Closed-loop policies for curing epidemics on graphs

John N. Tsitsiklis

(with K. Drakopoulos and A. Ozdaglar)

January 2016

Outline

- Motivation
- The SIS contagion/epidemic model
- Static versus dynamic policies
  - line and mesh examples

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  - fast extinction if budget $\geq c \cdot$CutWidth
- A lower bound
  - slow extinction if budget $\leq c' \cdot$CutWidth

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- Extensions, open problems, future work

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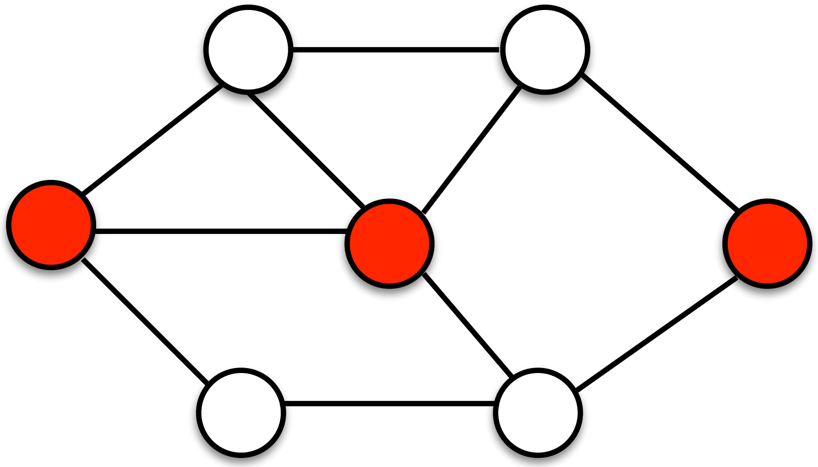
- Develop dynamic strategies
  - use information about current state

    Borgs et al., 2010 (exact)
    Khanafer and Basar, 2014 (mean field approximation)

The SIS (“susceptible $\to$ infected $\to$ susceptible”) model

- Undirected graph: node set V; n nodes; max-degree Δ

- State I_t: set of infected nodes at time t; I_0: given



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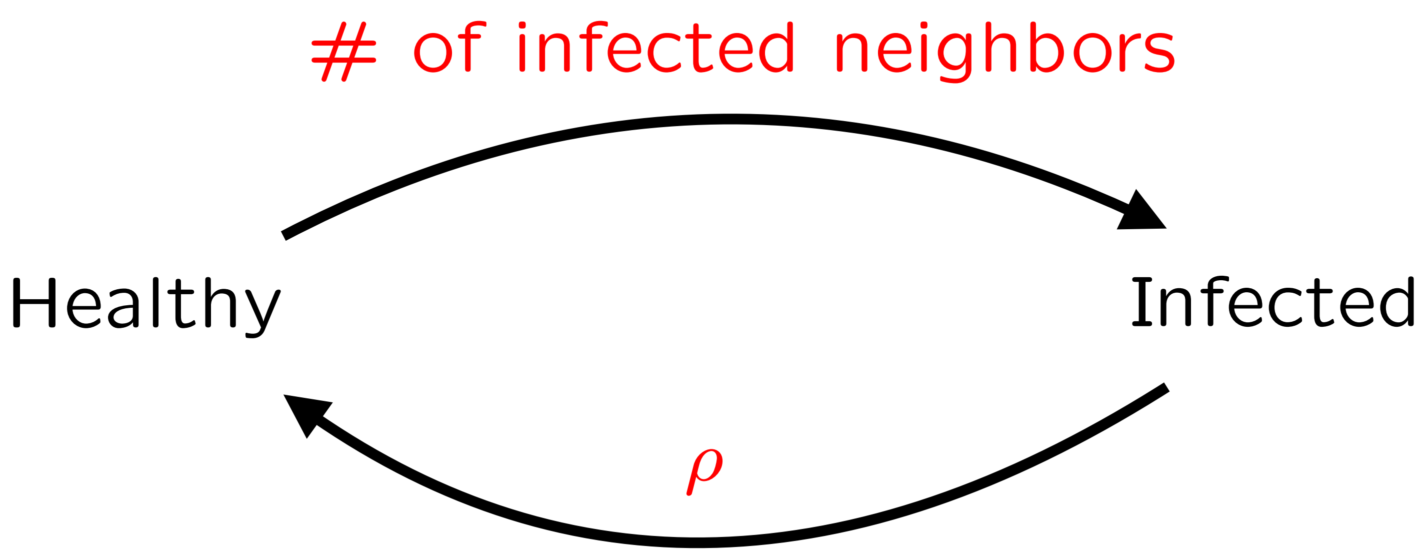

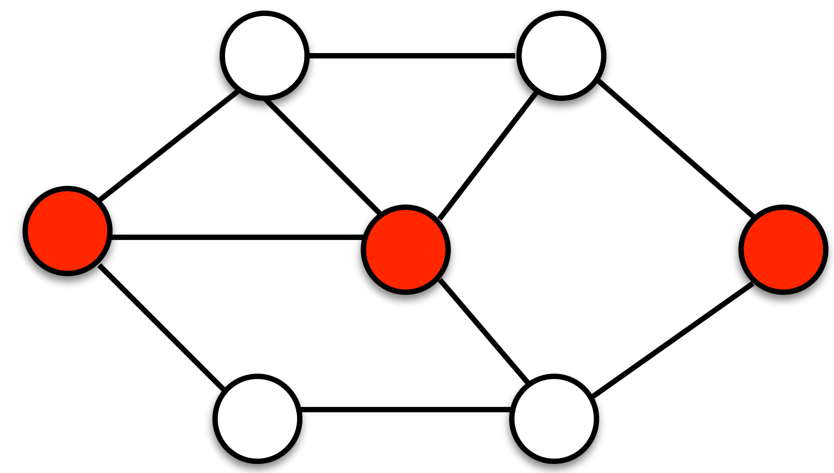

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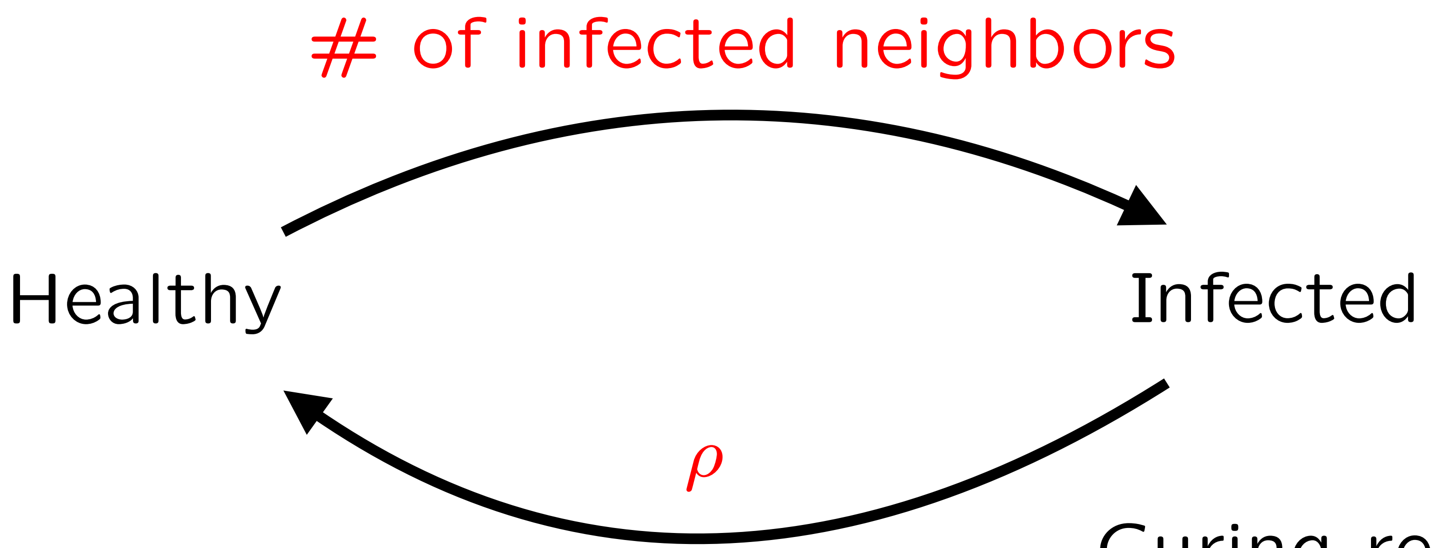

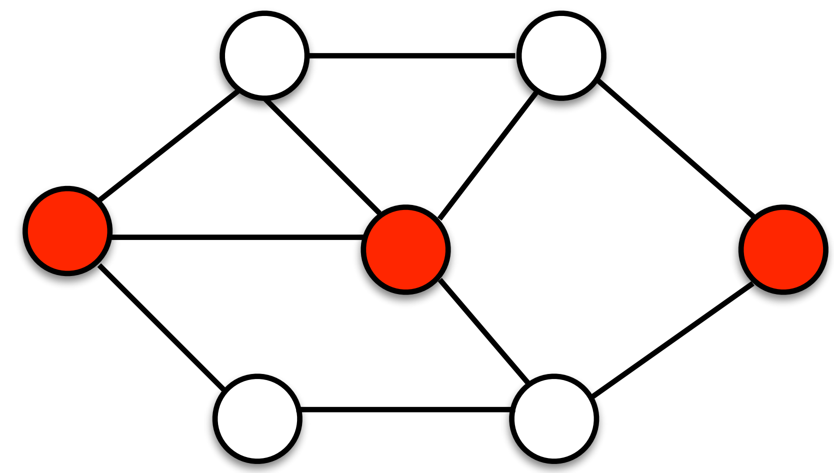

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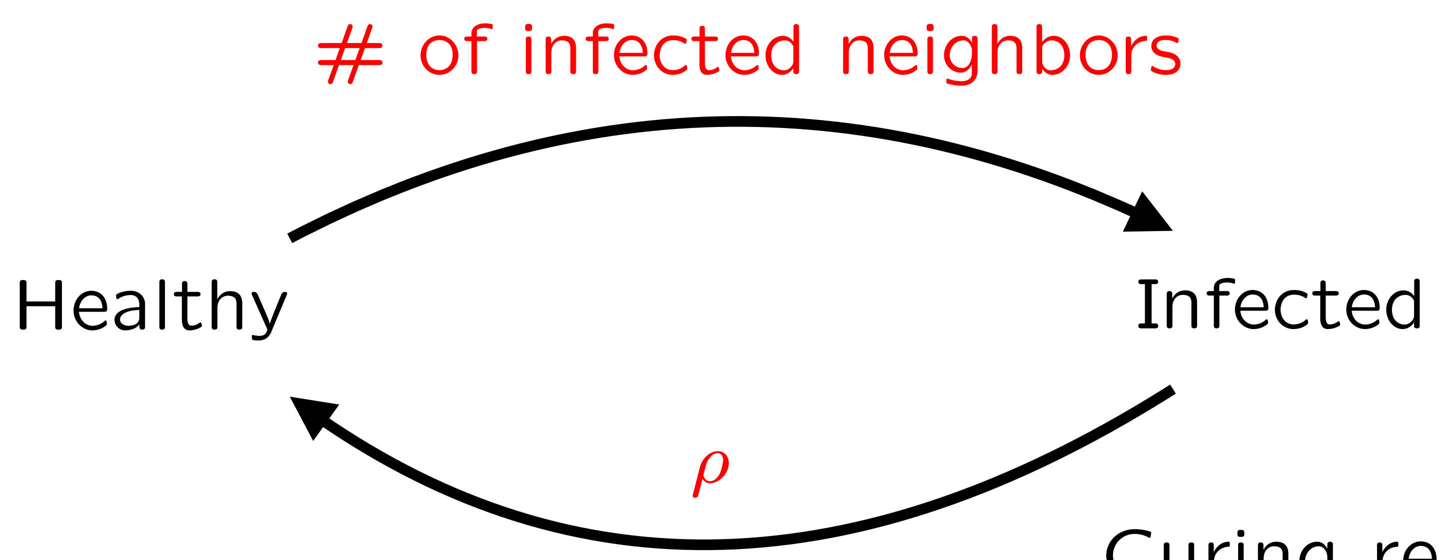

