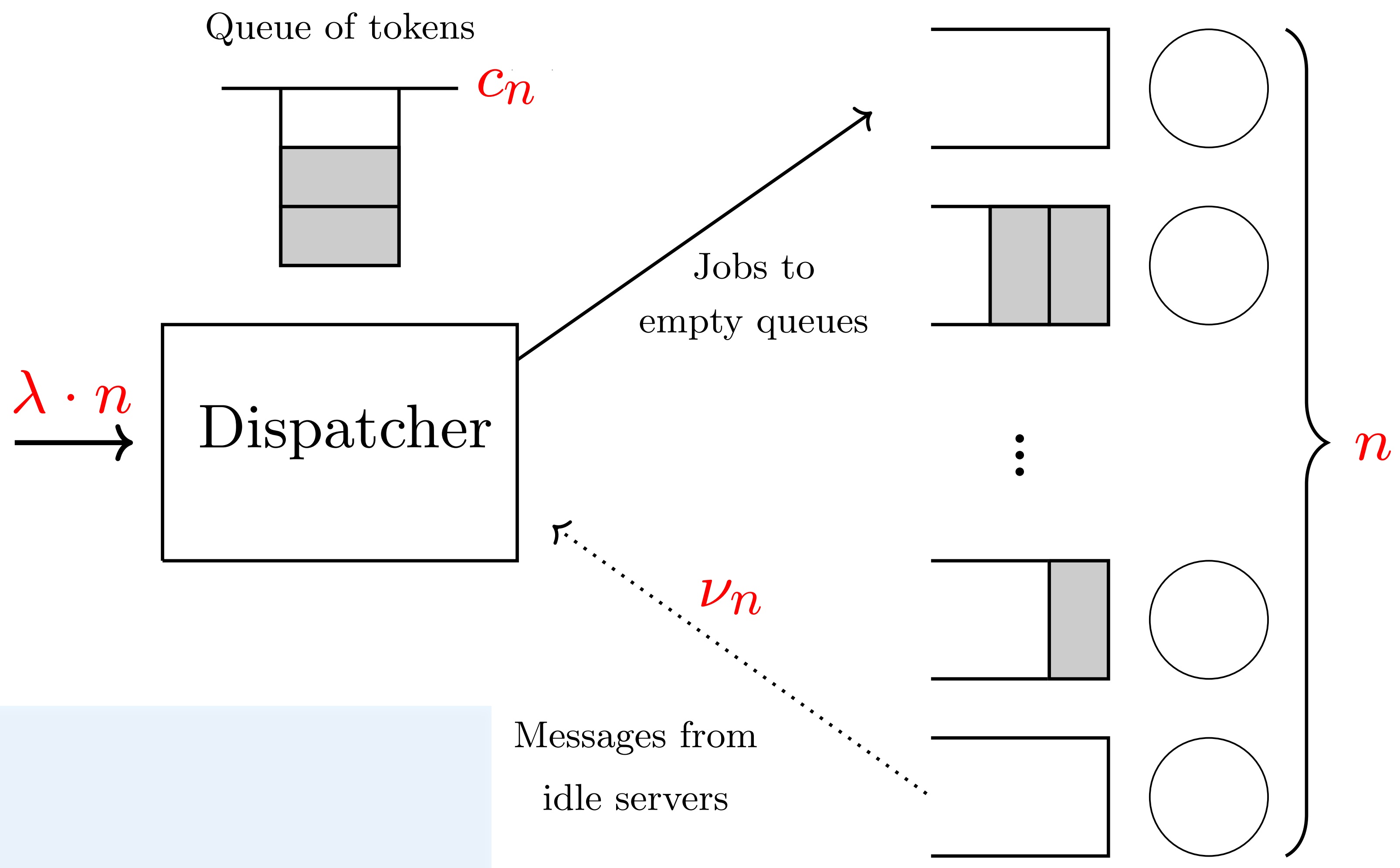

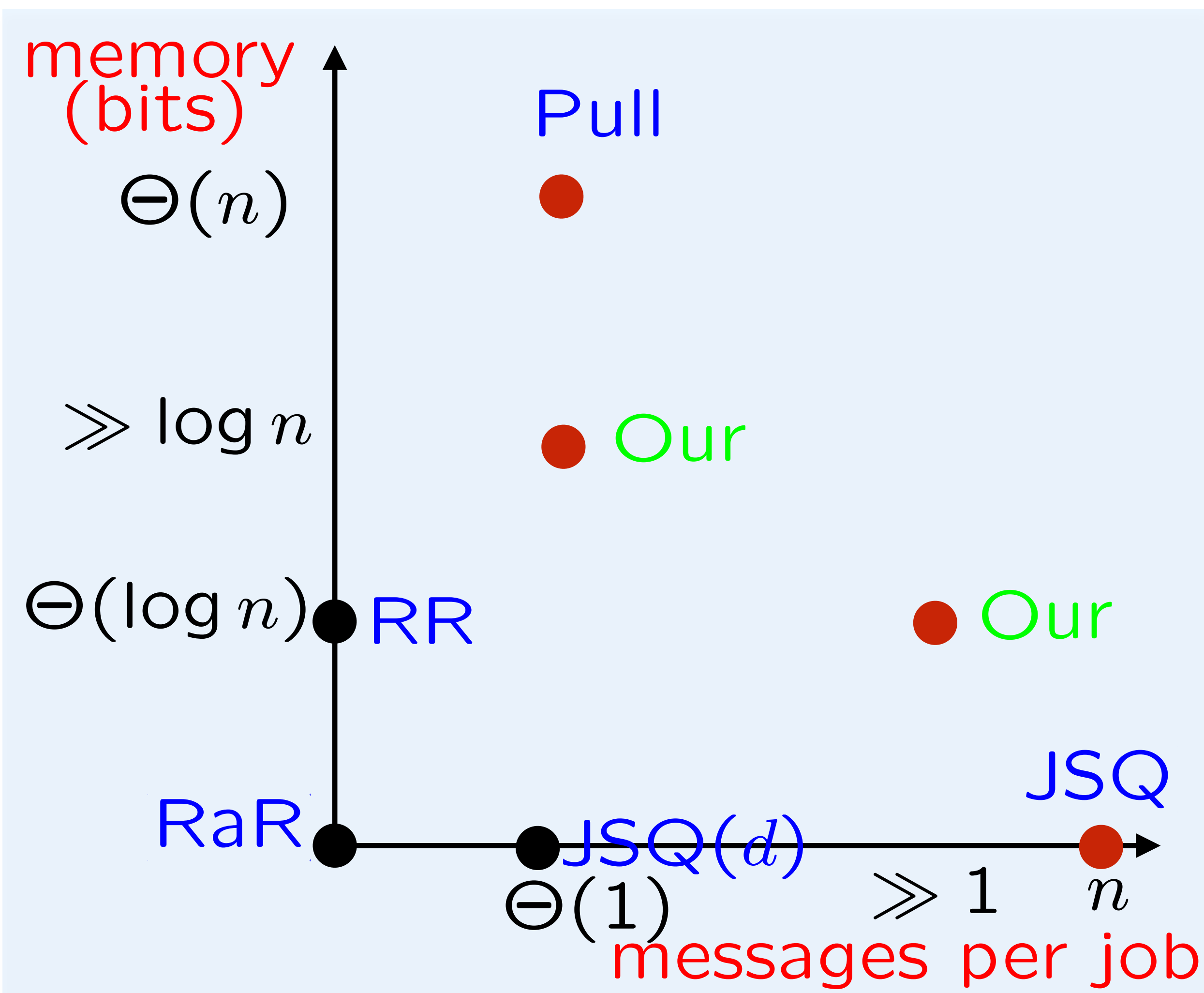

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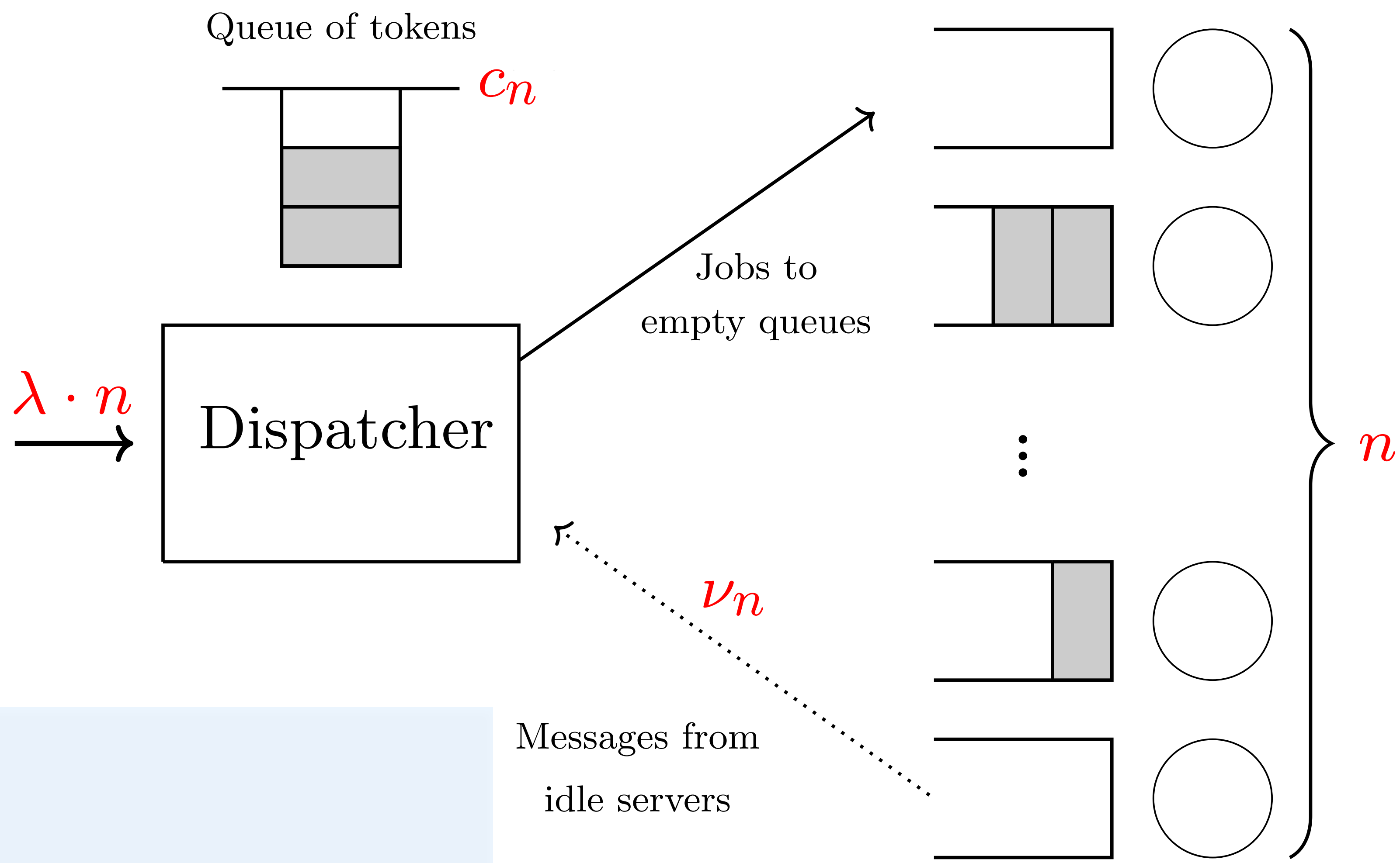