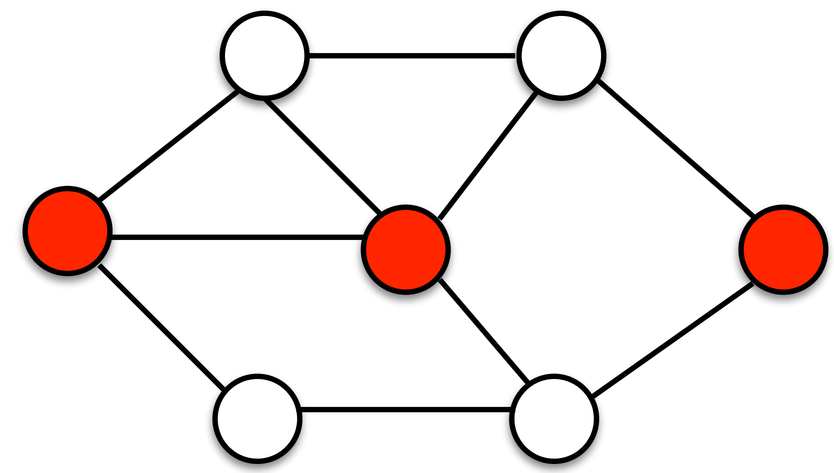

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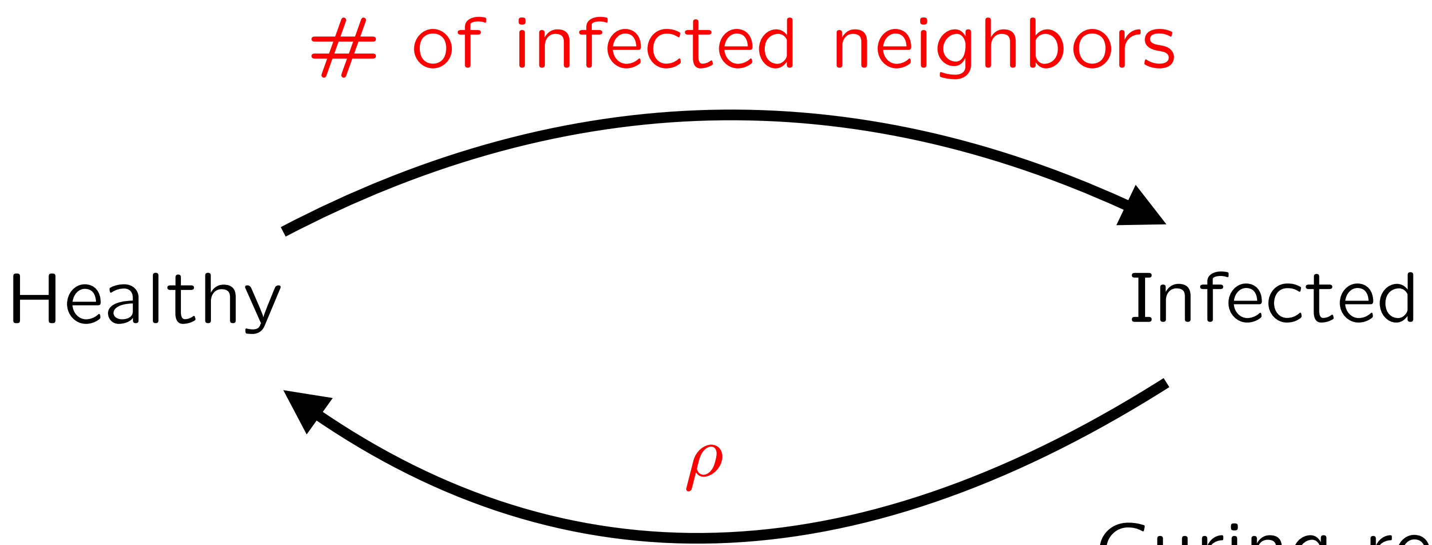

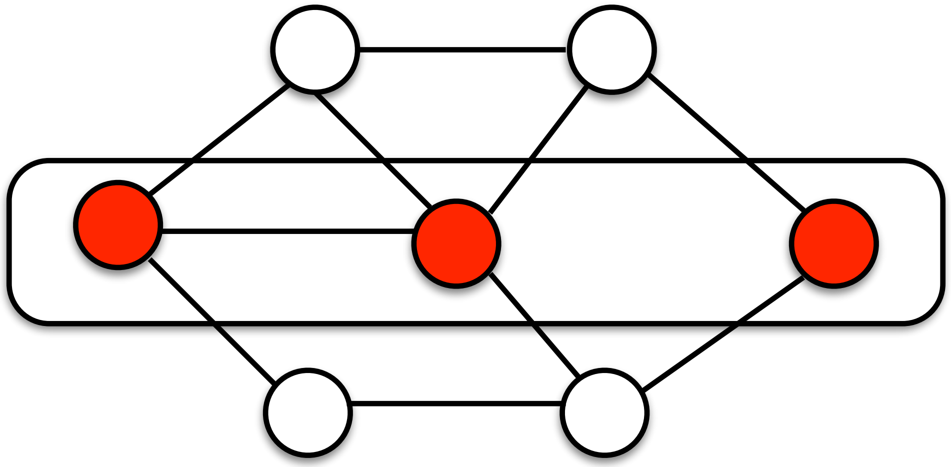

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  - infection: # of arcs joining healthy to infected nodes

    cut(I_t)

Curing rate allocation — Controlled SIS model

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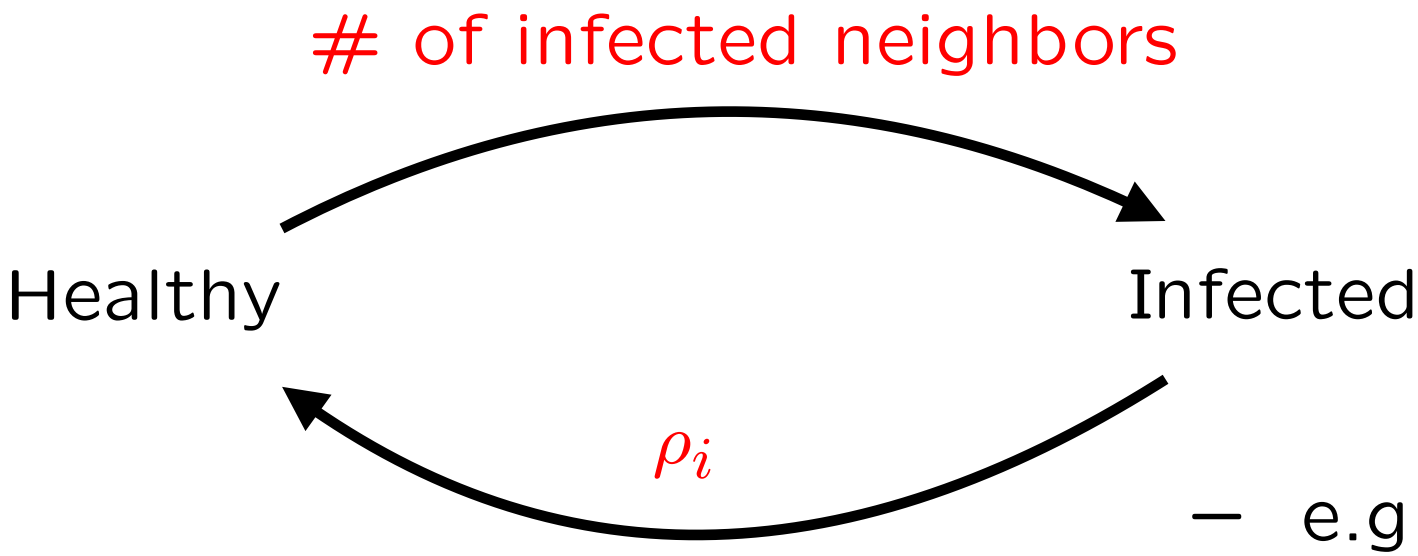

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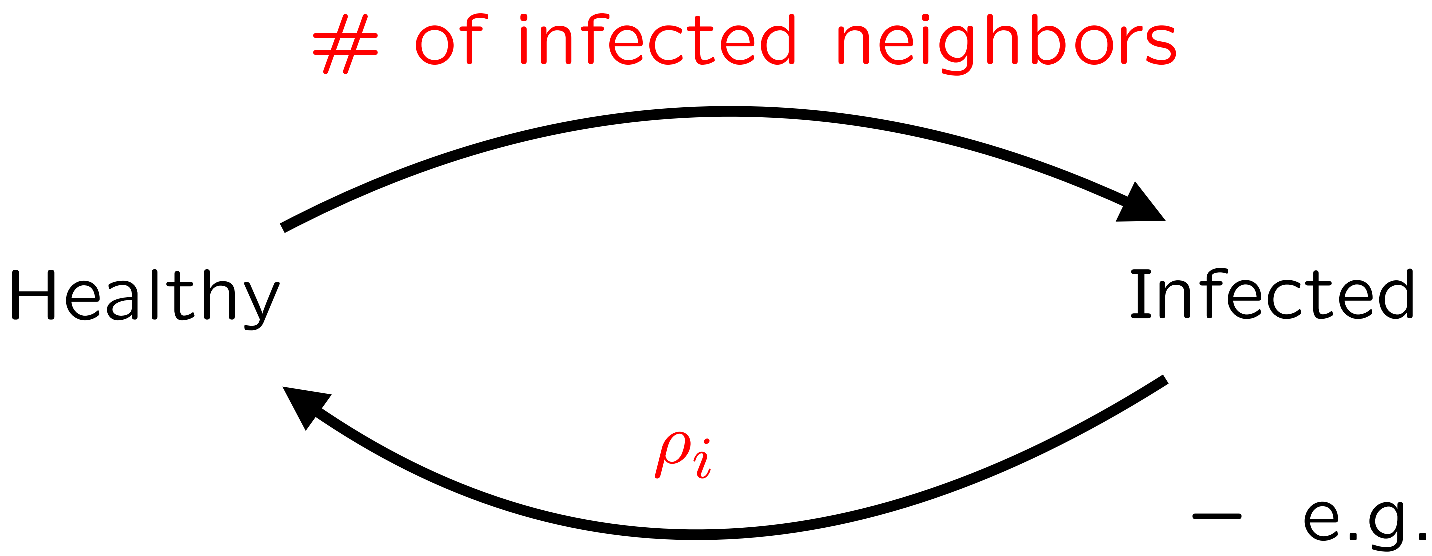

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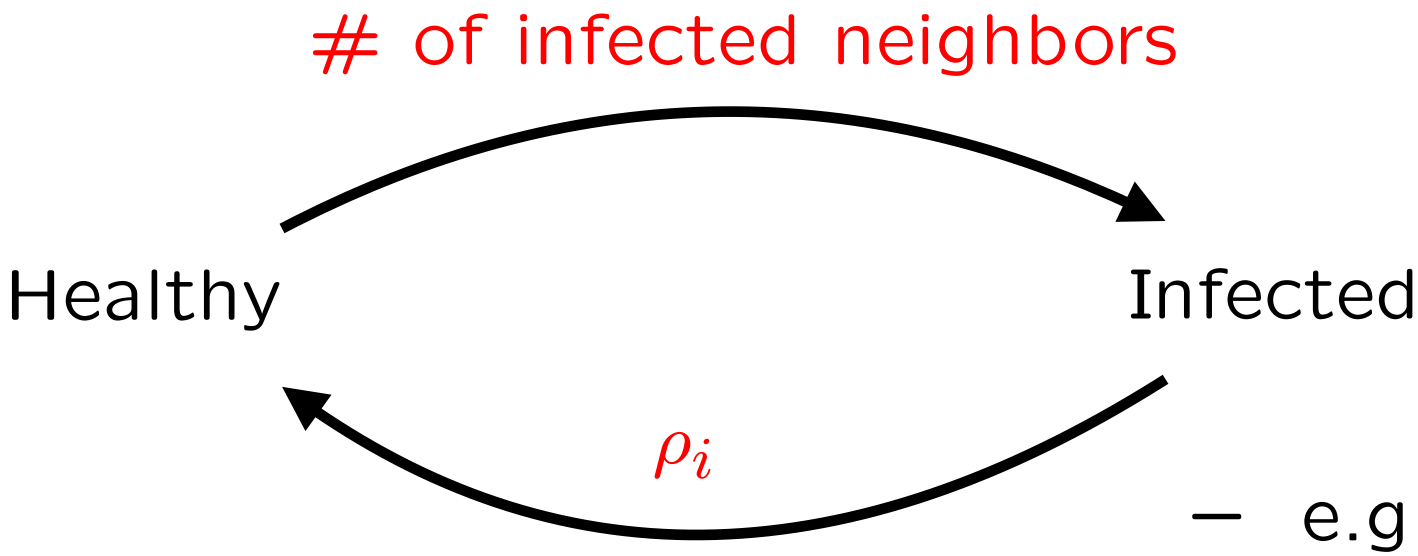