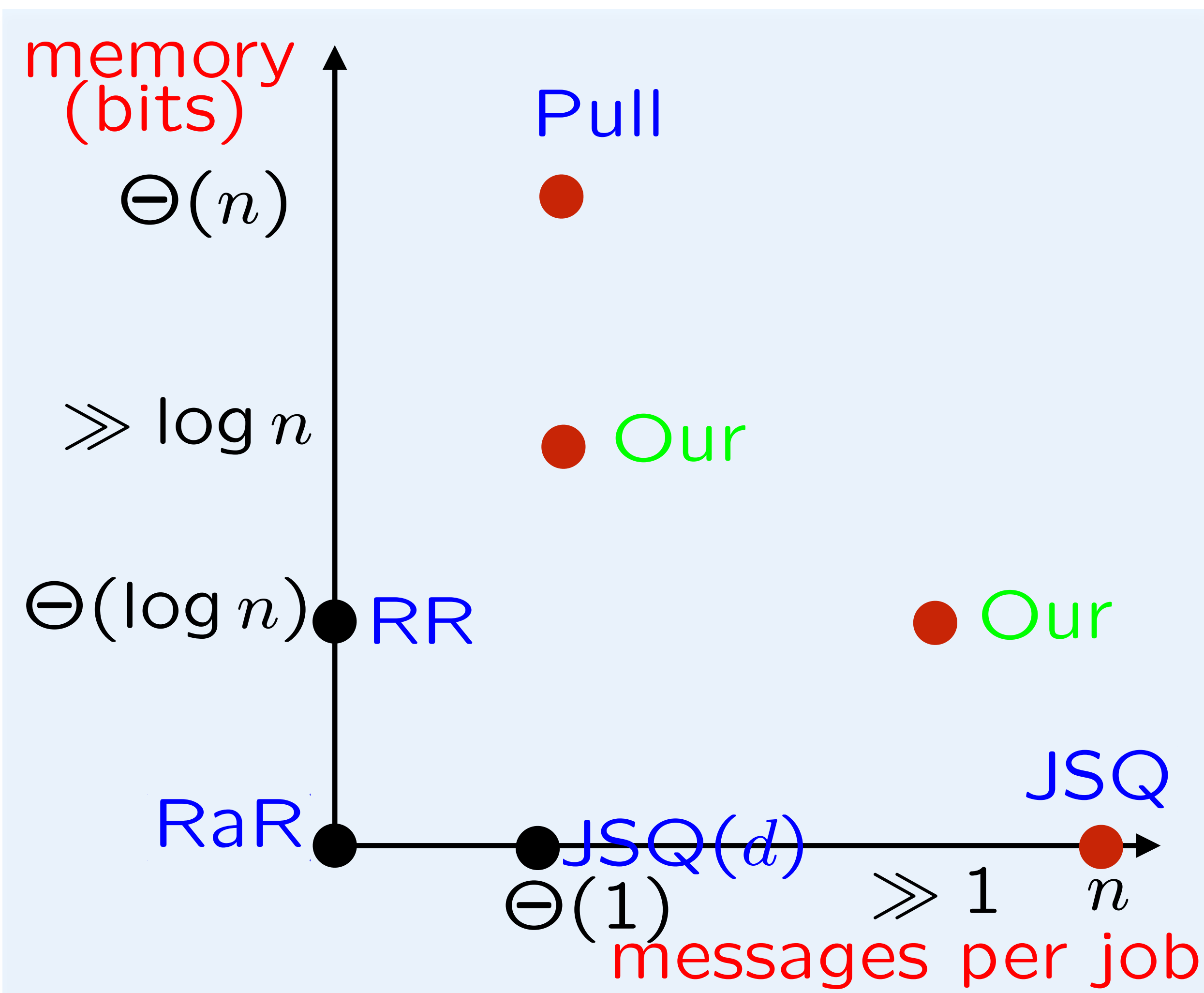

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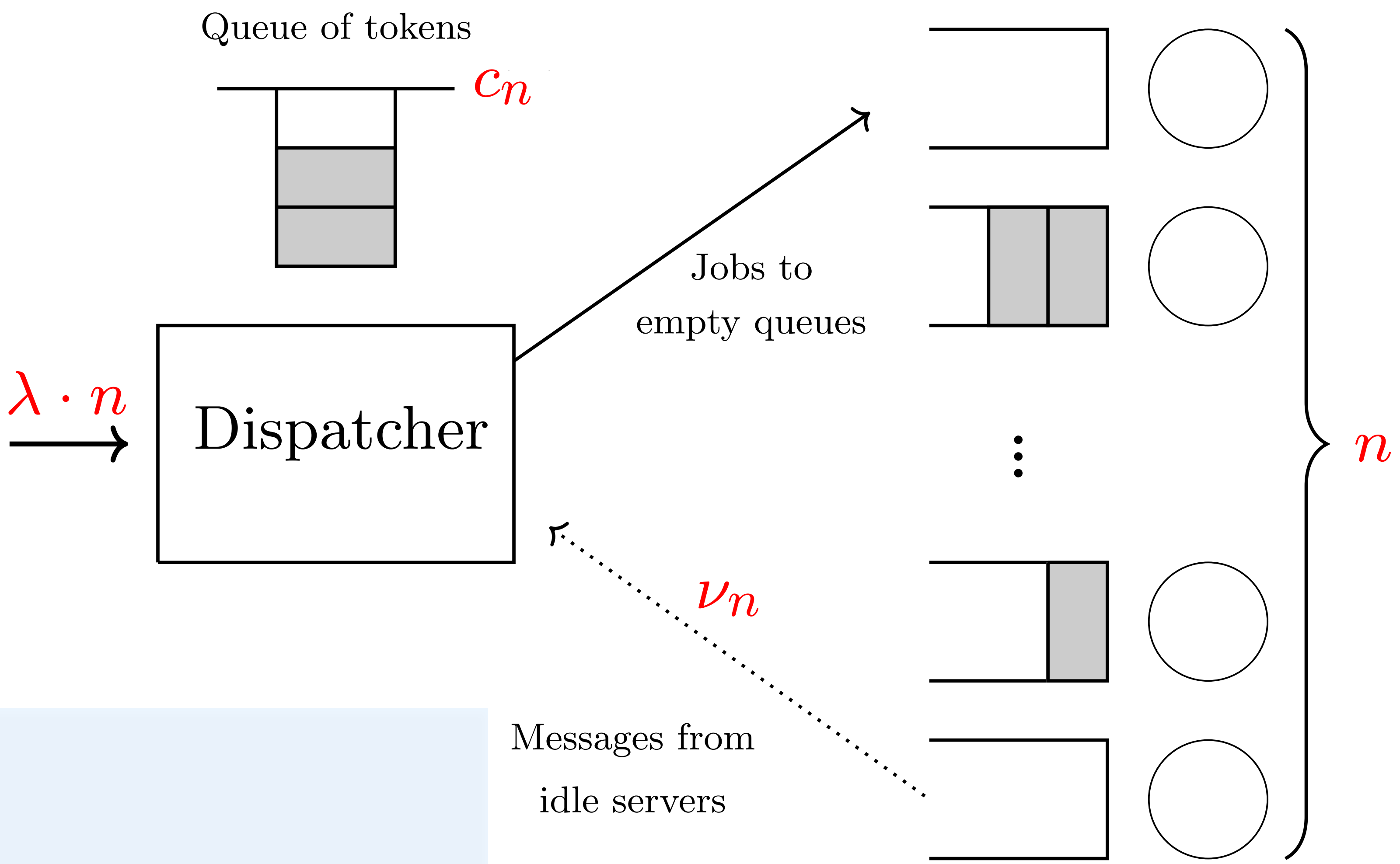

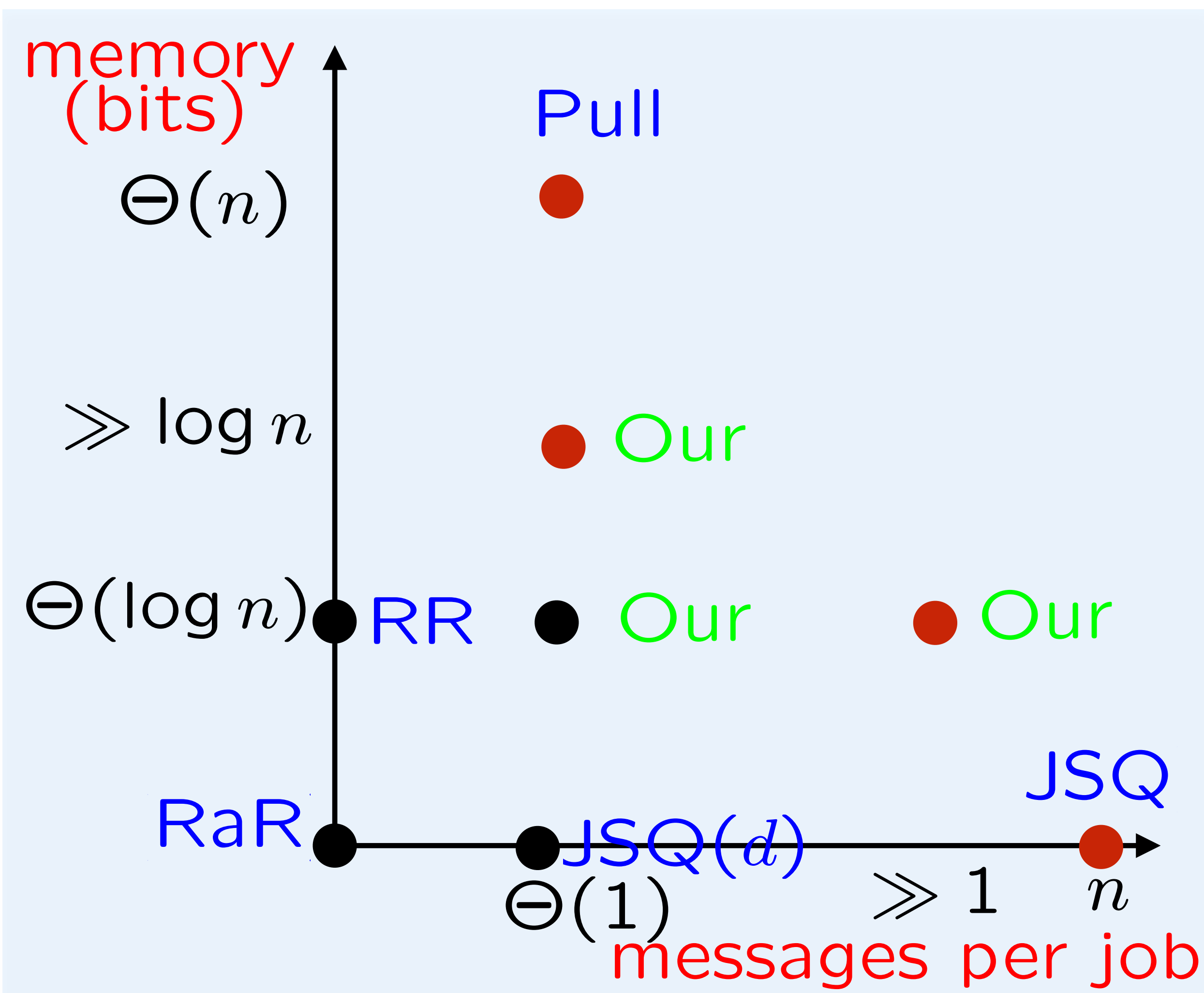

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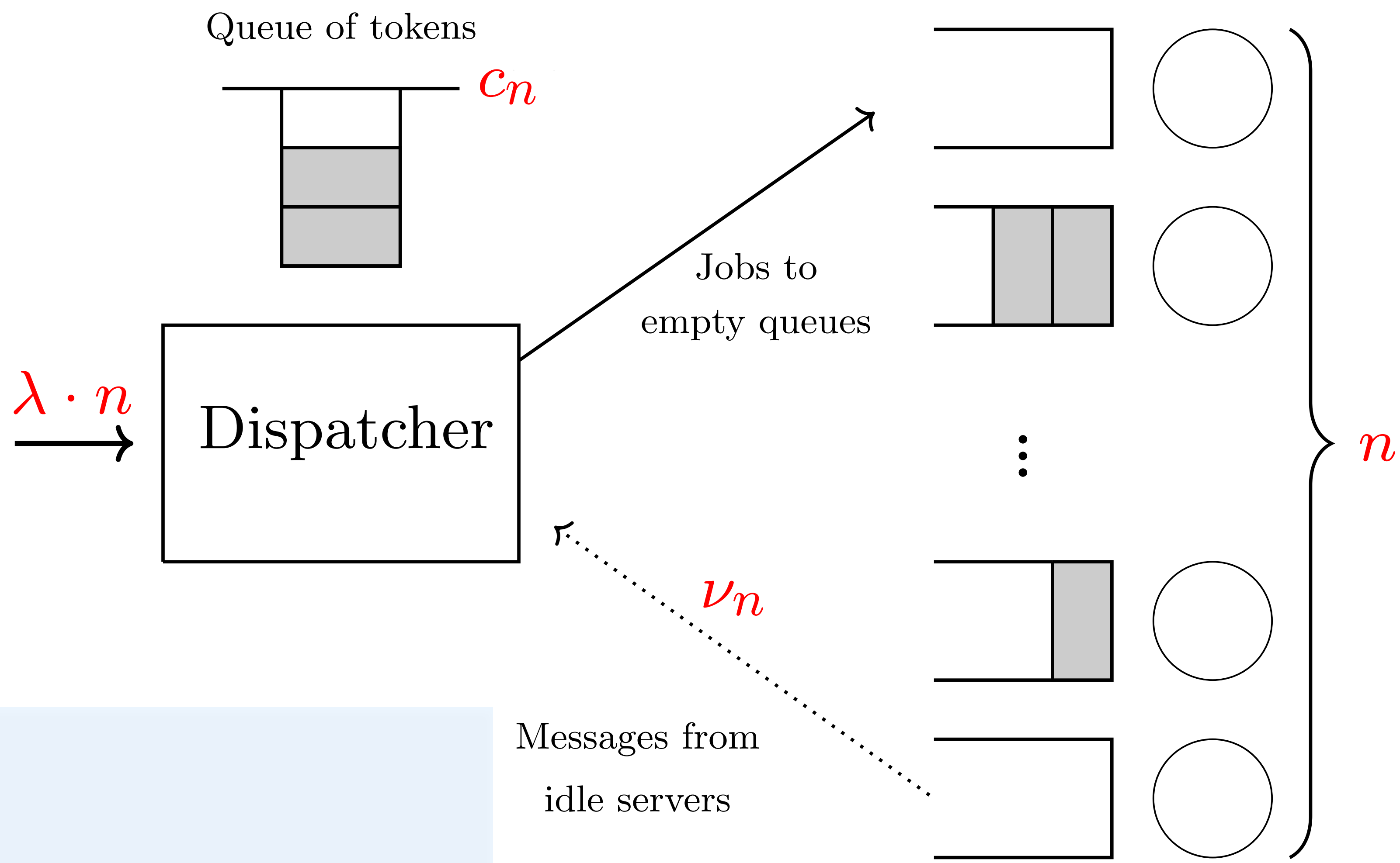
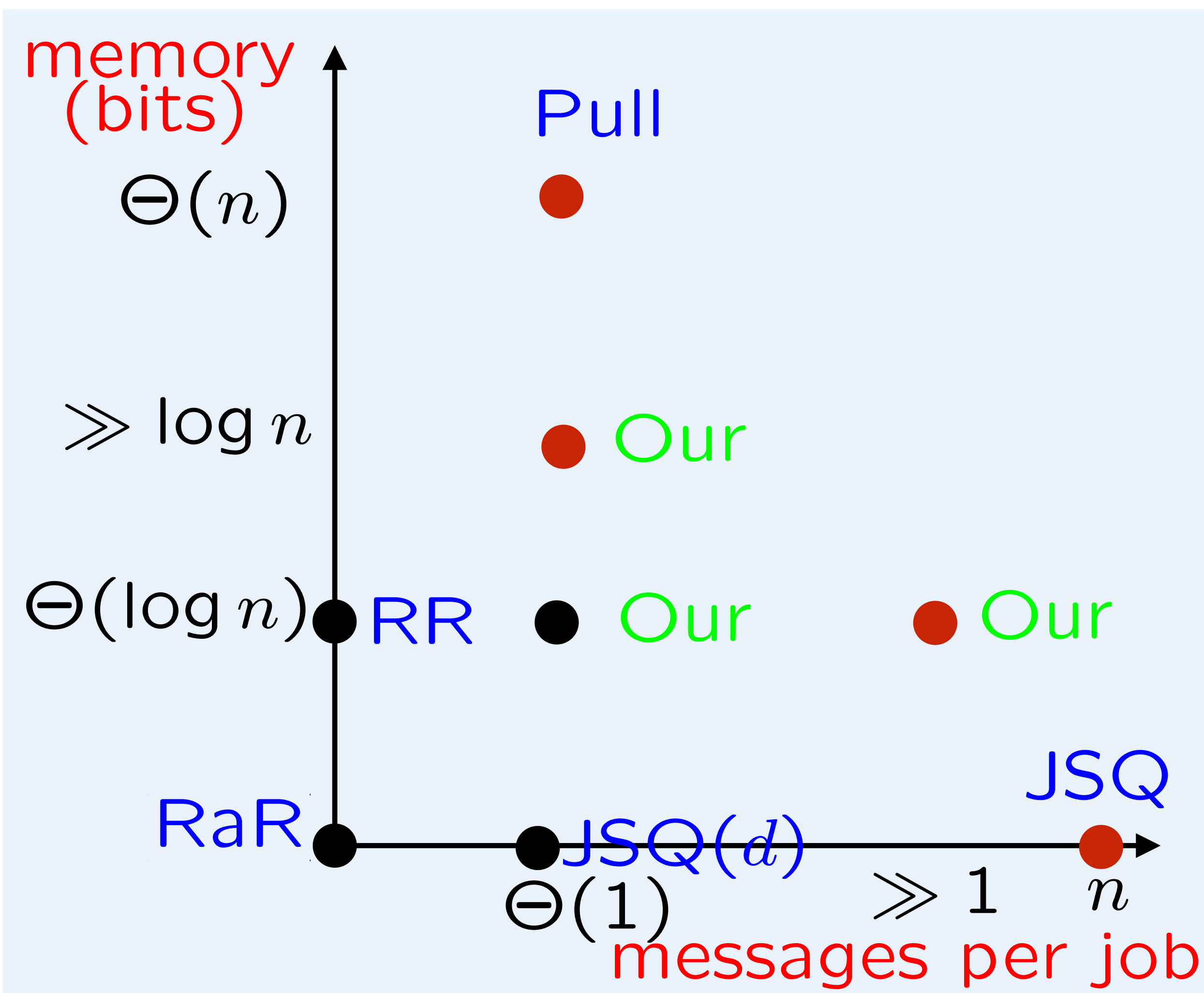


- Expressions for delay via fluid models



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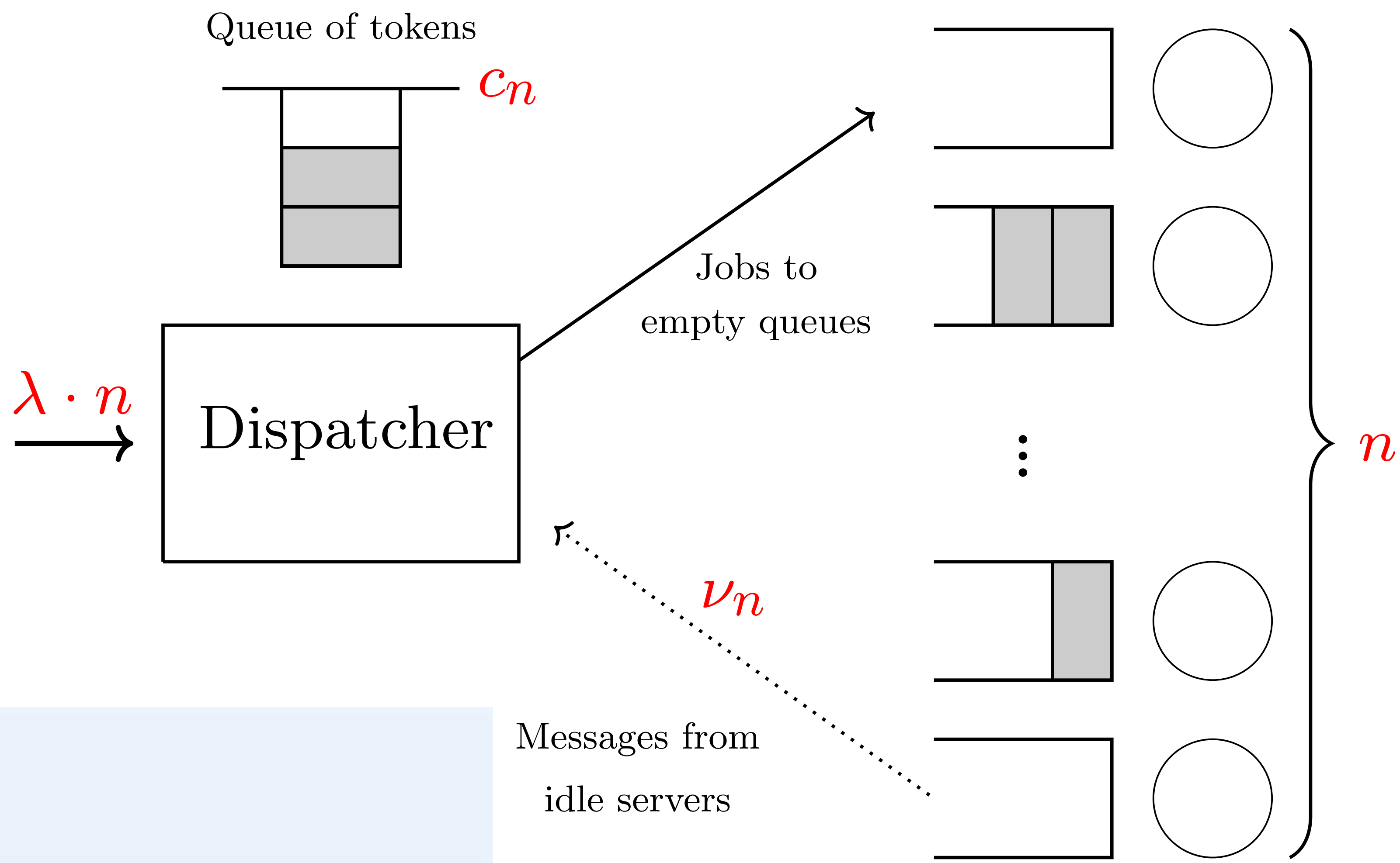

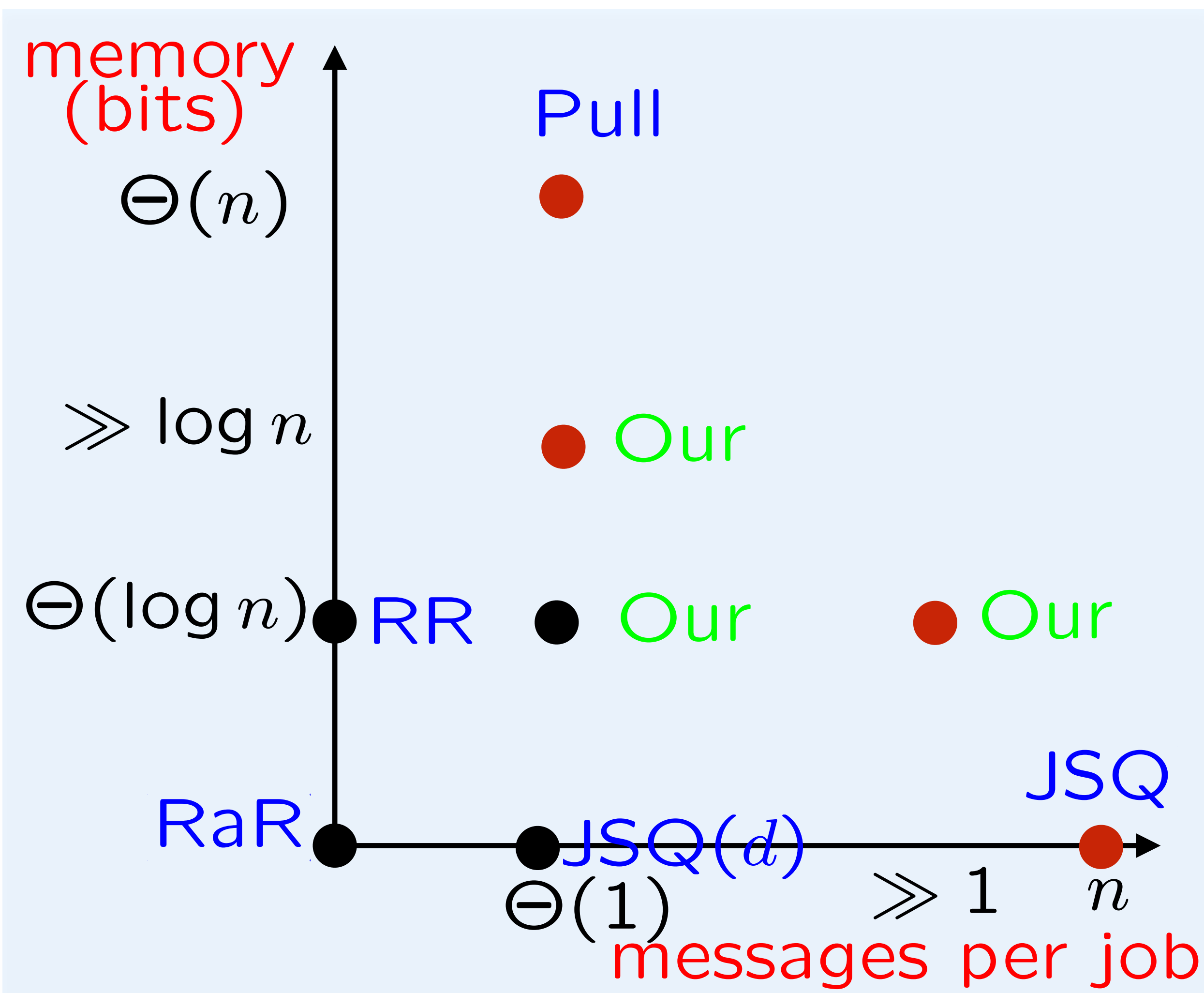

- Expressions for delay via fluid models
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Parsimonious regime

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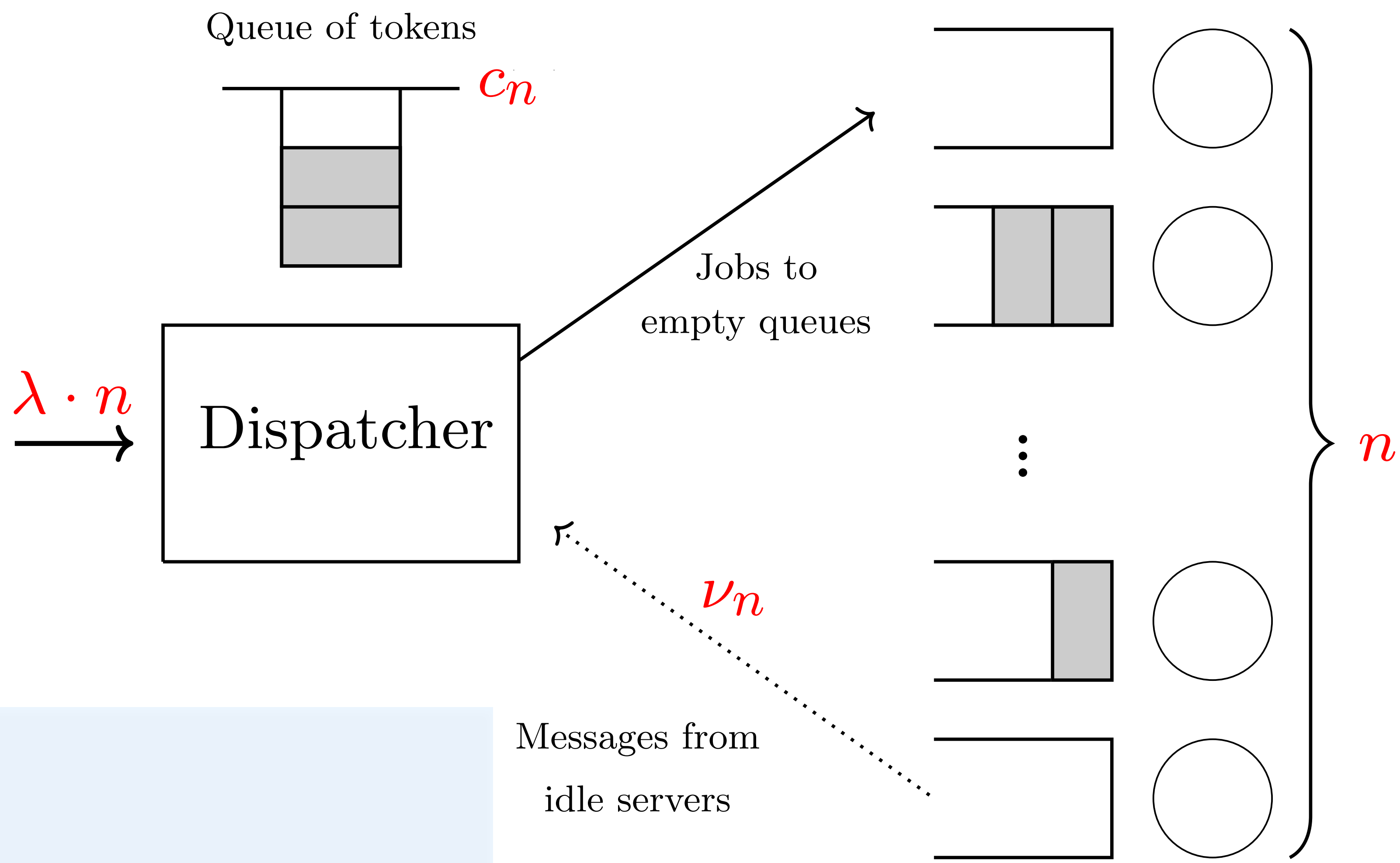