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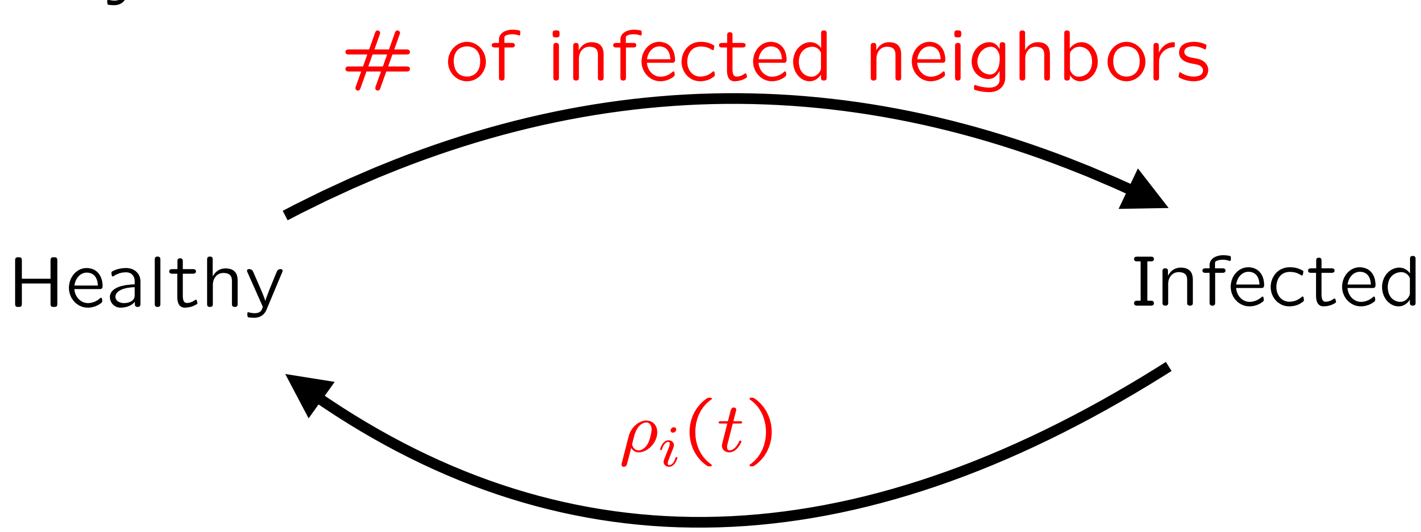

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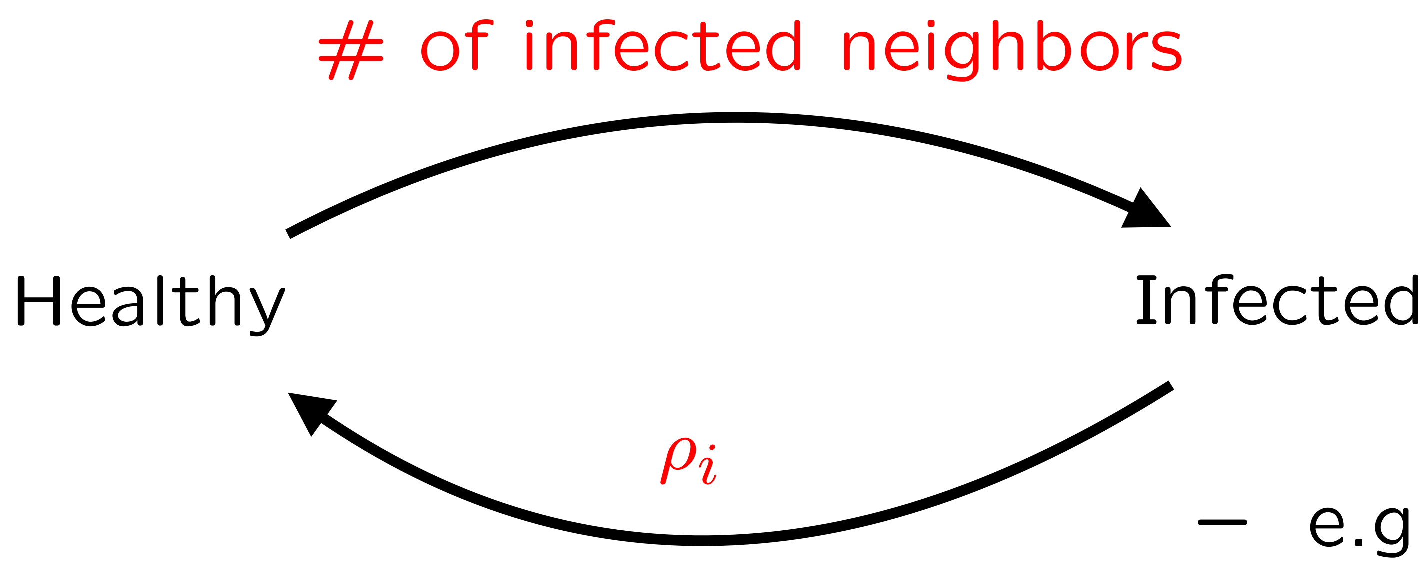

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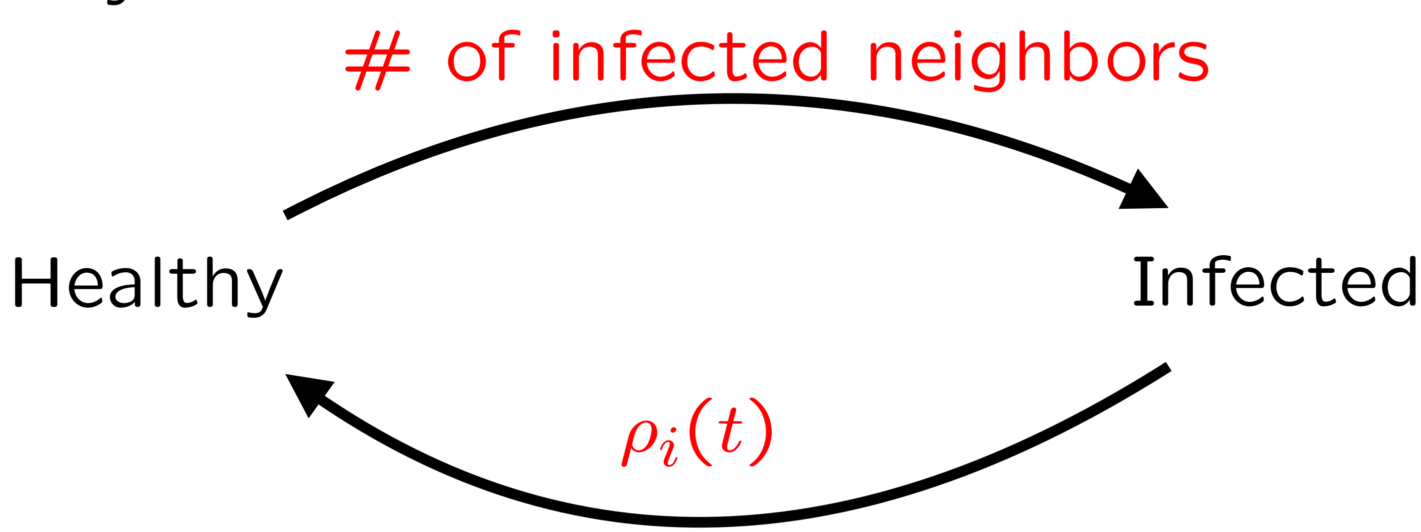

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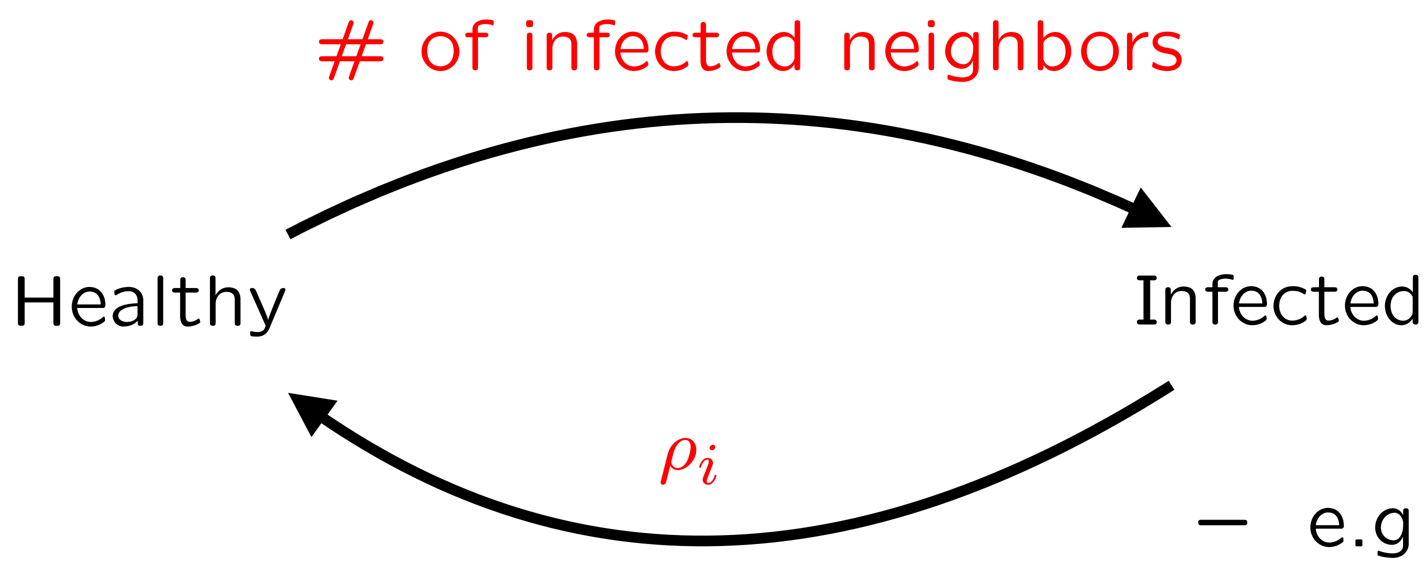

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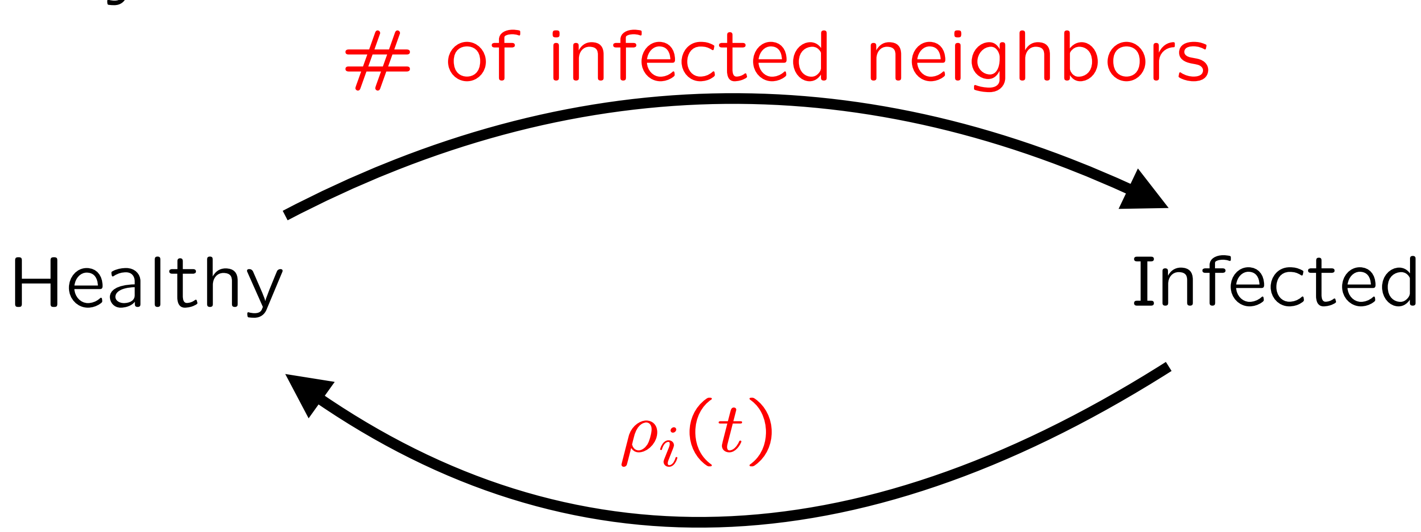

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- What B is needed to guarantee "fast extinction"?