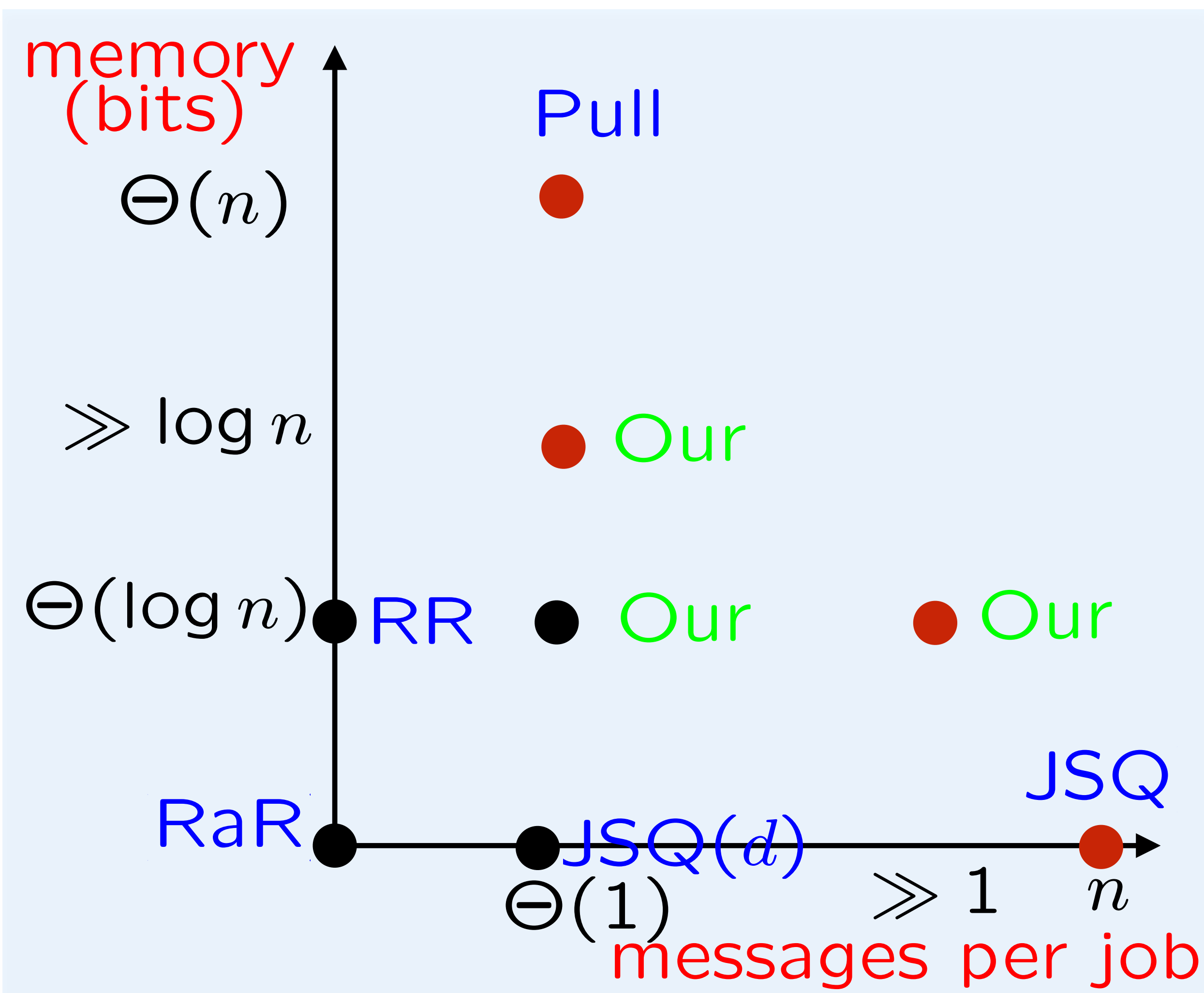

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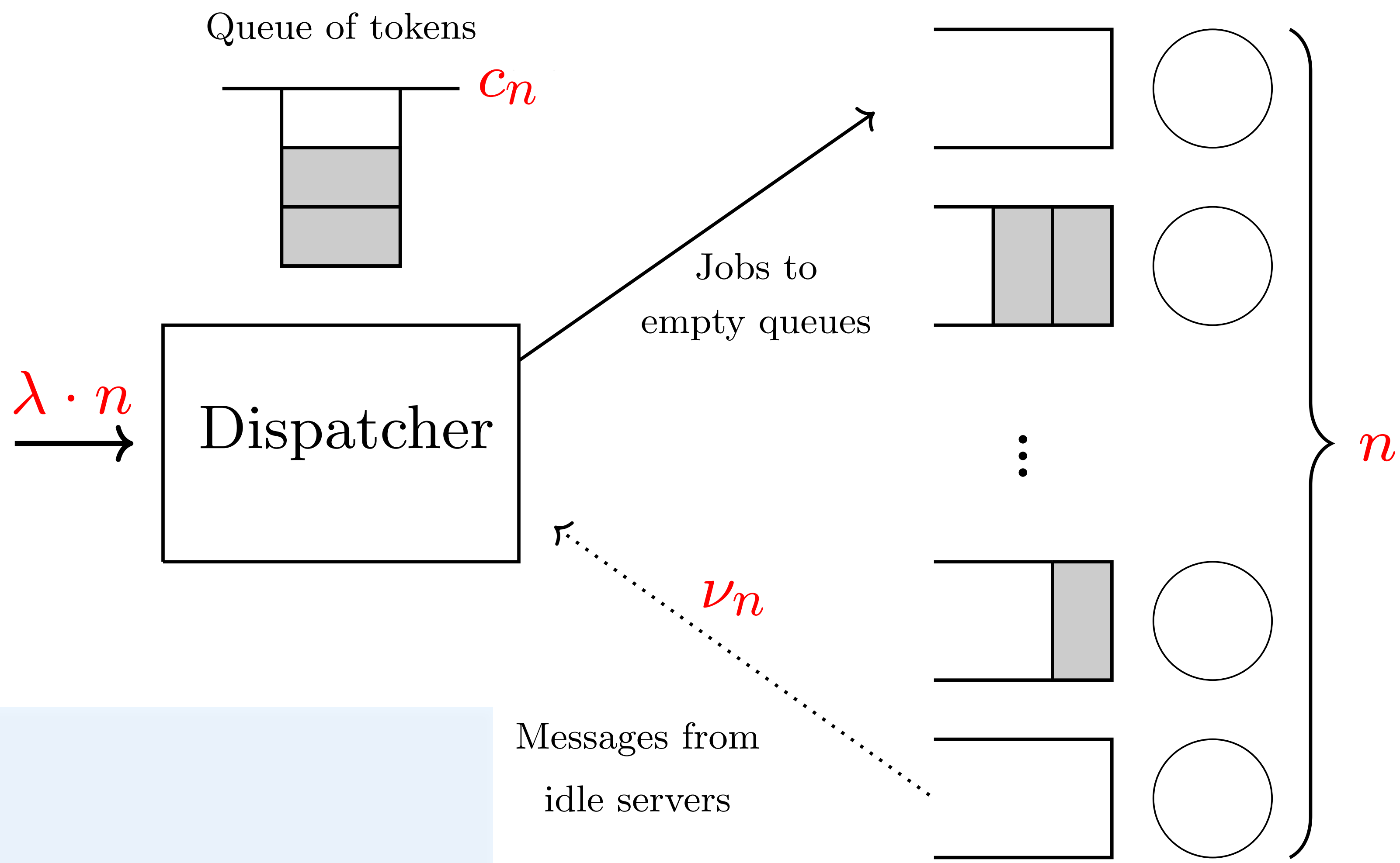

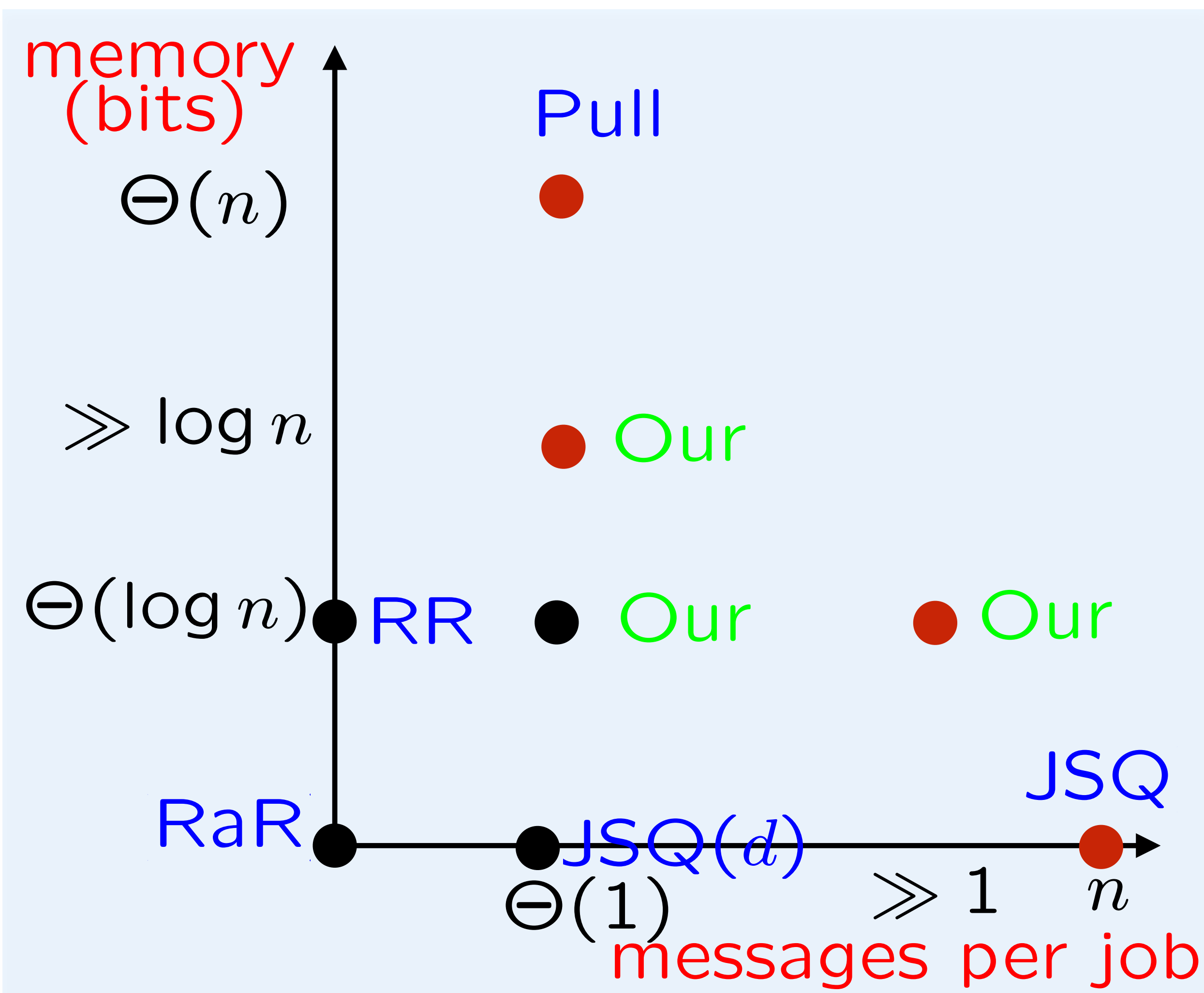

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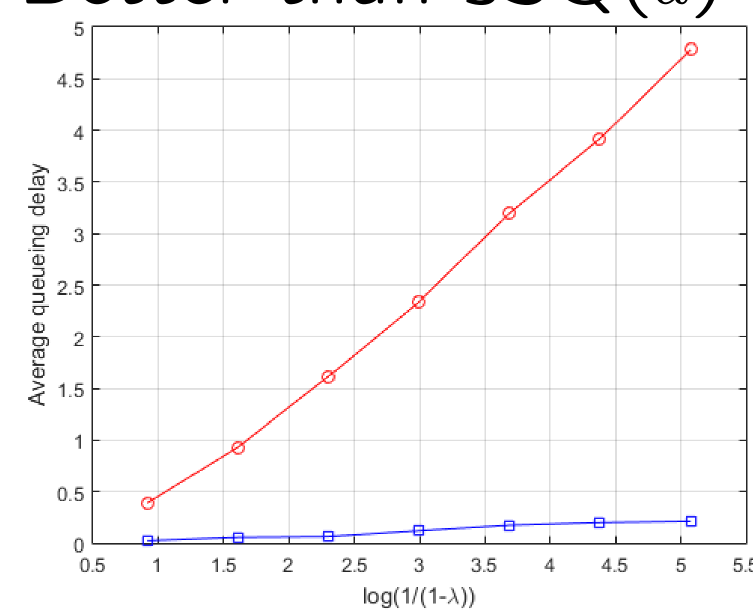

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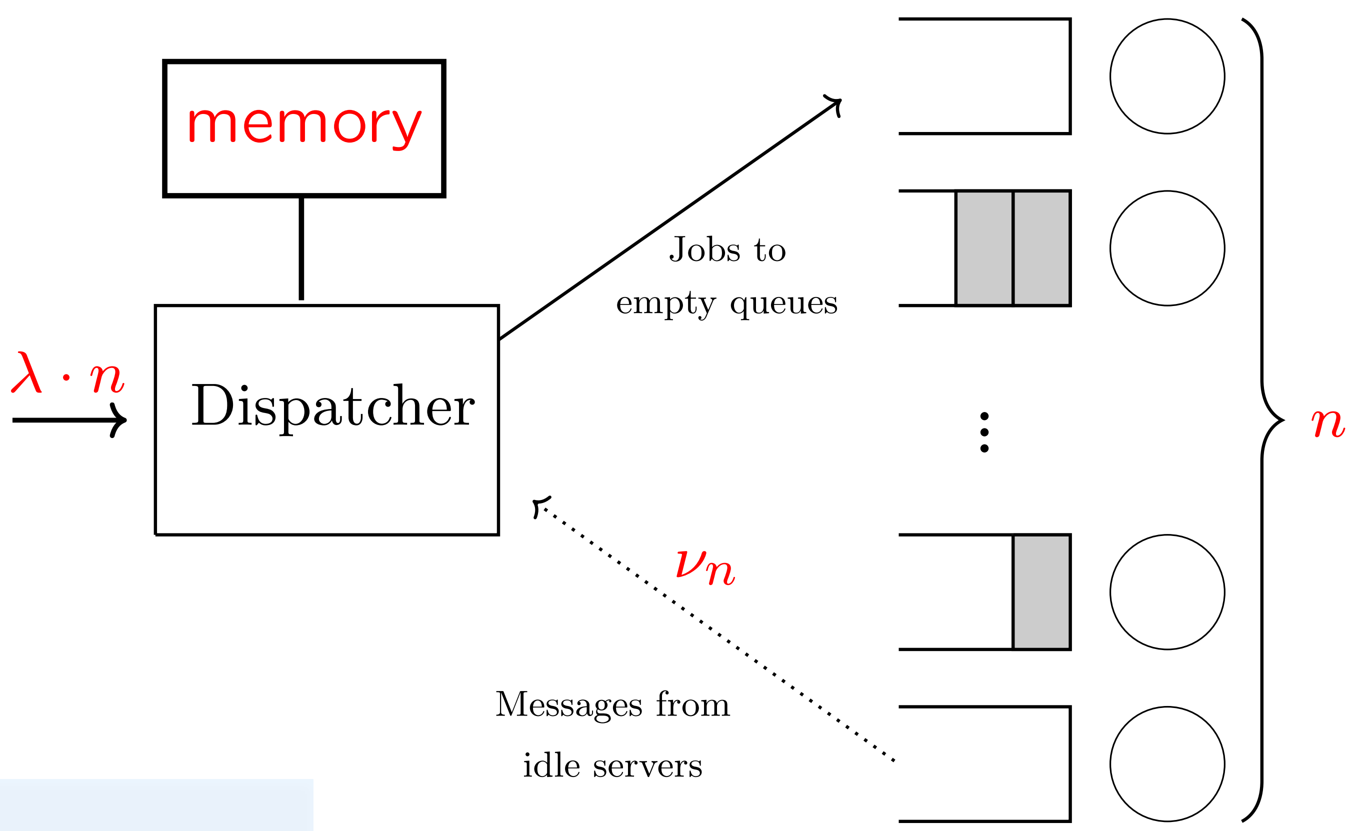

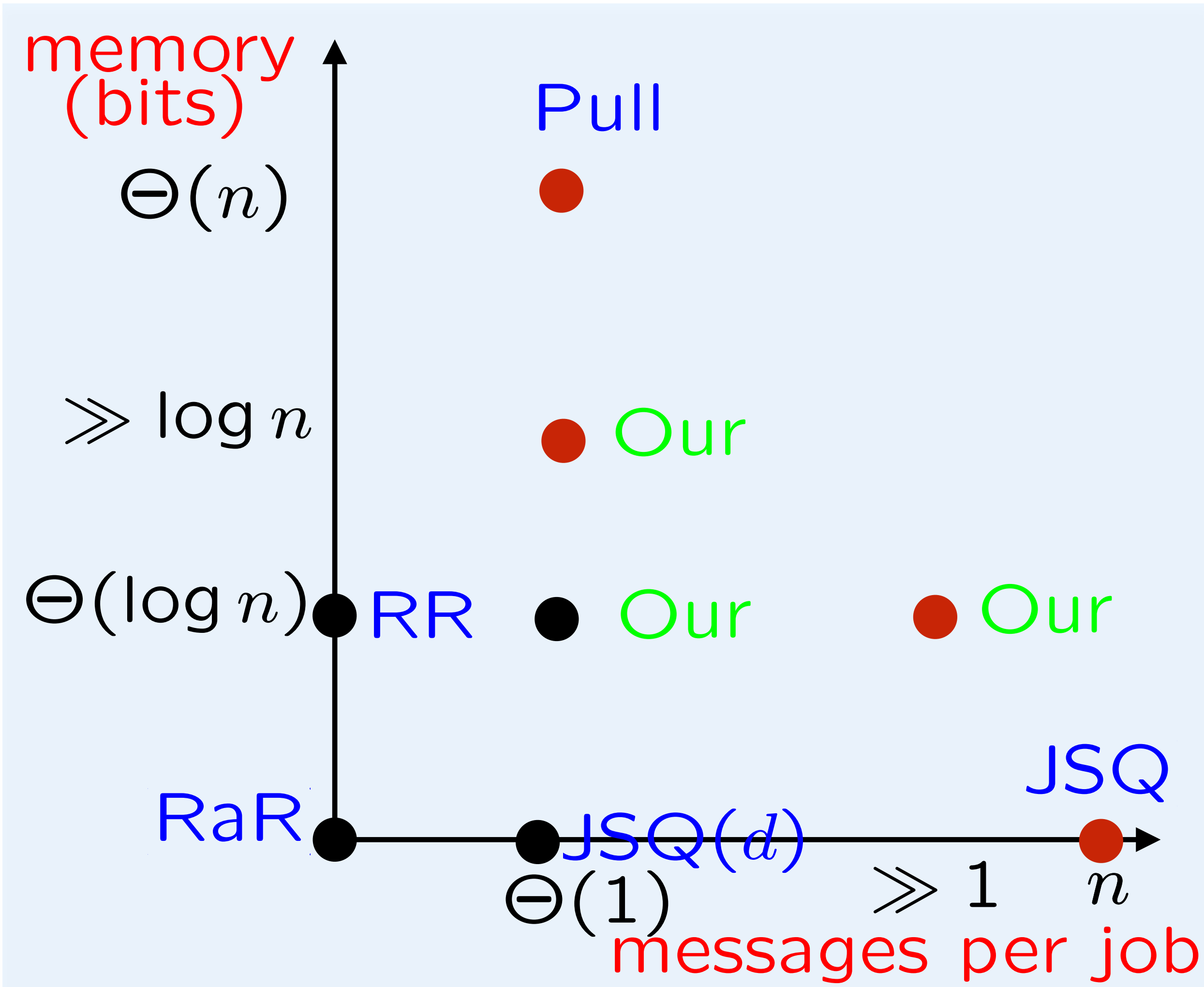

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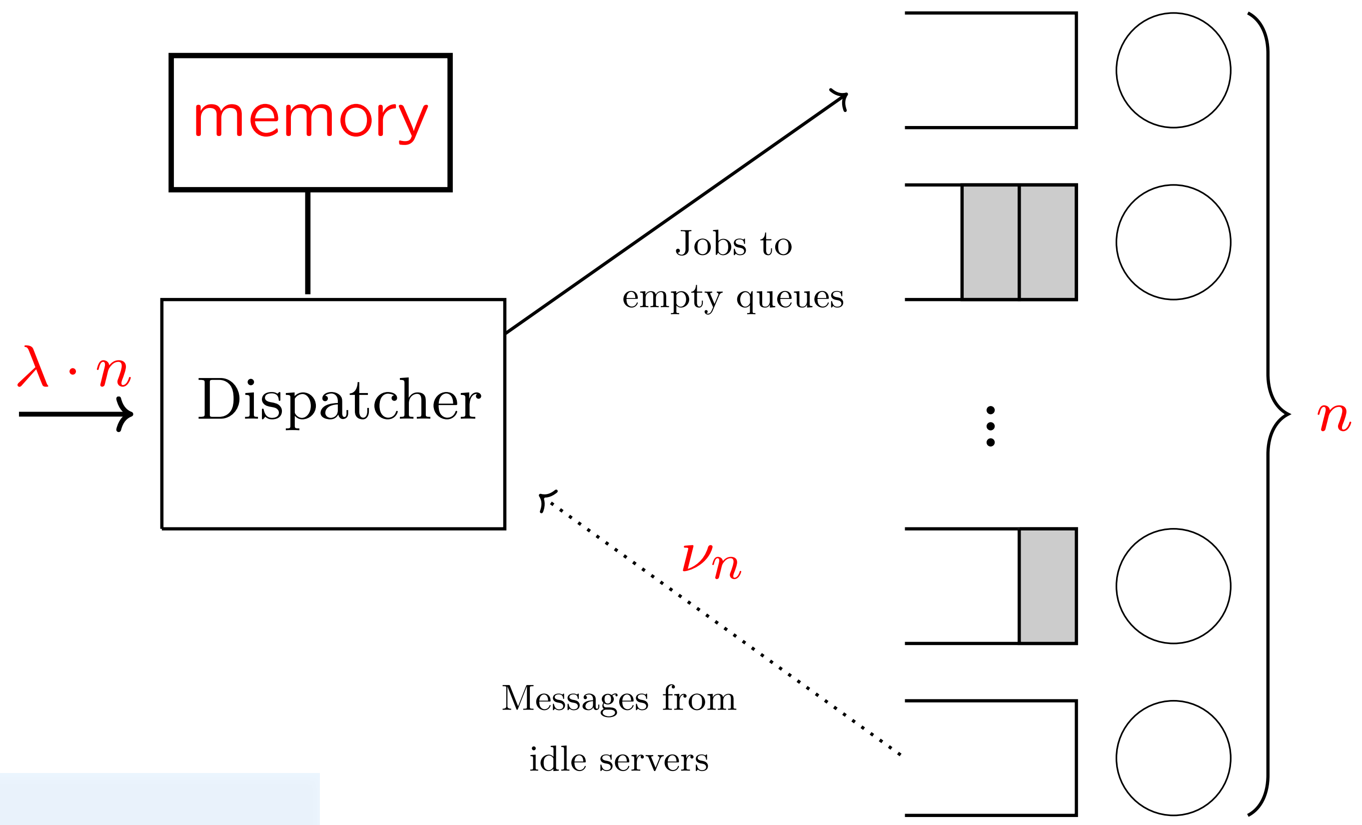

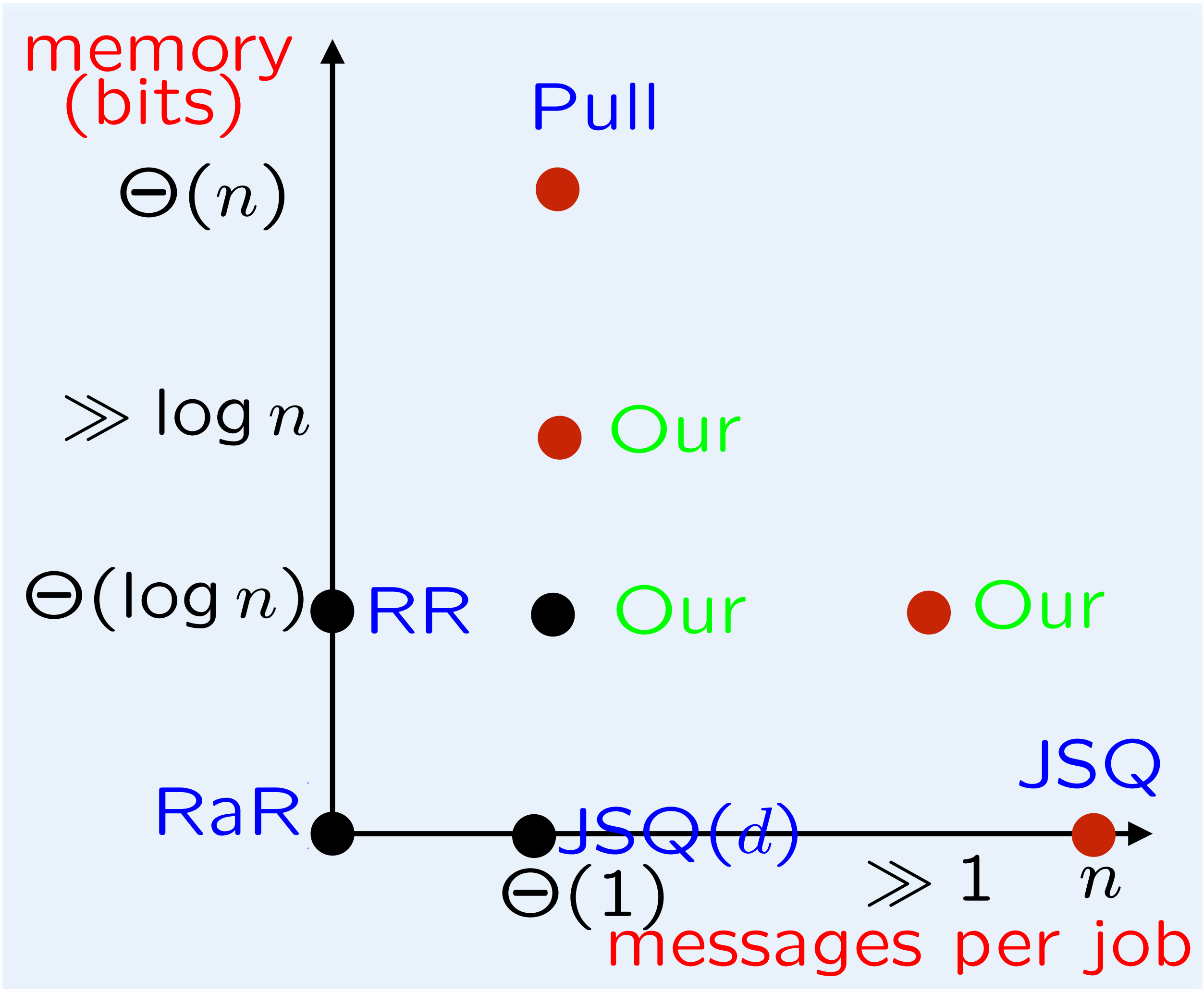

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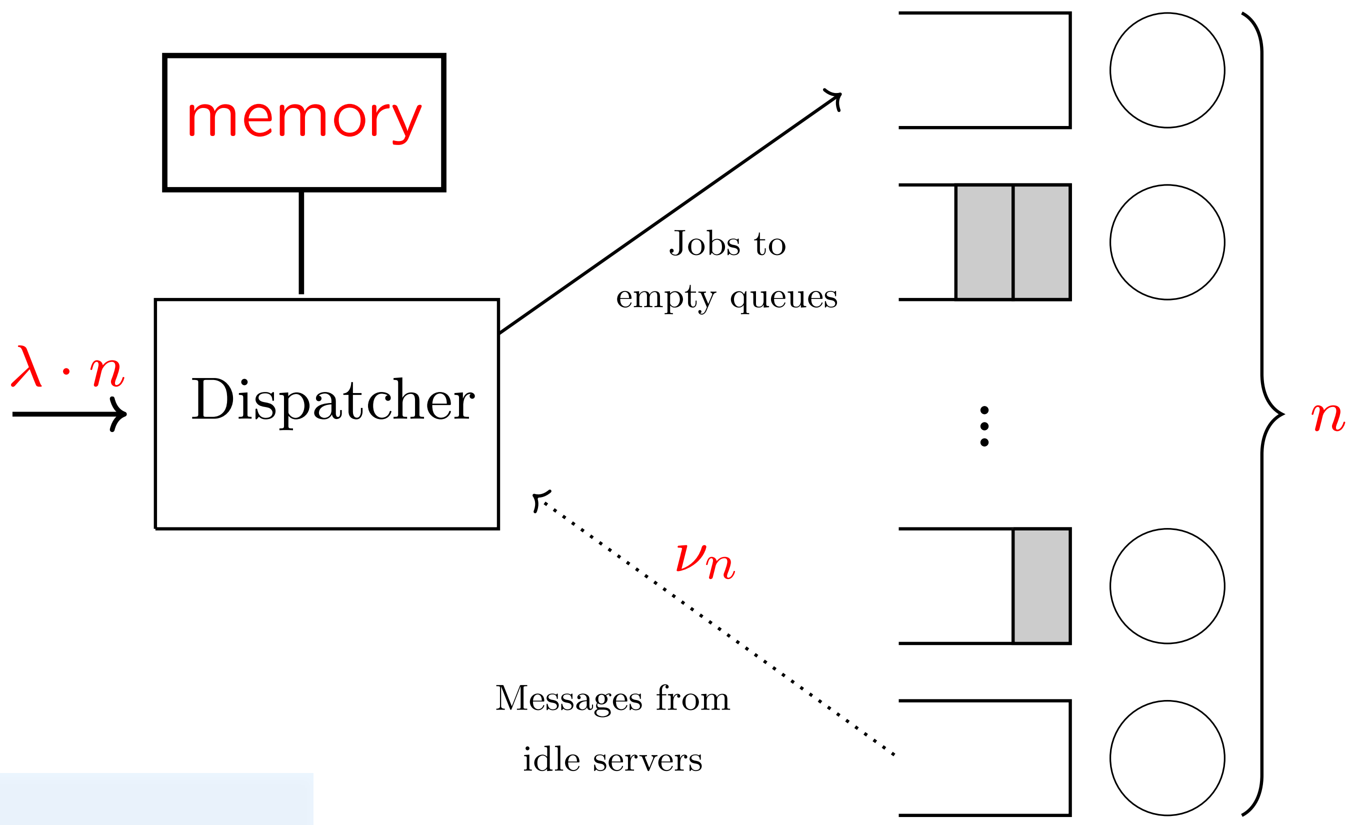