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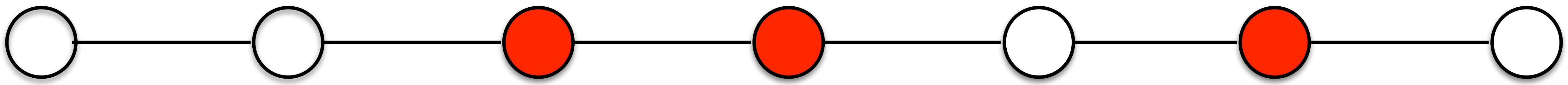
  - $|I_t|$ has downward (expected) drift
  - time to extinction is linear in n, or less

Example: Line graph



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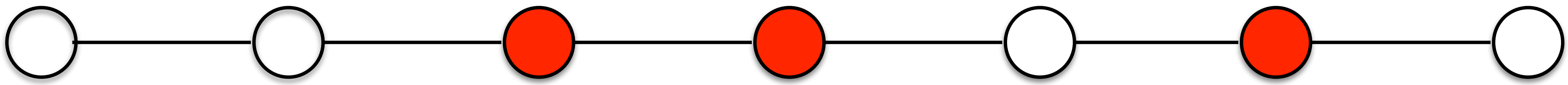
  - fast extinction needs $\rho_i > 1$, for most i $\Rightarrow$ Budget $= \Omega(n)$



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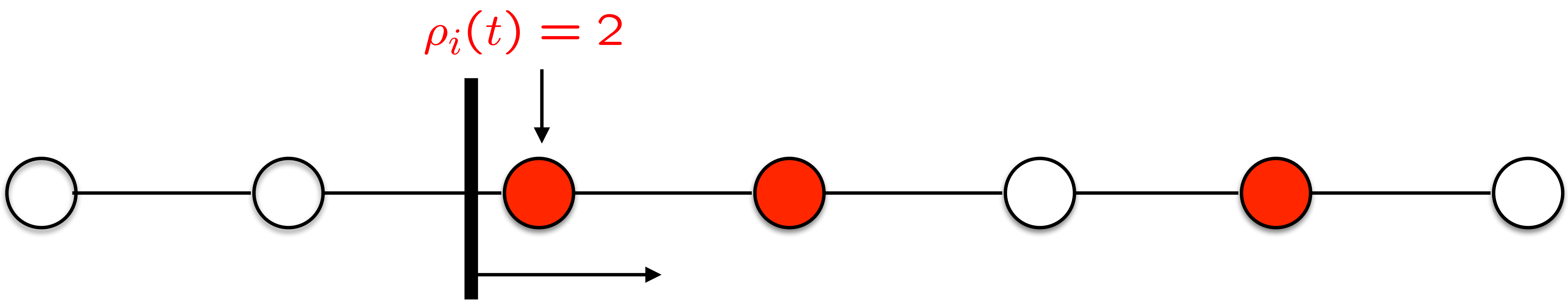

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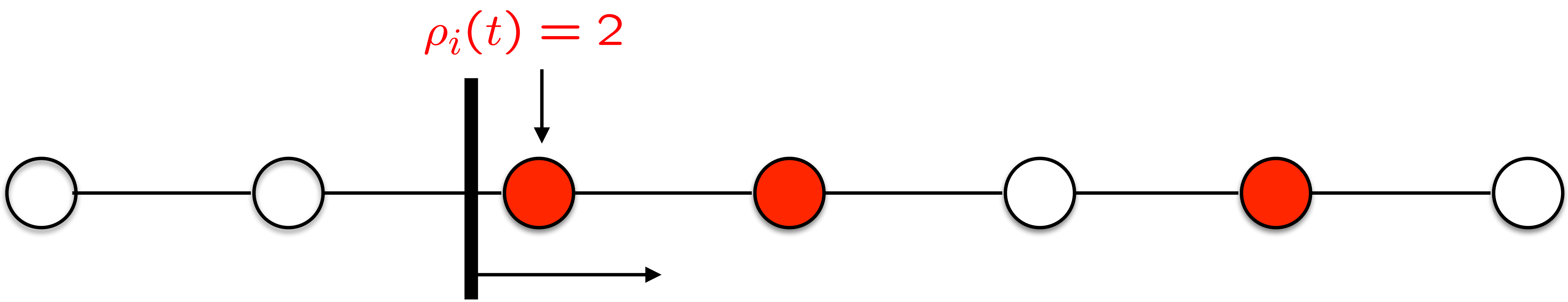

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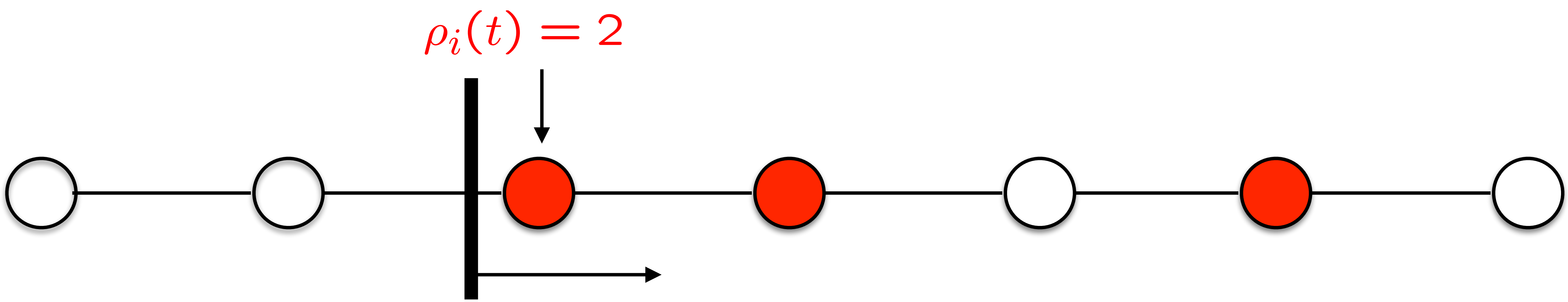

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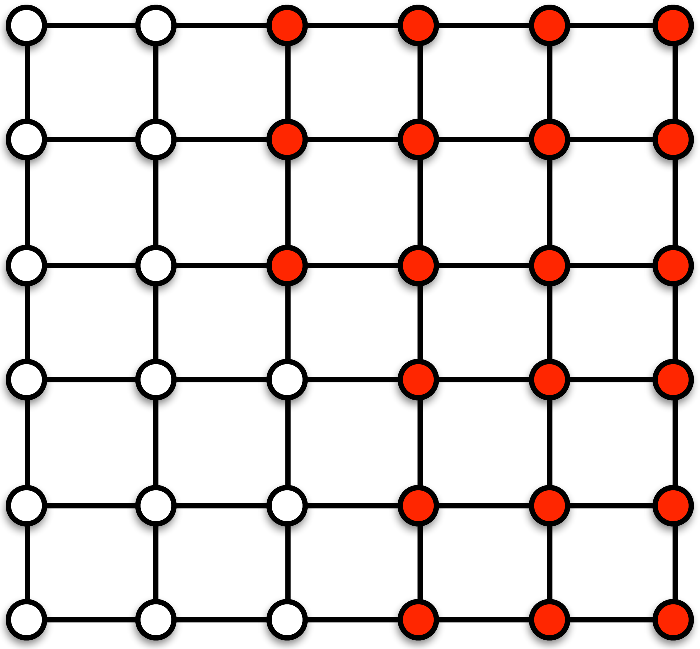

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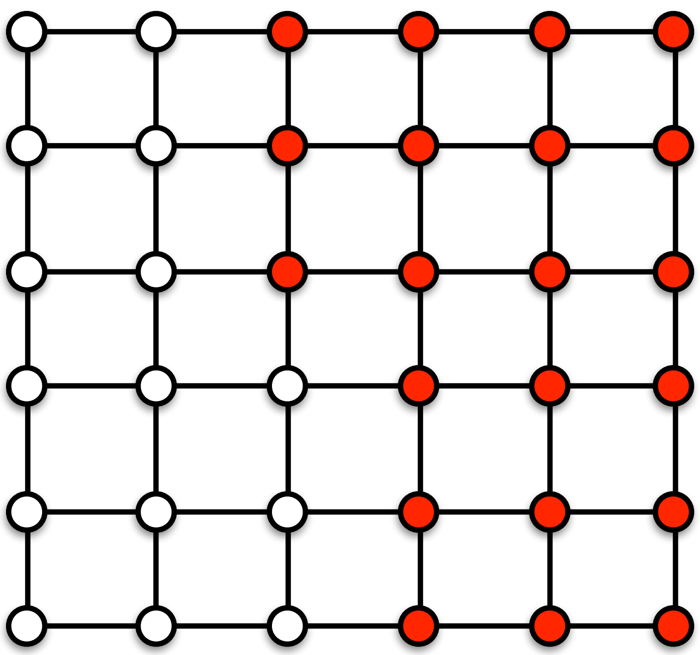

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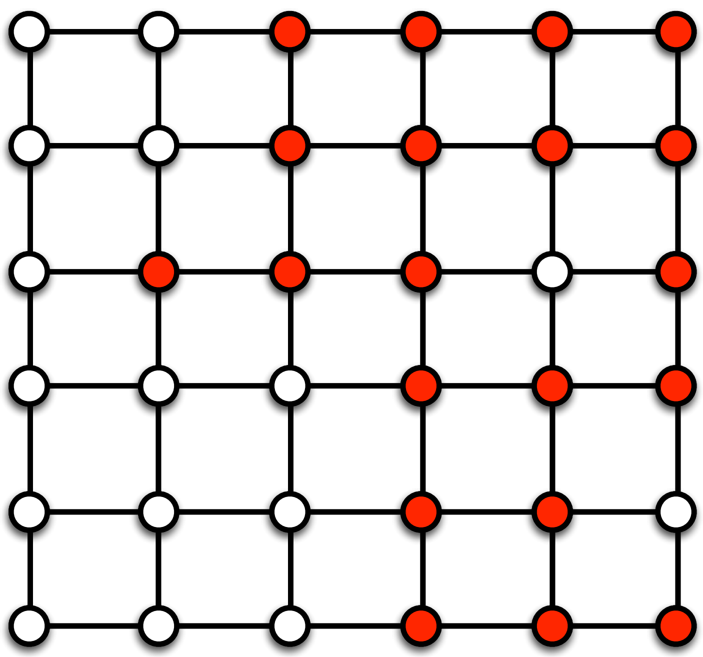

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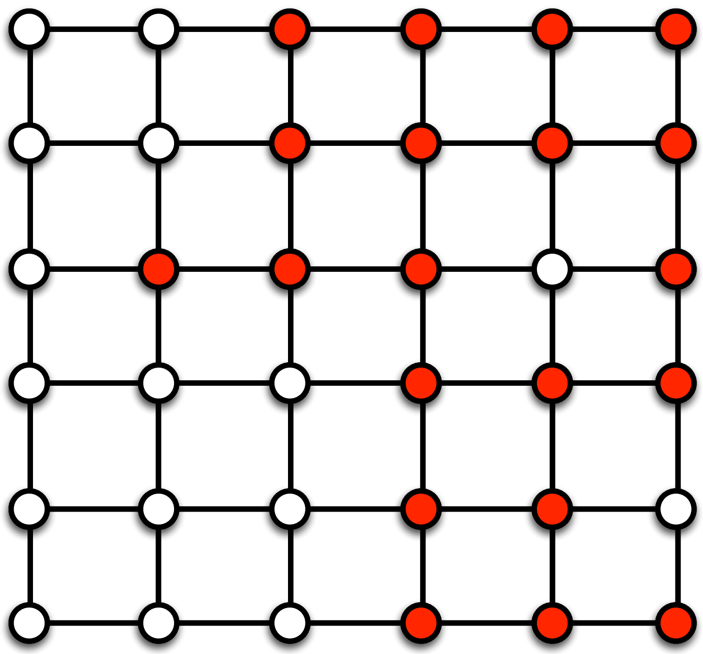

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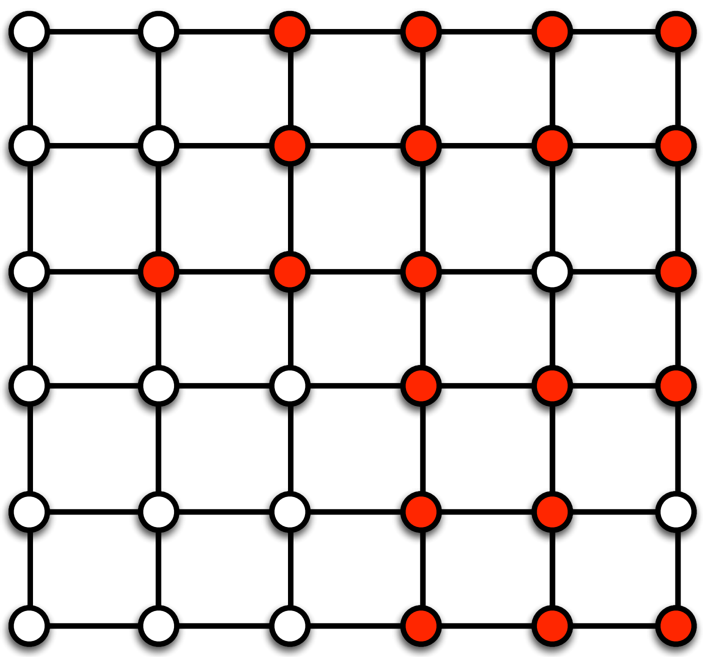