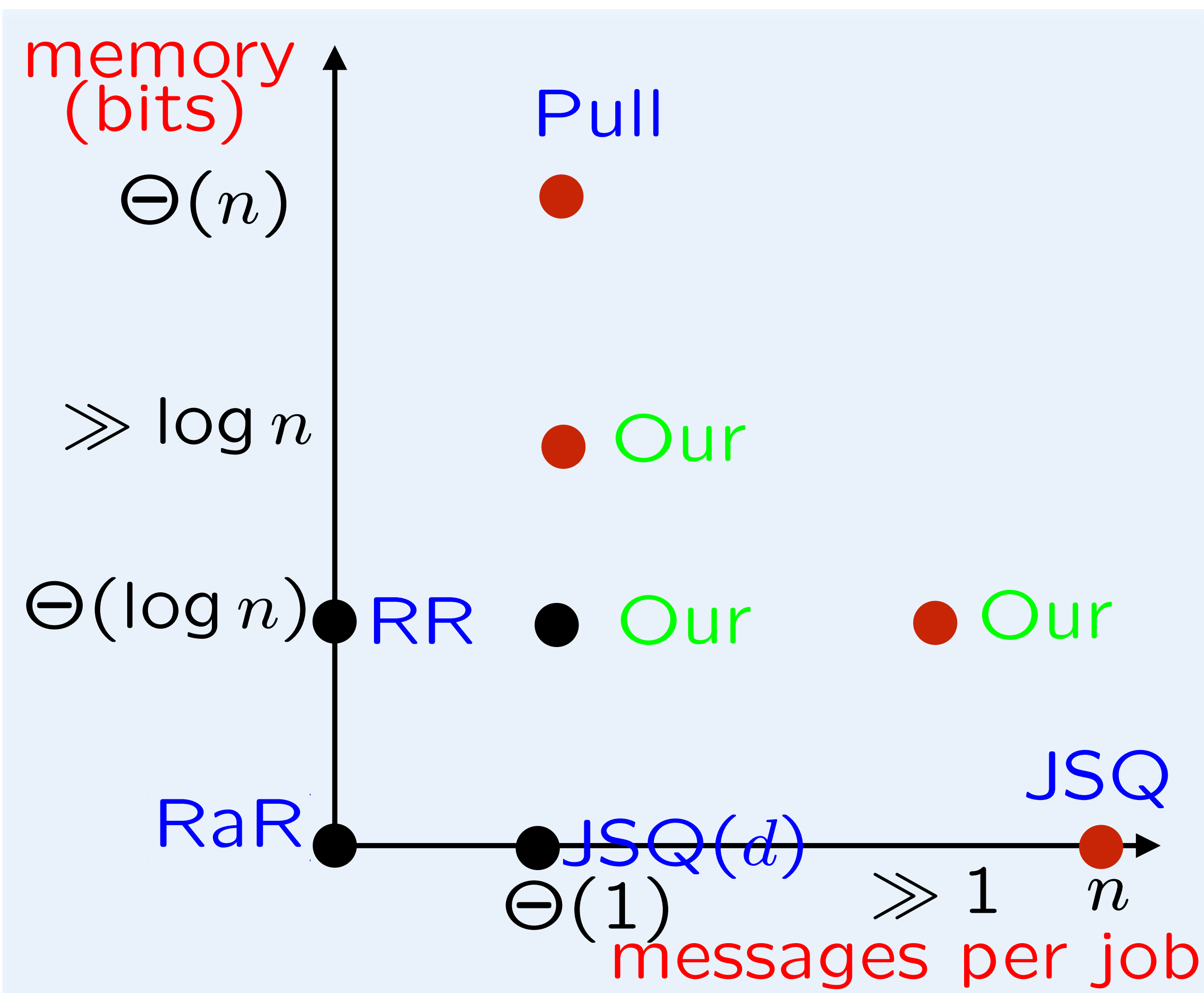

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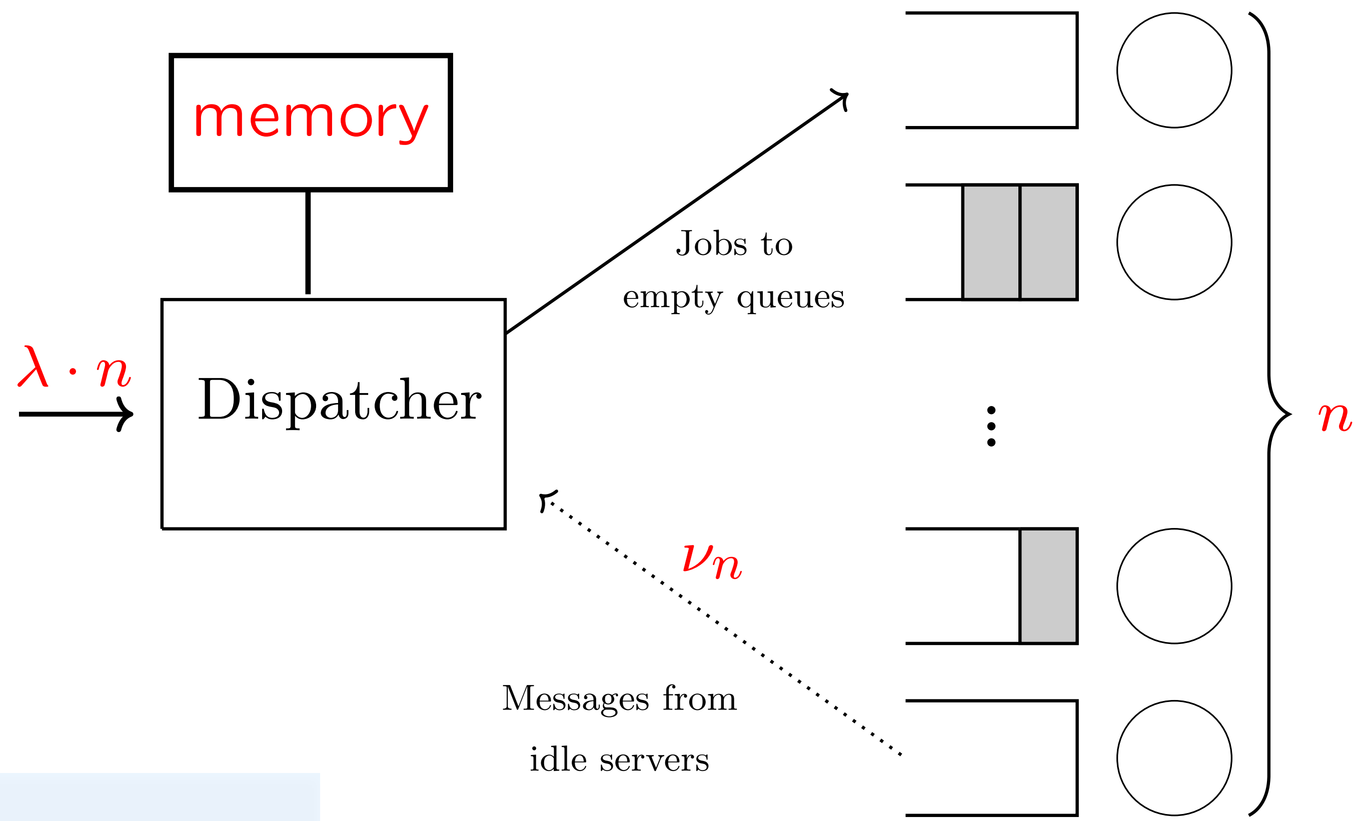
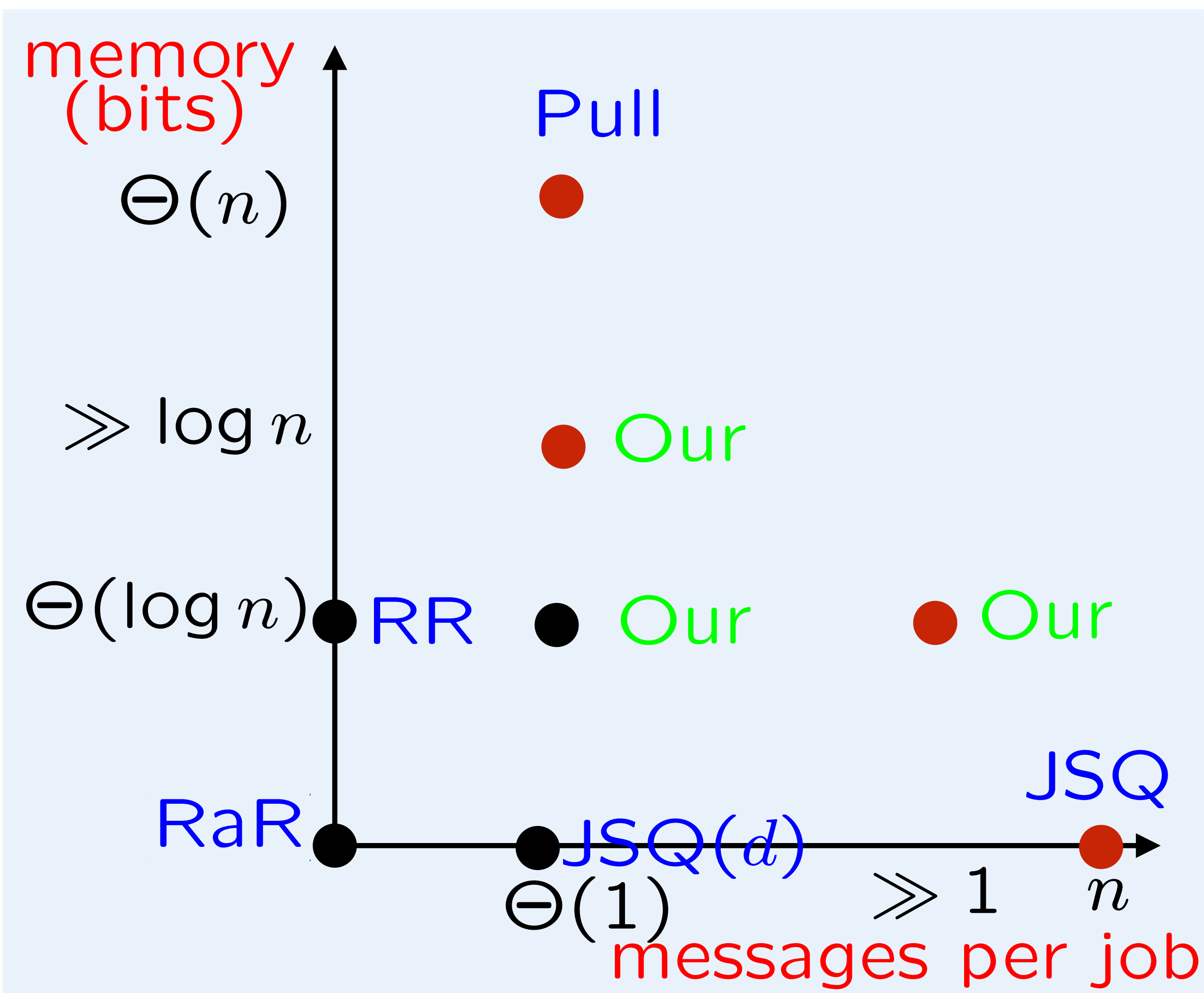
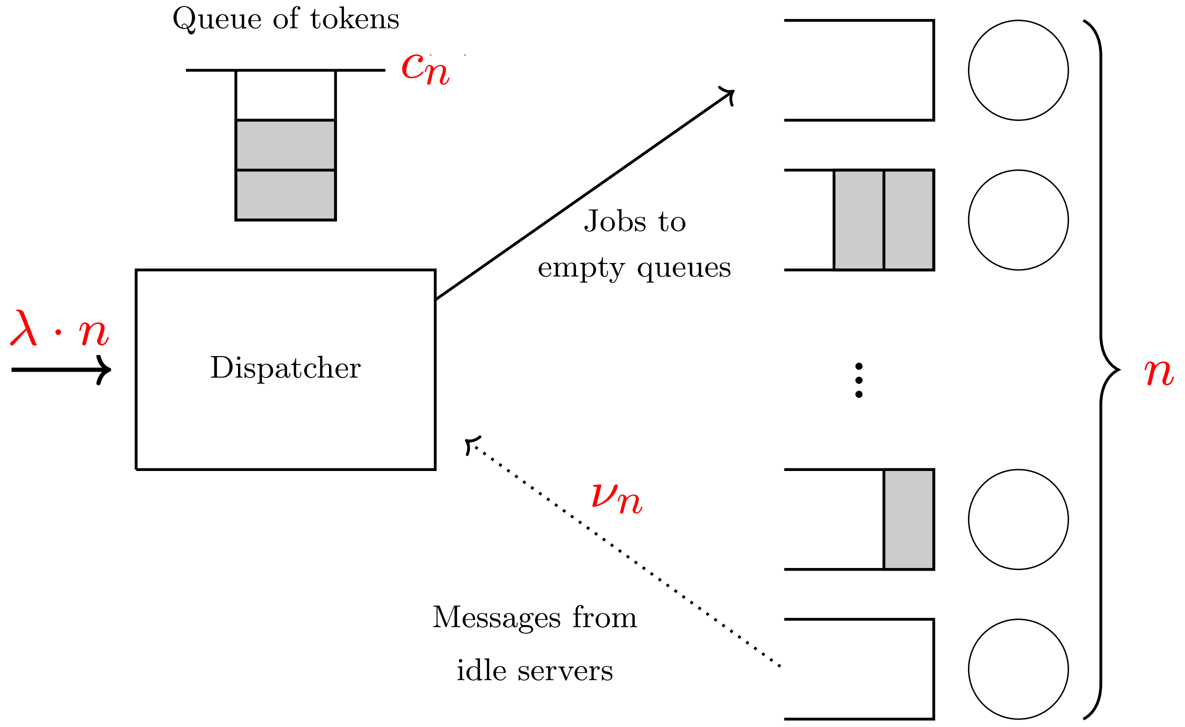




- **Thm:** queueing delay $\not\to 0$
- Assumptions:
  - no queueing at dispatcher
  - "symmetric" policy
  - not too many back-and-forths in too little time

The technical side

Two time scale dynamics



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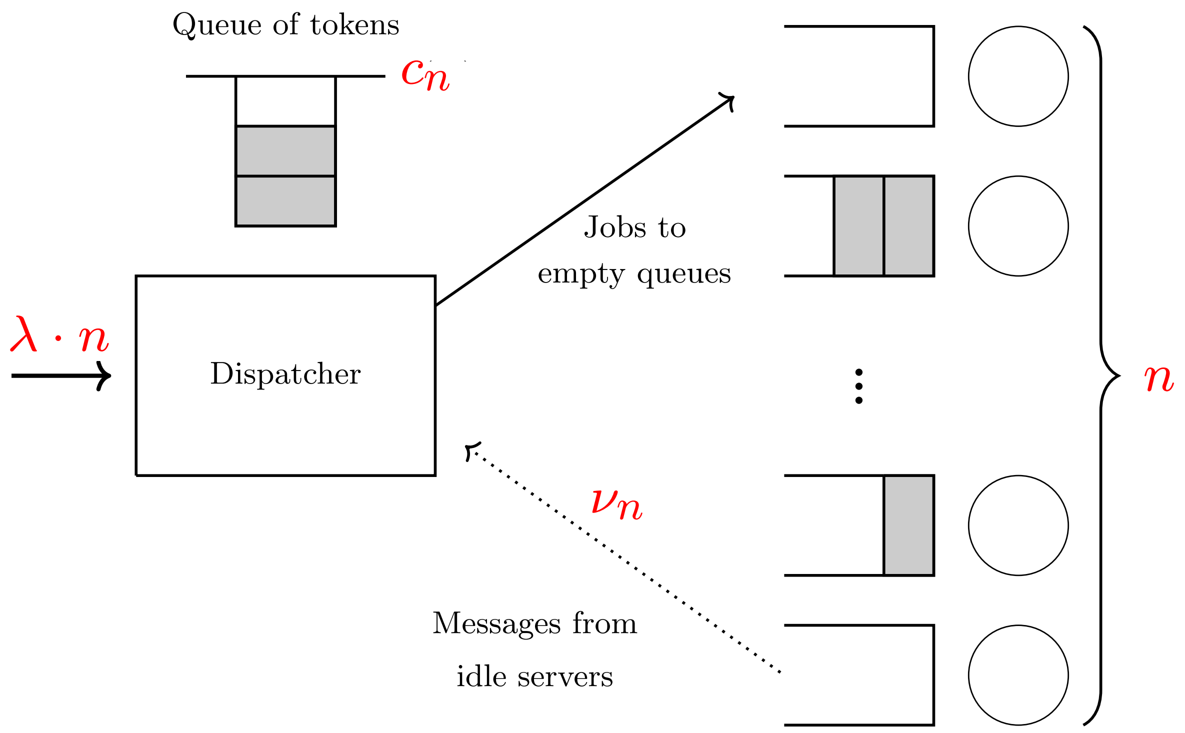
- $S_i^n(t)$: fraction of servers with at least i jobs (in queue or in service)



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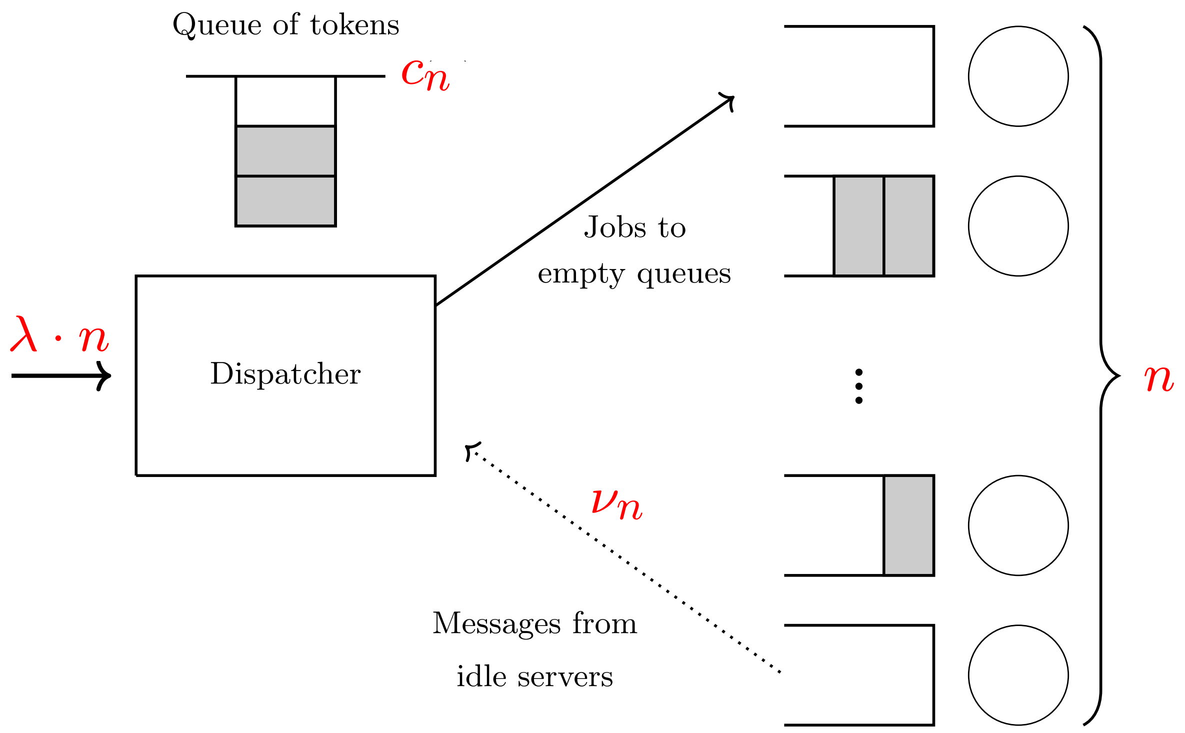

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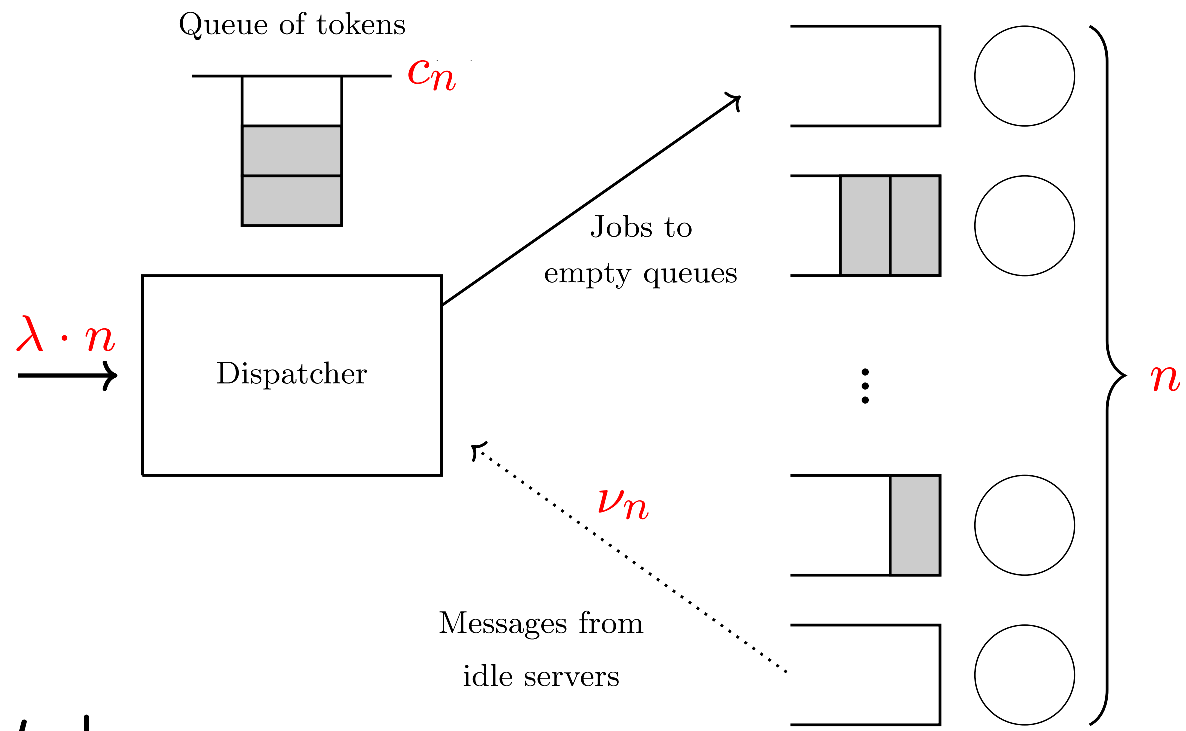

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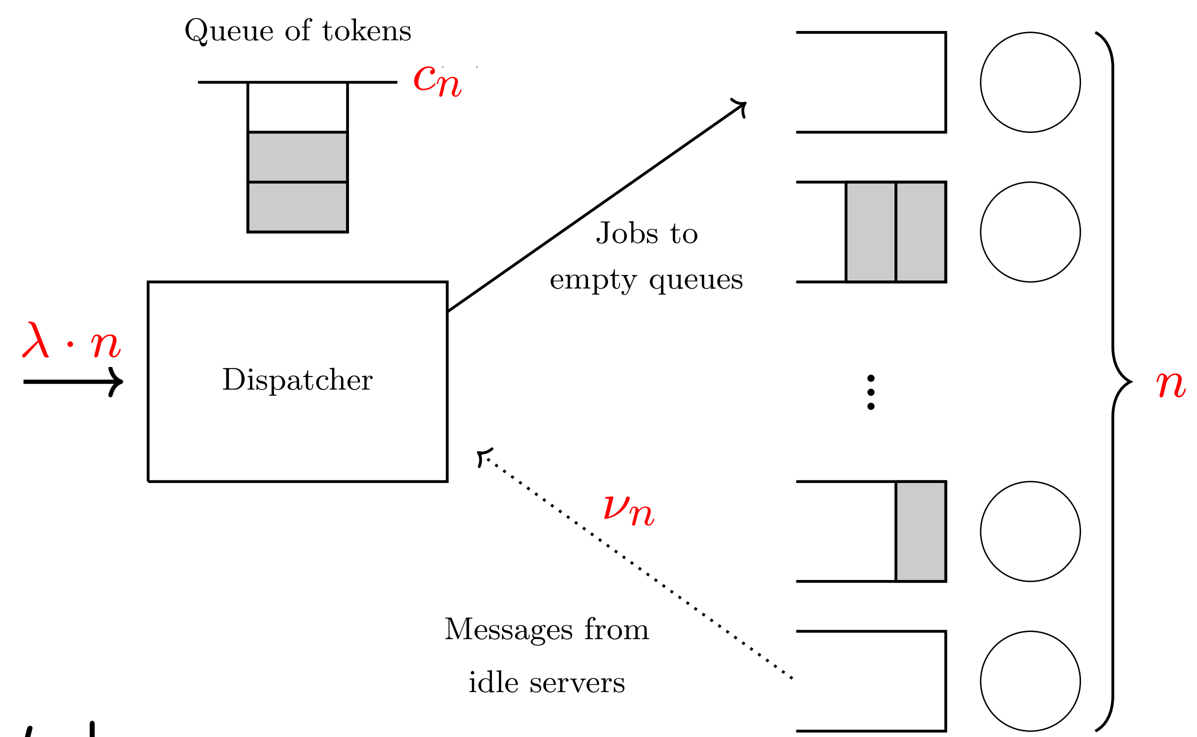

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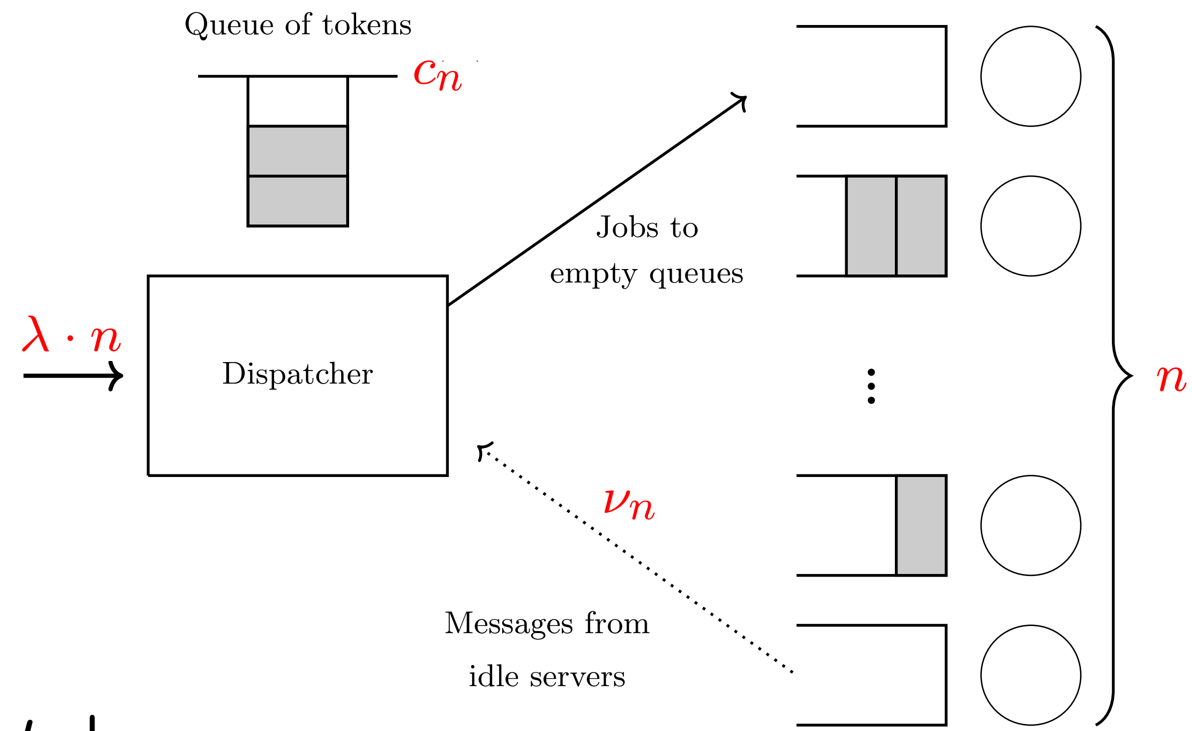

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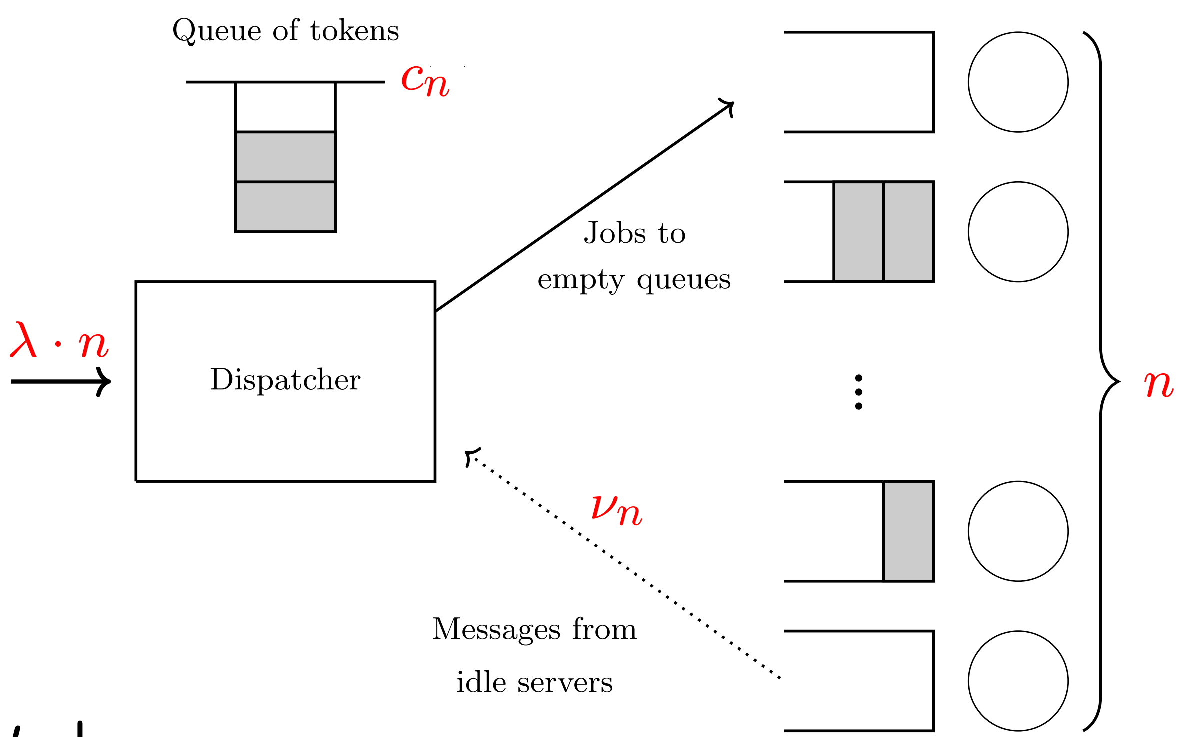

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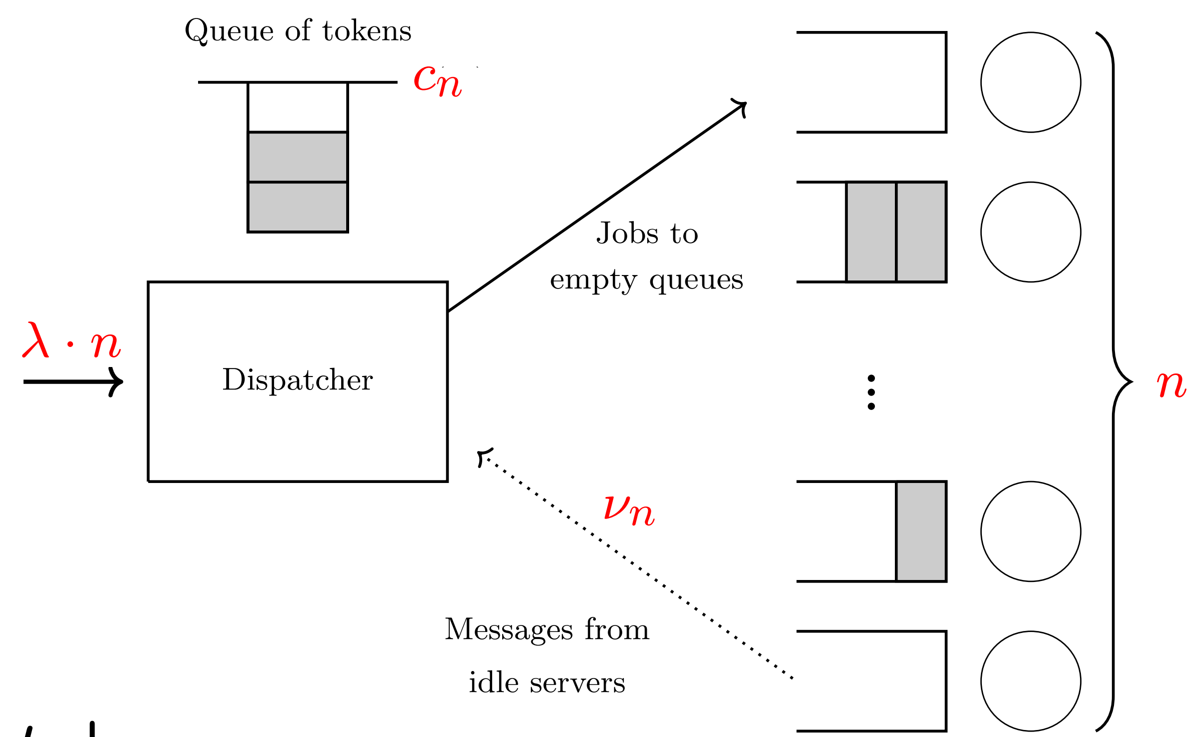

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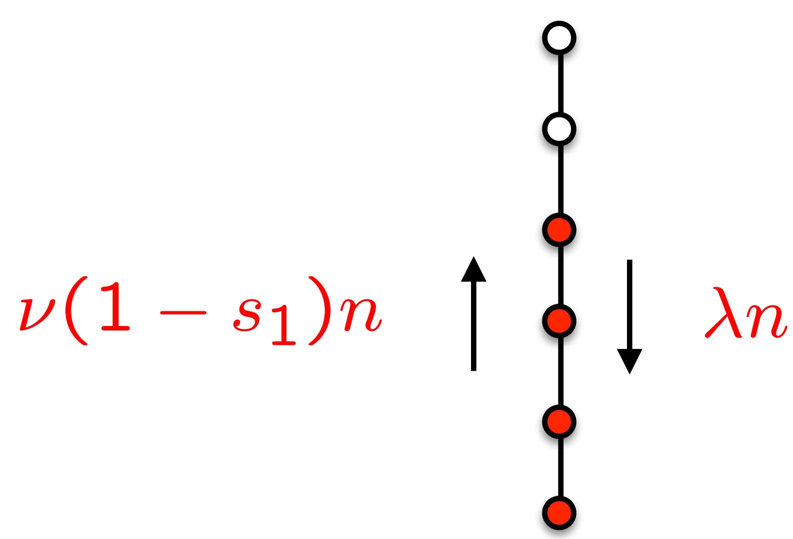

The “easy” cases

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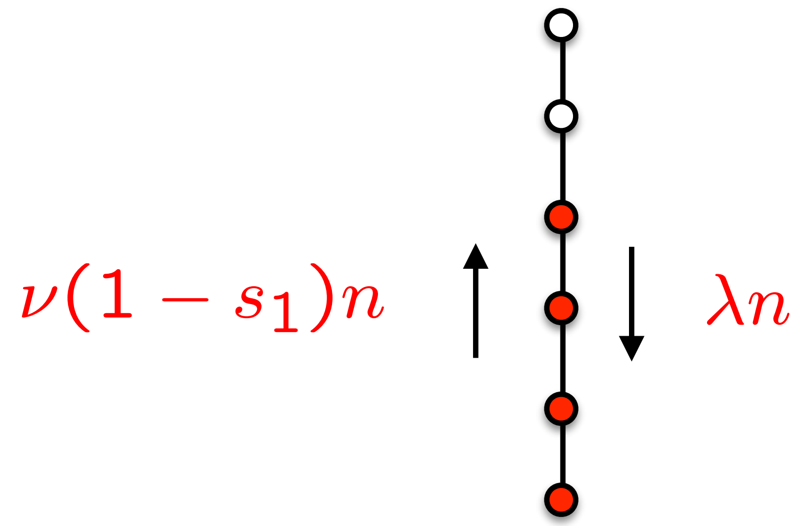