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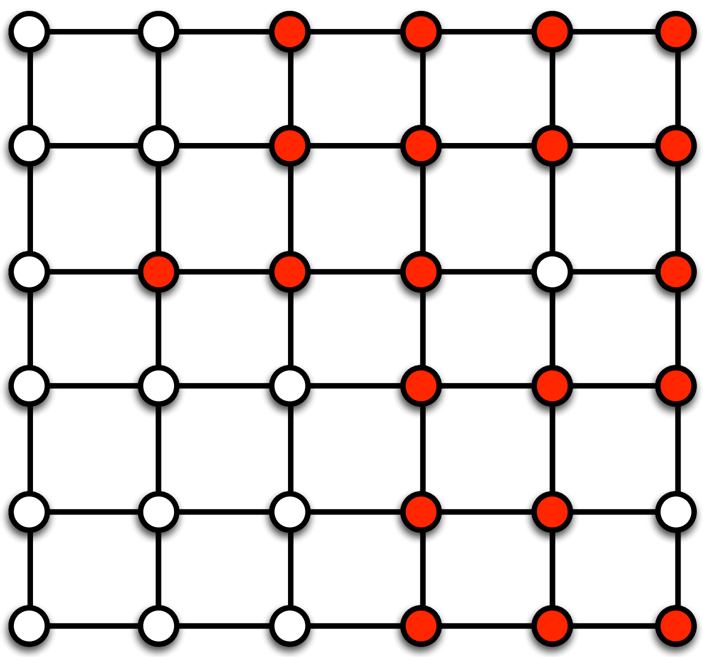

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  - to make progress: $\Omega(\sqrt{n})$ budget necessary

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- Budget should be large enough to counter large cut(I_t)
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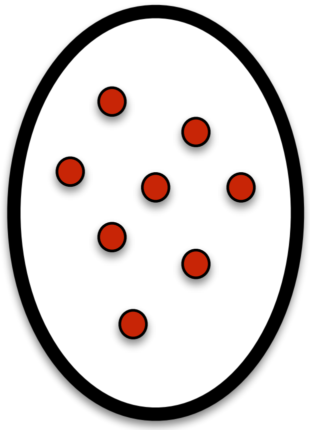

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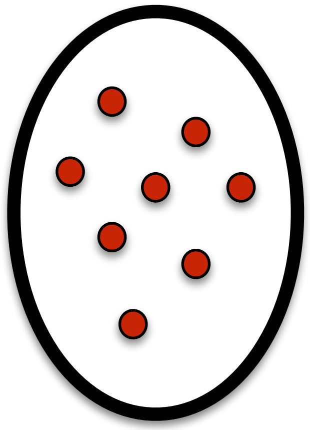

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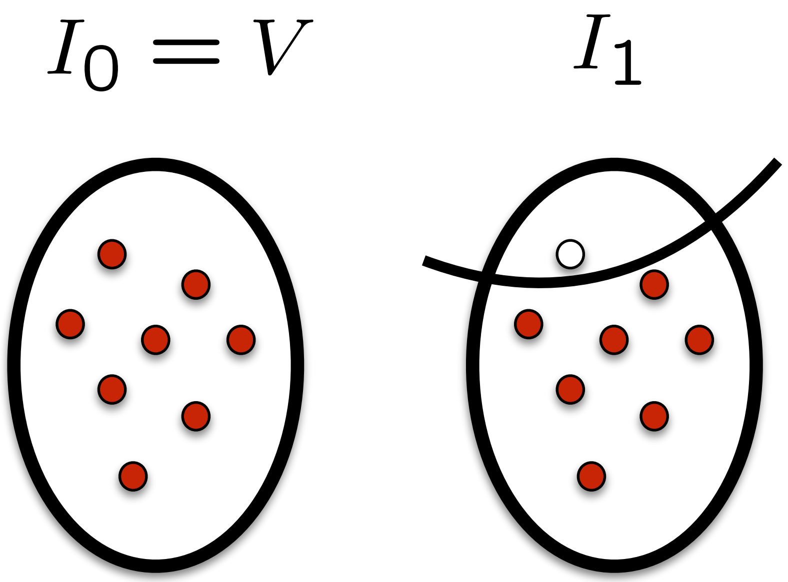

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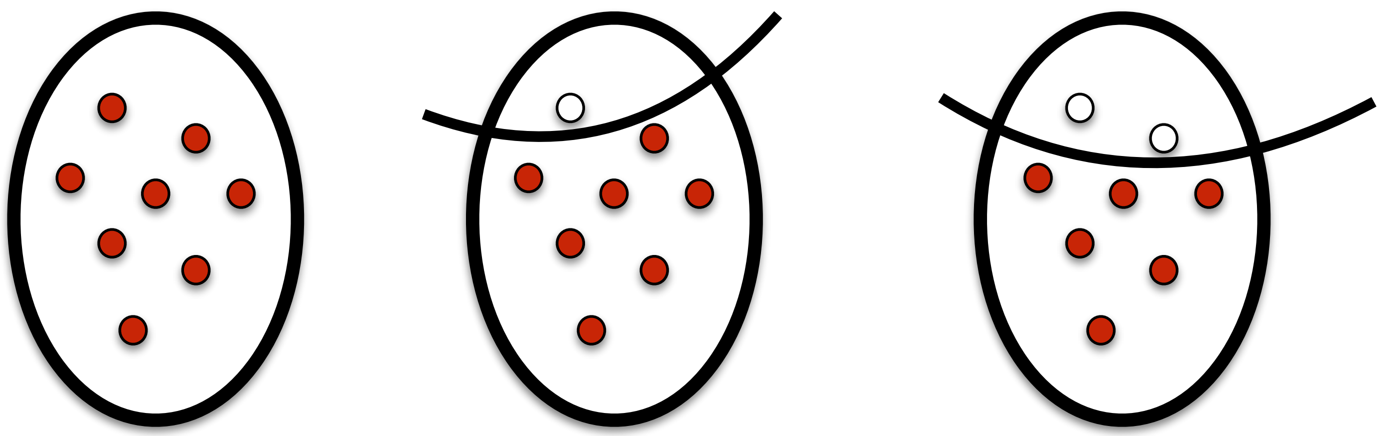

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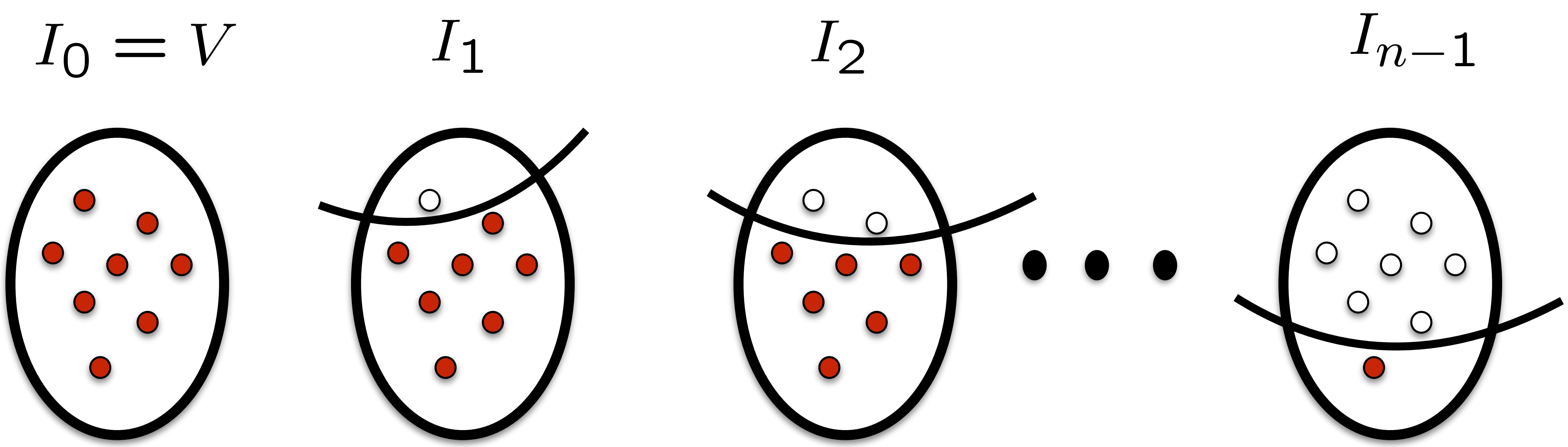

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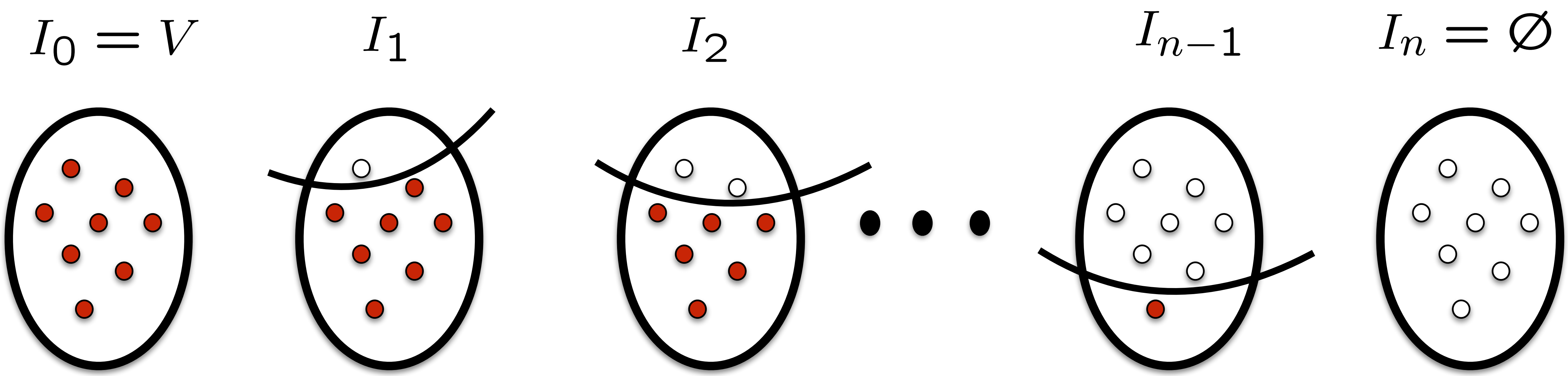

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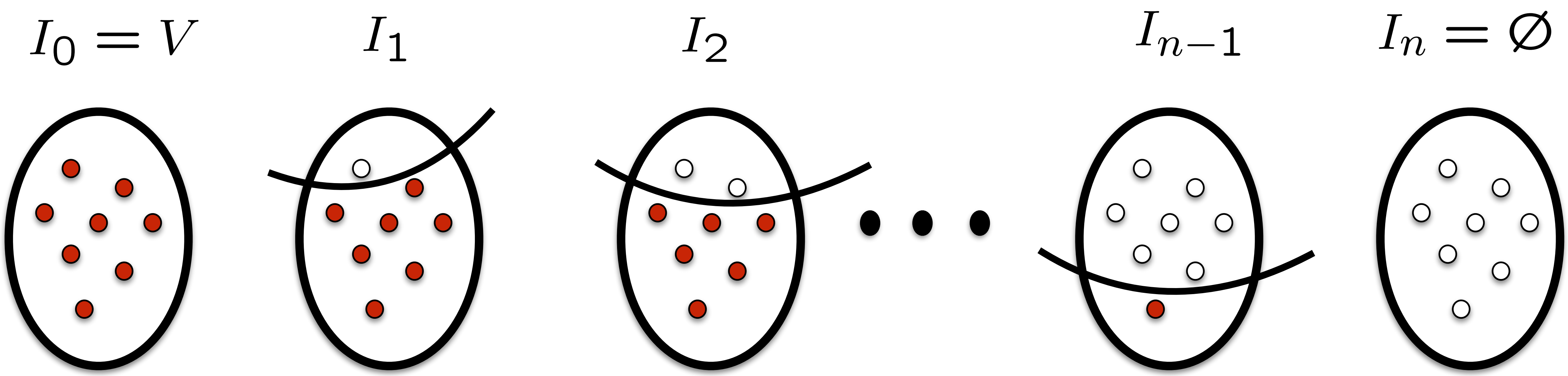

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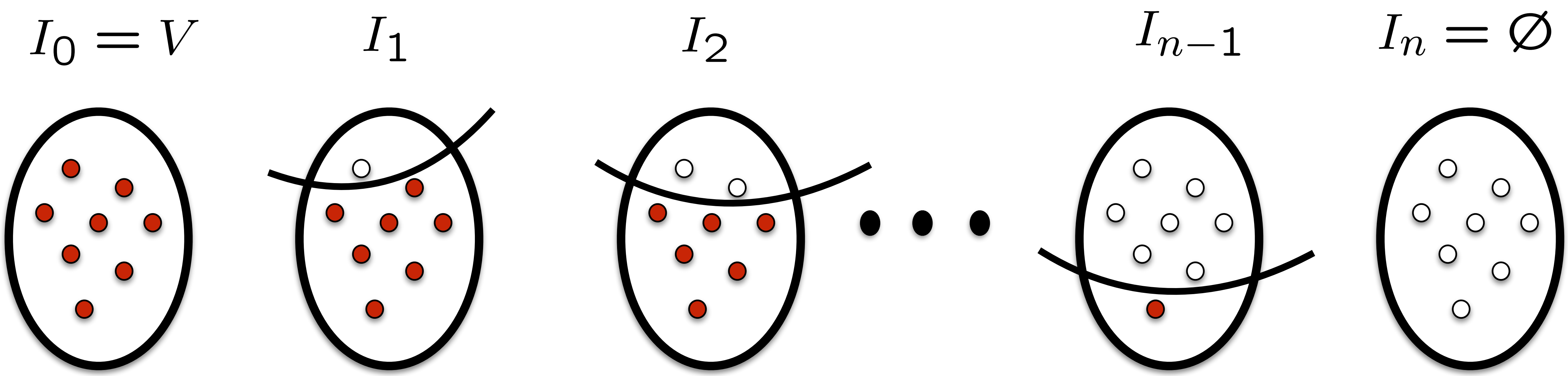