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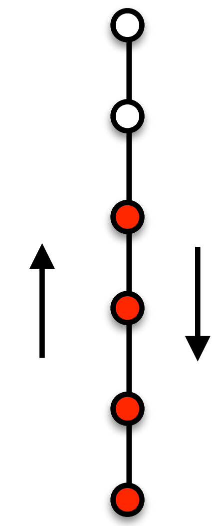

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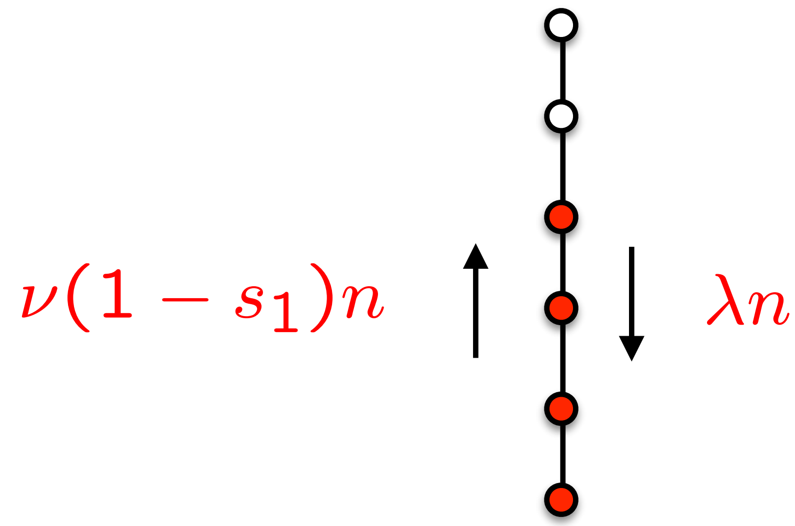

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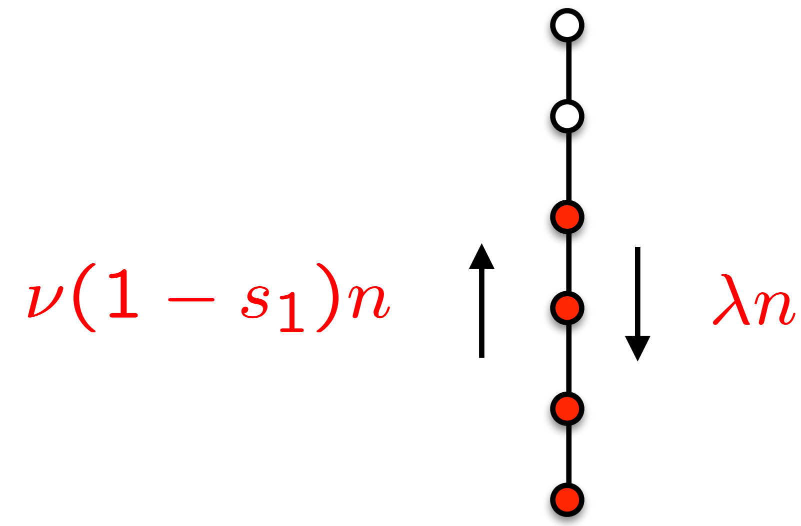

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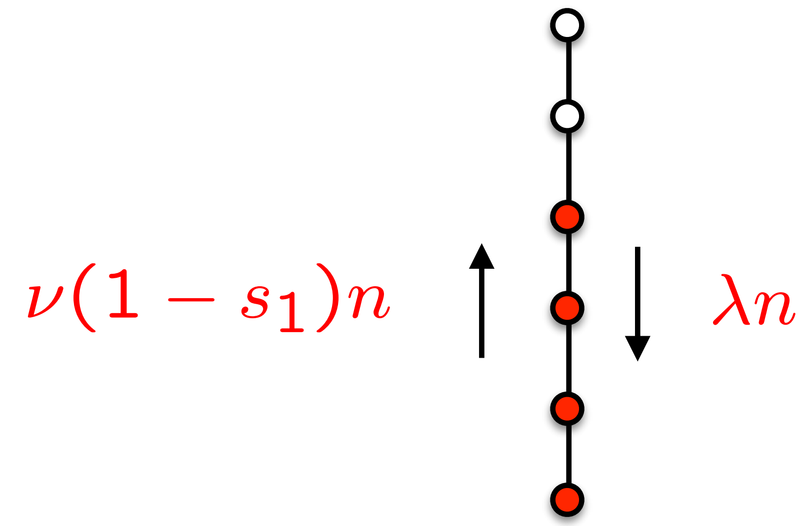

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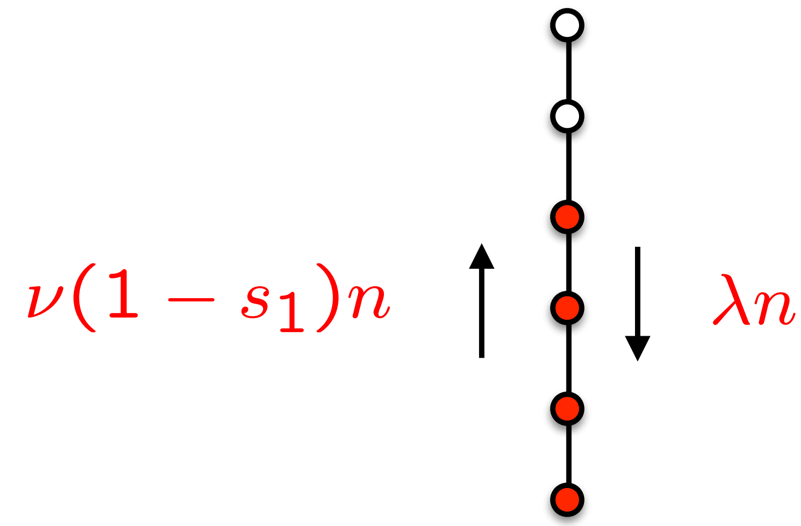

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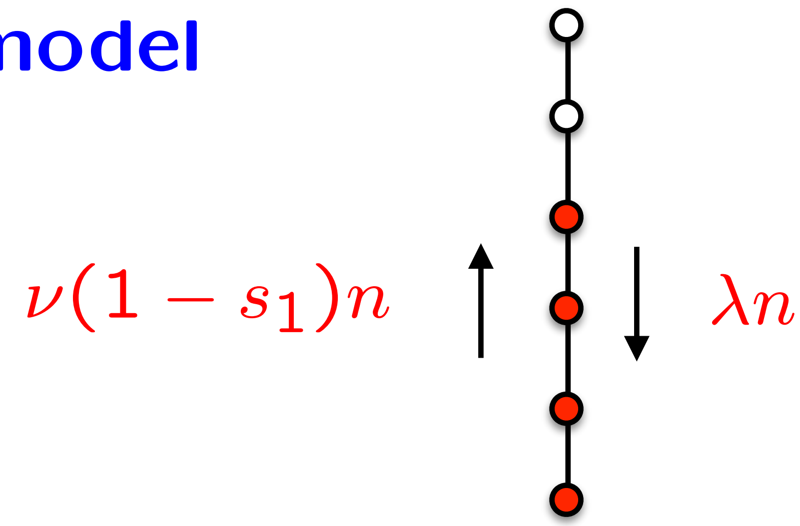
  - In steady state: $s_1 = \lambda$ (Little’s law)
  - delay $\to 0$ iff $\nu(1 - \lambda) \geq \lambda$

The resource constrained case - fluid model

$$P_0(s) = \left[\sum_{k=0}^{C} \left(\frac{\nu(1 - s_1)}{\lambda} \right)^k \right]^{-1}$$



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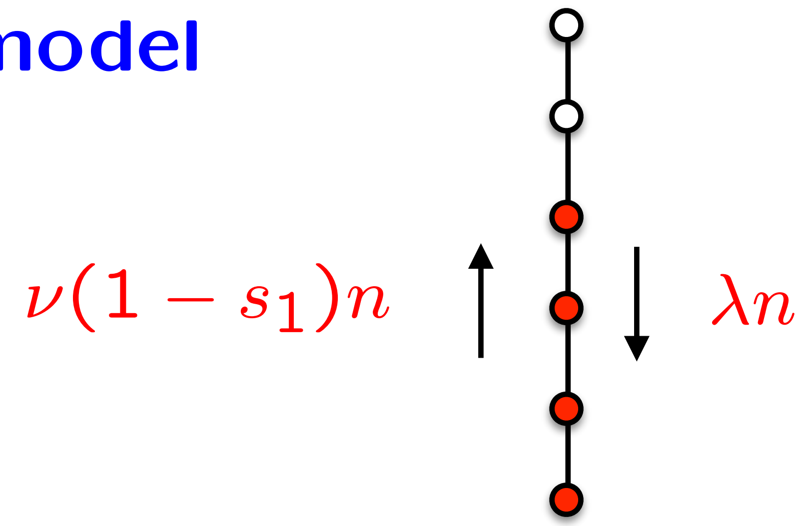

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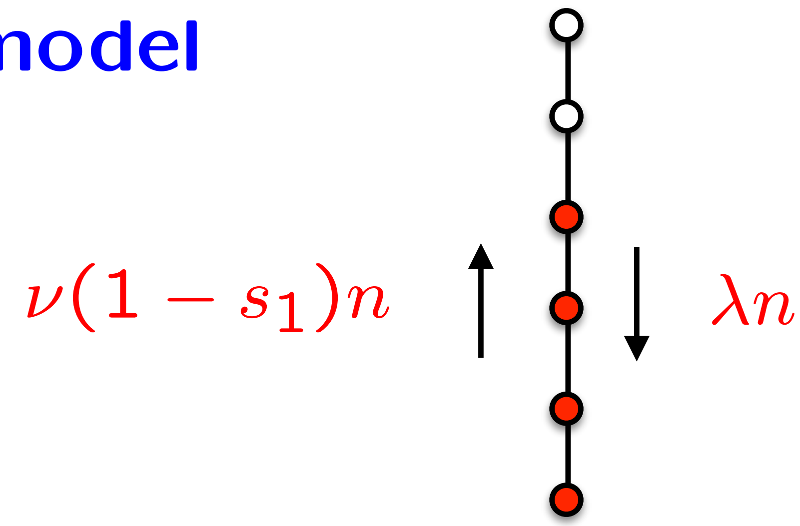

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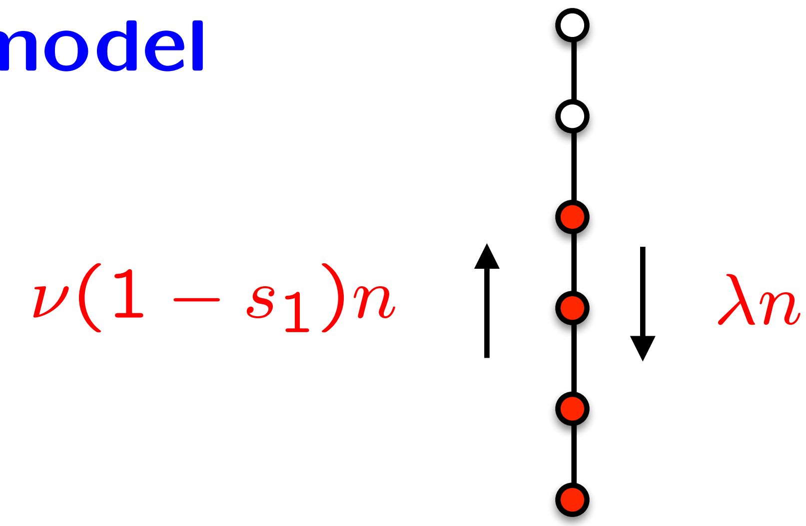

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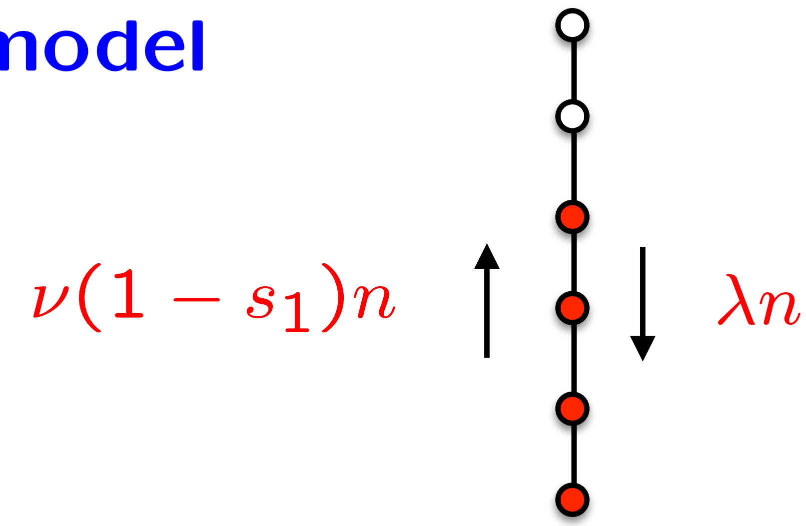

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- Unique equilibrium point s^* (algebra) which is asymptotically stable for all (interesting) initial conditions (sandwich between tractable solutions)

The theorems (ctd.)

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S^n(t) \arrow[r, "n \to \infty"] \arrow[d, "t \to \infty"'] & s(t) \arrow[d, "t \to \infty"] \\
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- standard “proof technology”
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  - Messages per unit time: $\lambda d n$
  - P_0^* (and therefore, delay): independent of λ

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- Choose ν so that $\nu(1-\lambda) = \lambda d$
  - Messages per unit time: λdn
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Delay analysis (resource constrained case)

- $s_1^* = \lambda$
- $P_0^* = P_0(s^*) = \left[\sum_{k=0}^{C} \left(\frac{\nu(1-\lambda)}{\lambda}\right)^k\right]^{-1}$
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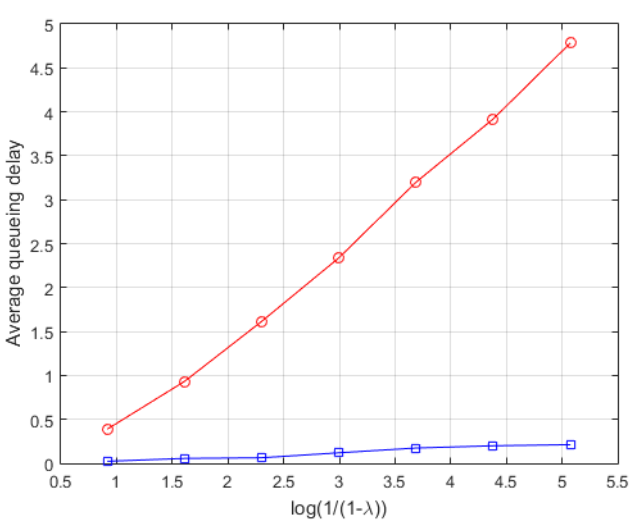

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- If we get $C+1$ arrivals in a row, and no messages from idle servers, at least one job will be sent to a "random" server

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- Conjecture: the impossibility result holds for arbitrary (non-symmetric) policies