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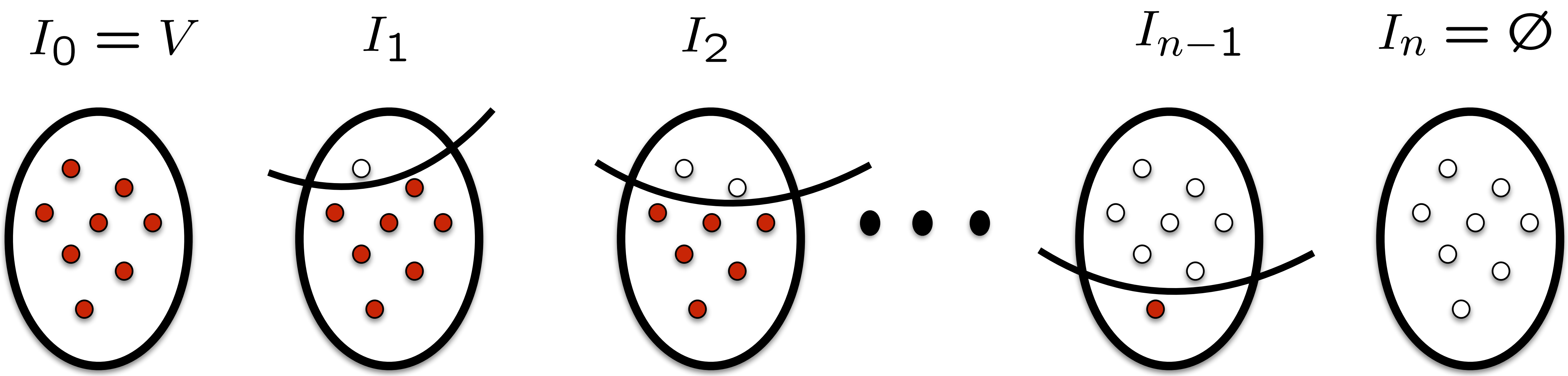

$$W = \text{CutWidth} = \min_{\text{crusades}} \left[\max_k \text{cut}(I_k) \right]$$

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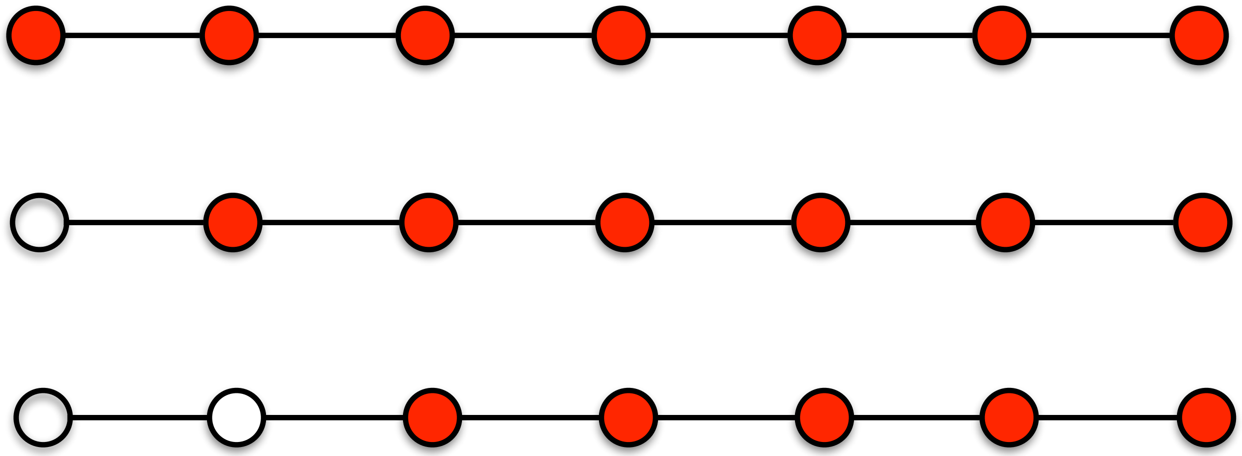

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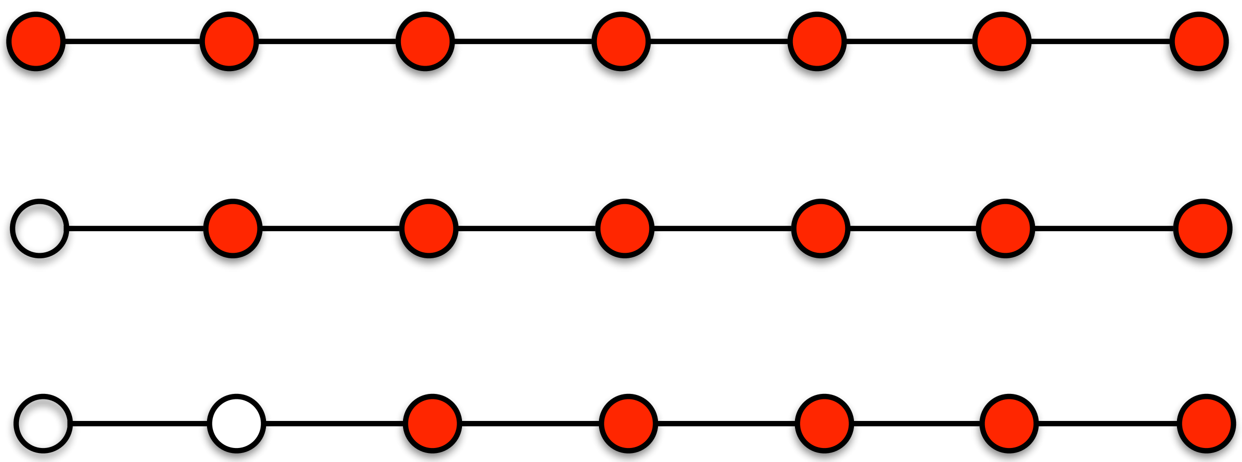

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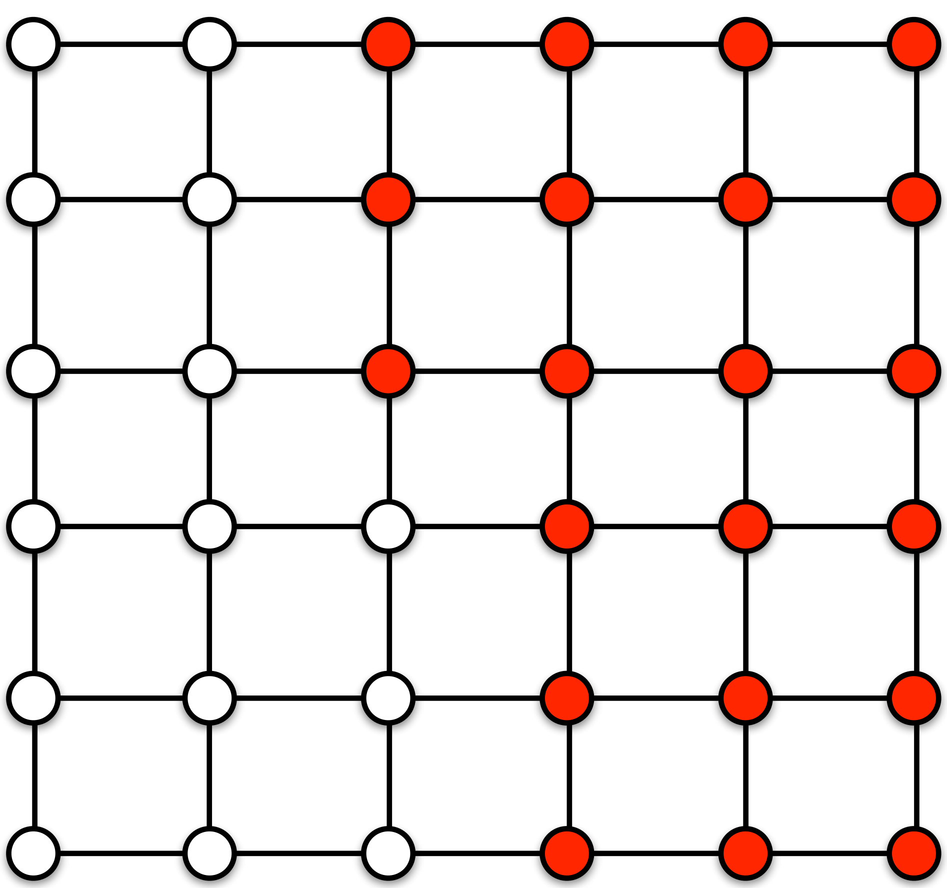

mesh: $W \approx \sqrt{n}$

Upper bound

Thm: If $B \geq 4W$, [and $B \geq \Delta \log_2 n$]

there is a policy for which: $\mathbf{E}[\text{time to extinction}] \leq 26 \cdot \frac{n}{B}$

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- **Corollary:** If W is sublinear in n [e.g., mesh], can get "fast extinction" (sublinear time), with sublinear budget.

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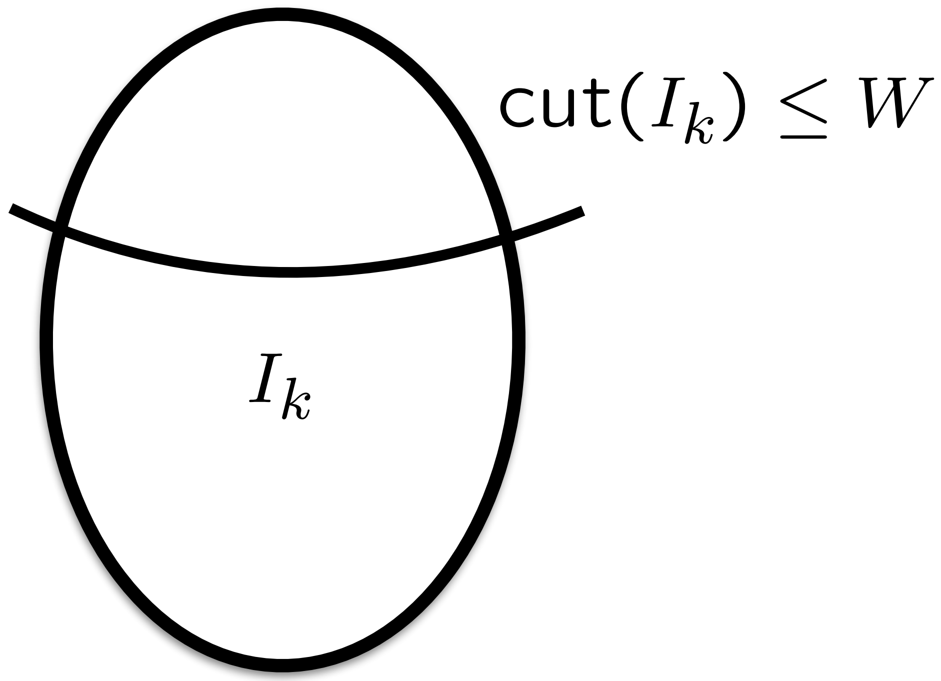

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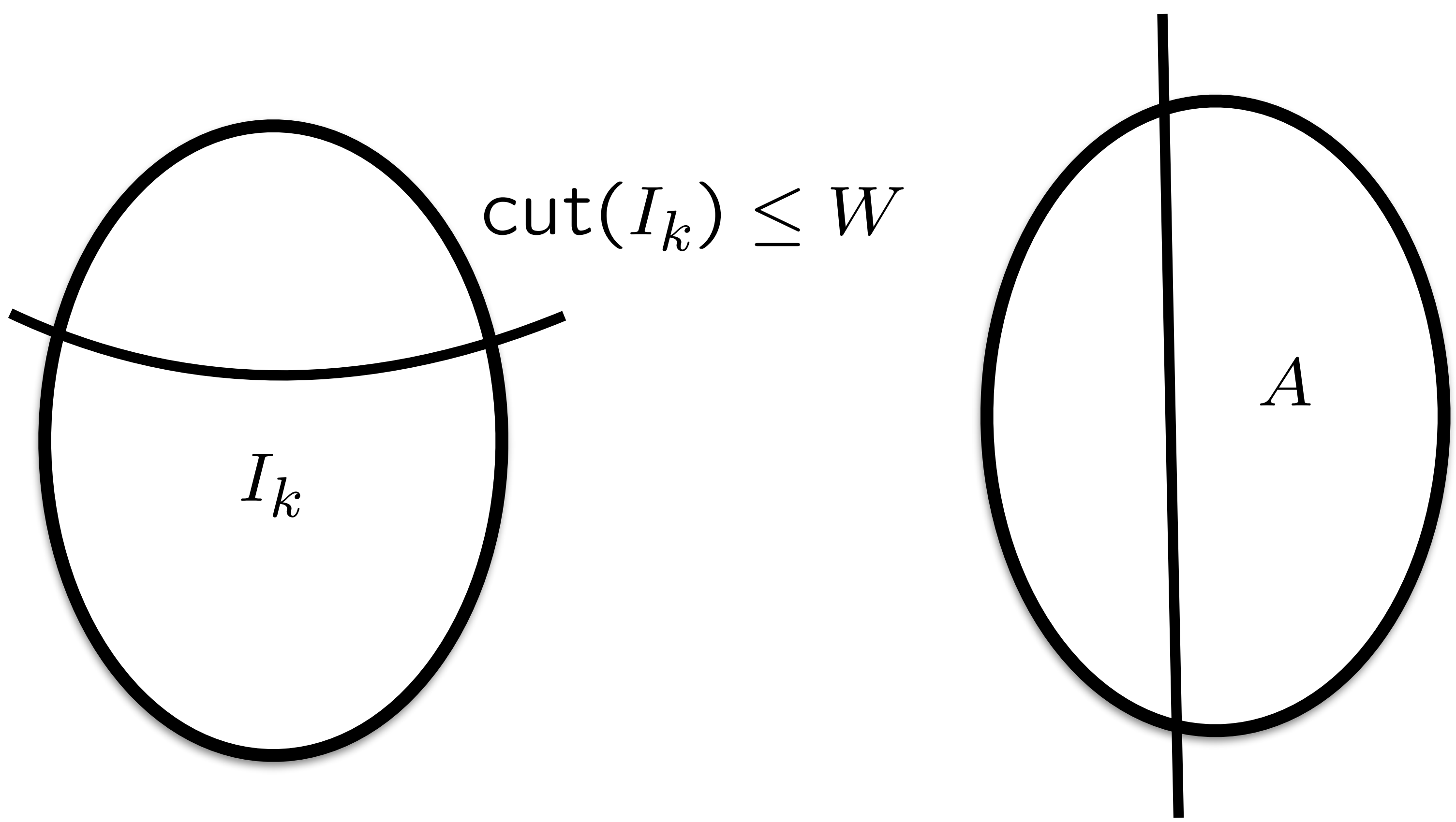

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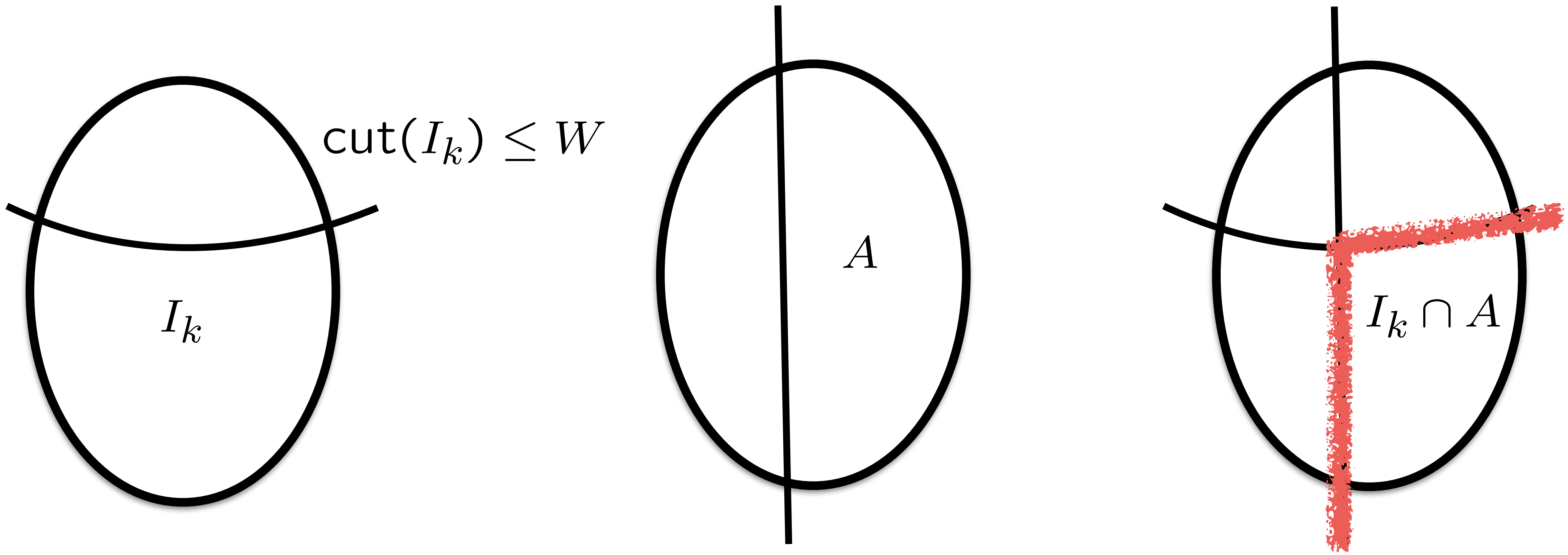
- **Proof:** Take "optimal" monotone crusade: $V = I_0, \ldots, I_n = \emptyset$

  Consider crusade $A = A \cap I_0,\ I_1 \cap A, \ldots I_n \cap A = \emptyset$



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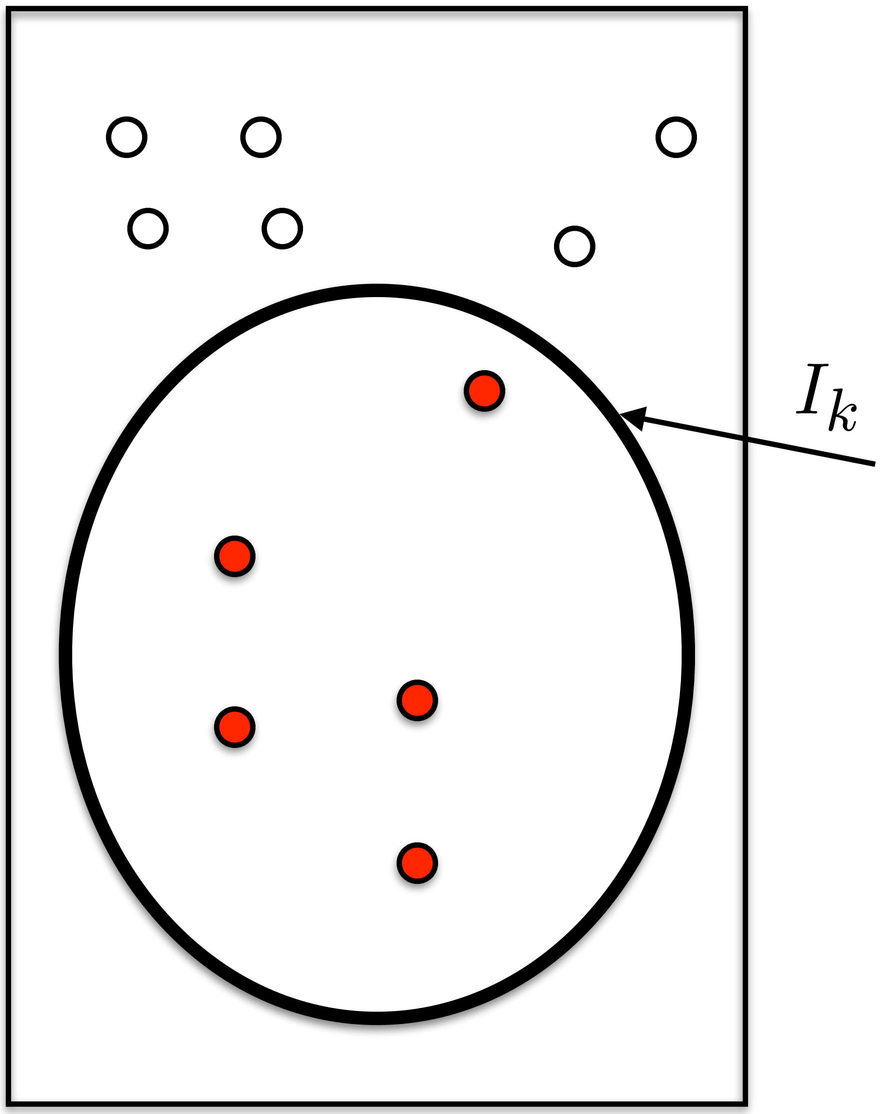

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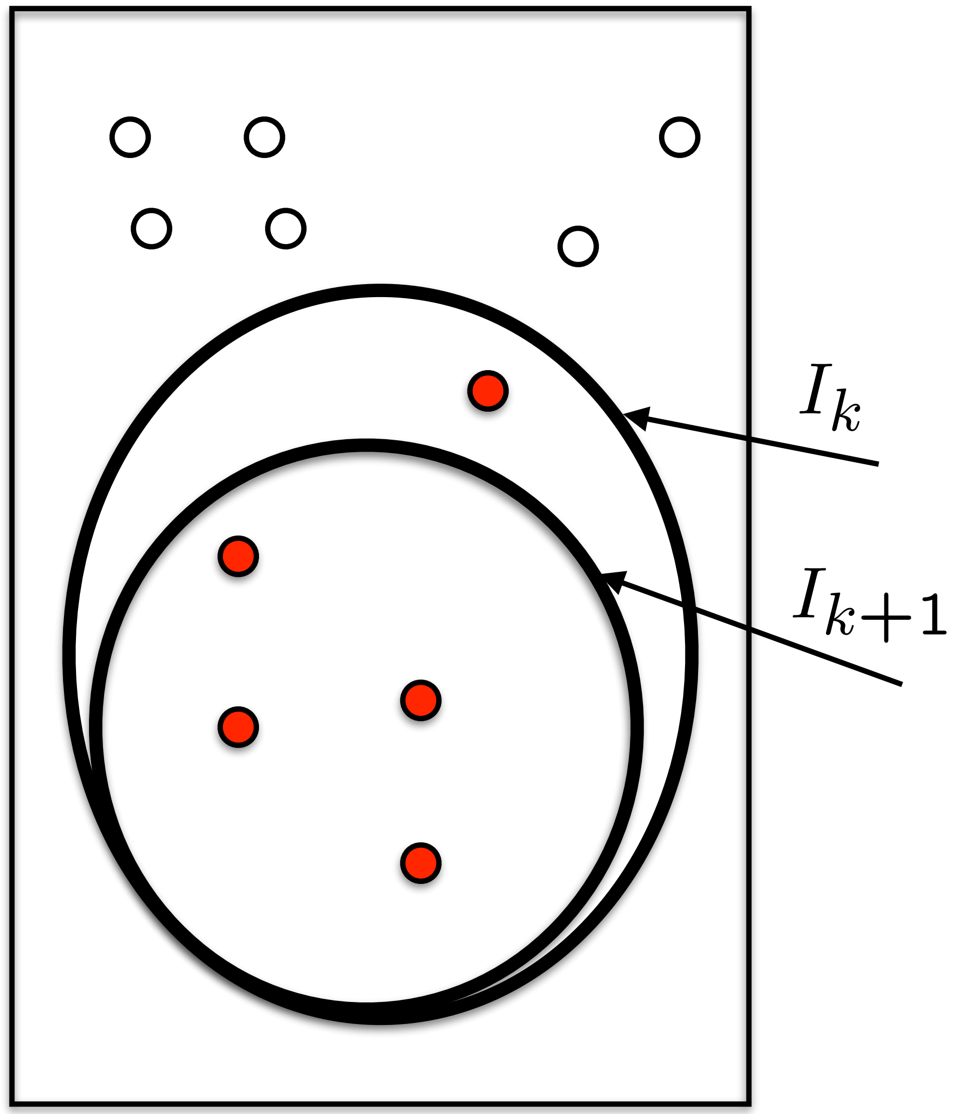

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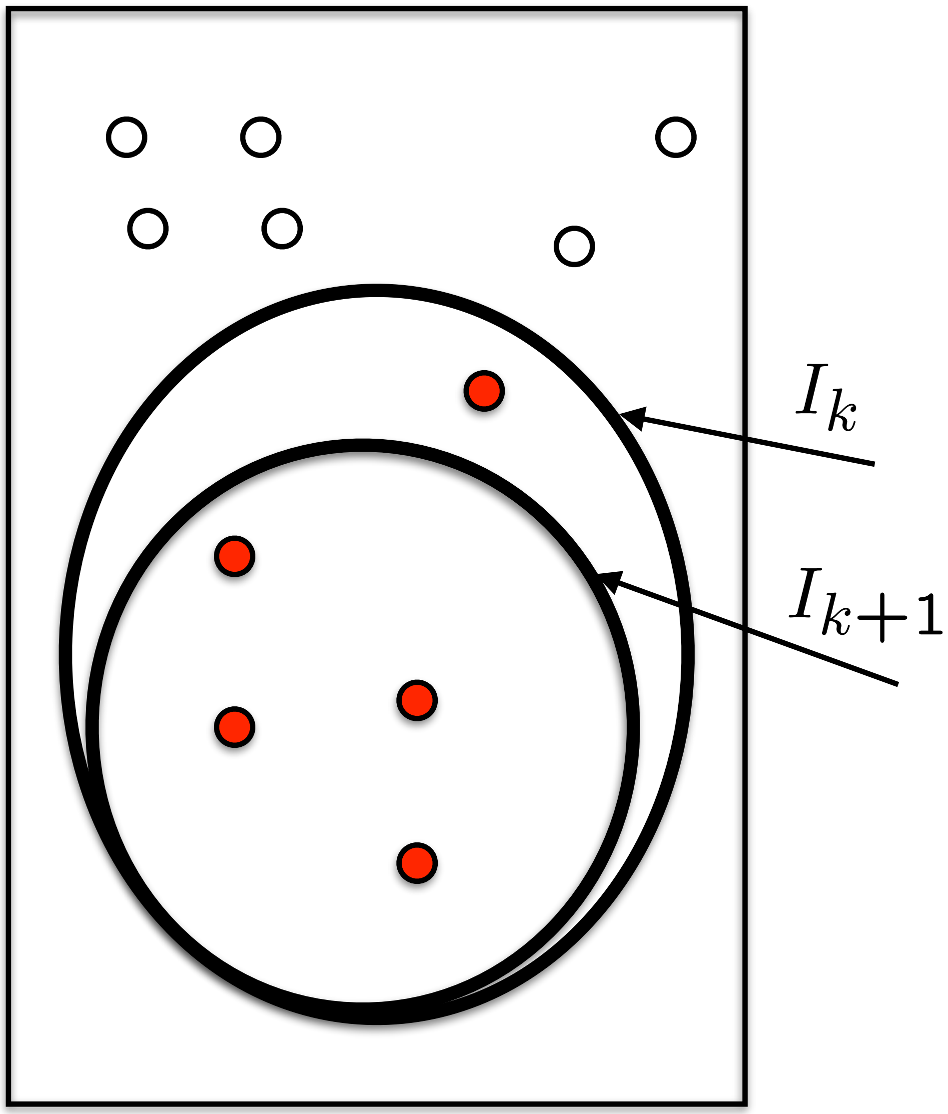

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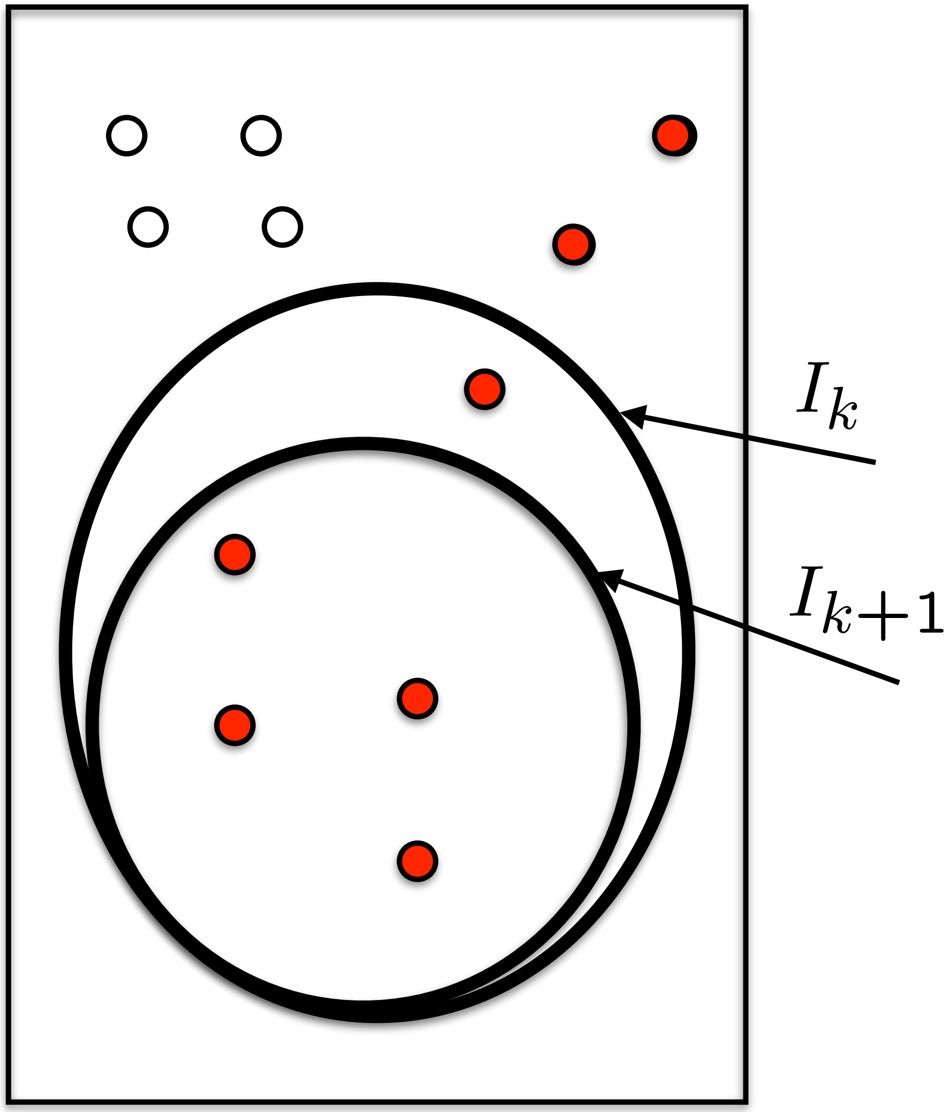

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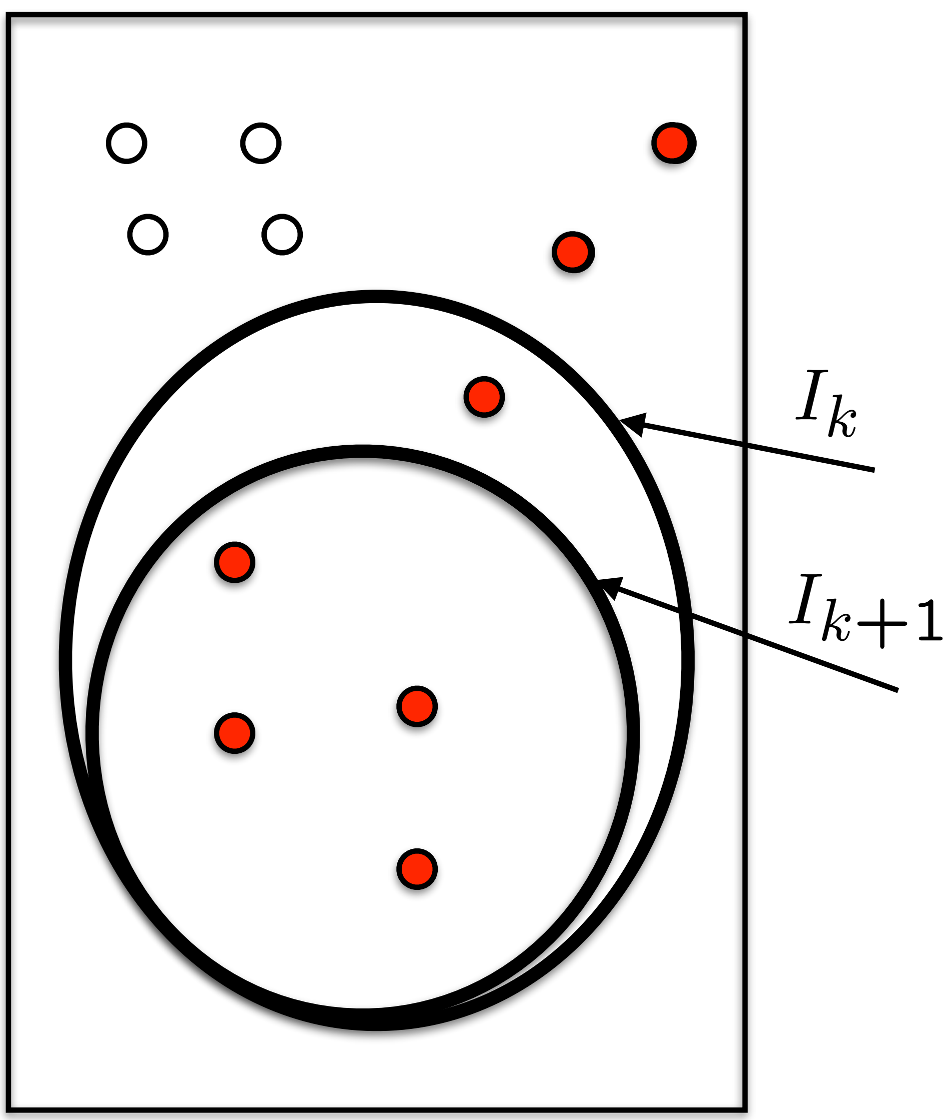

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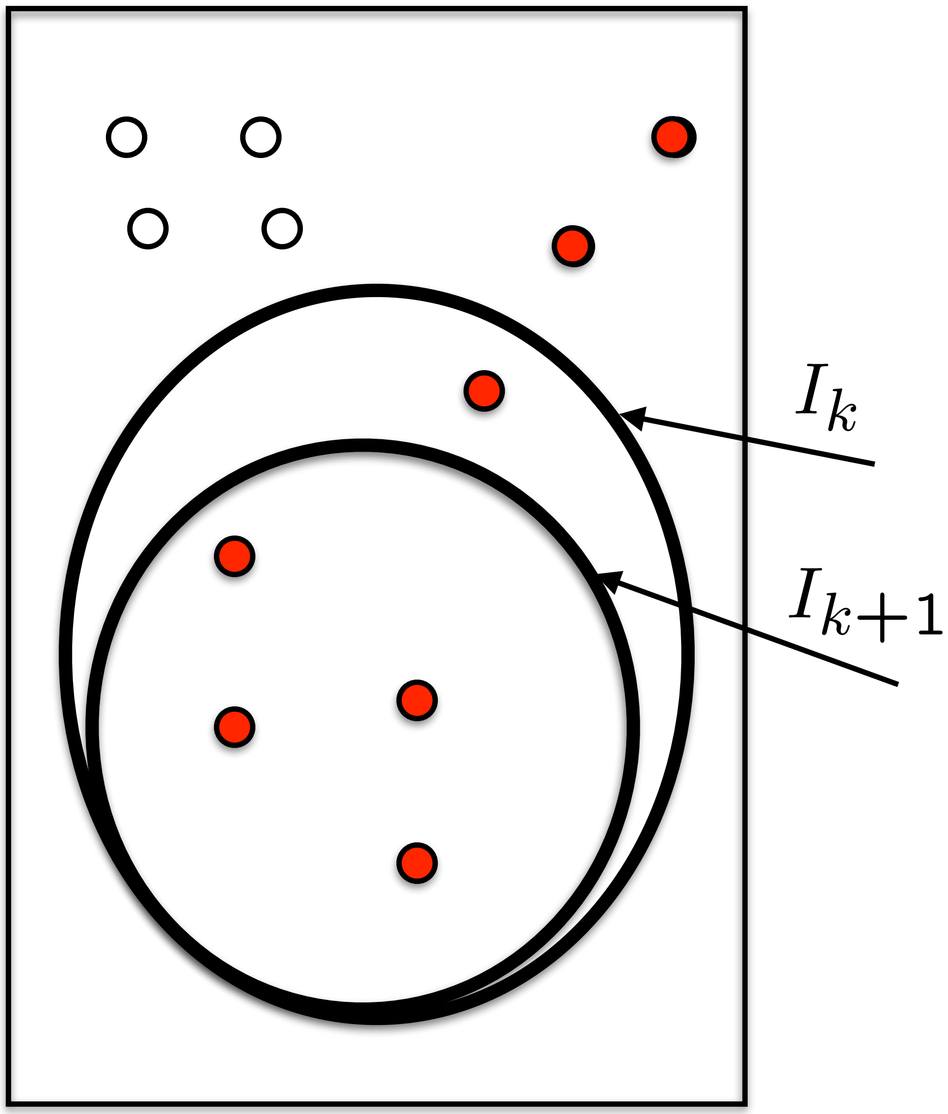

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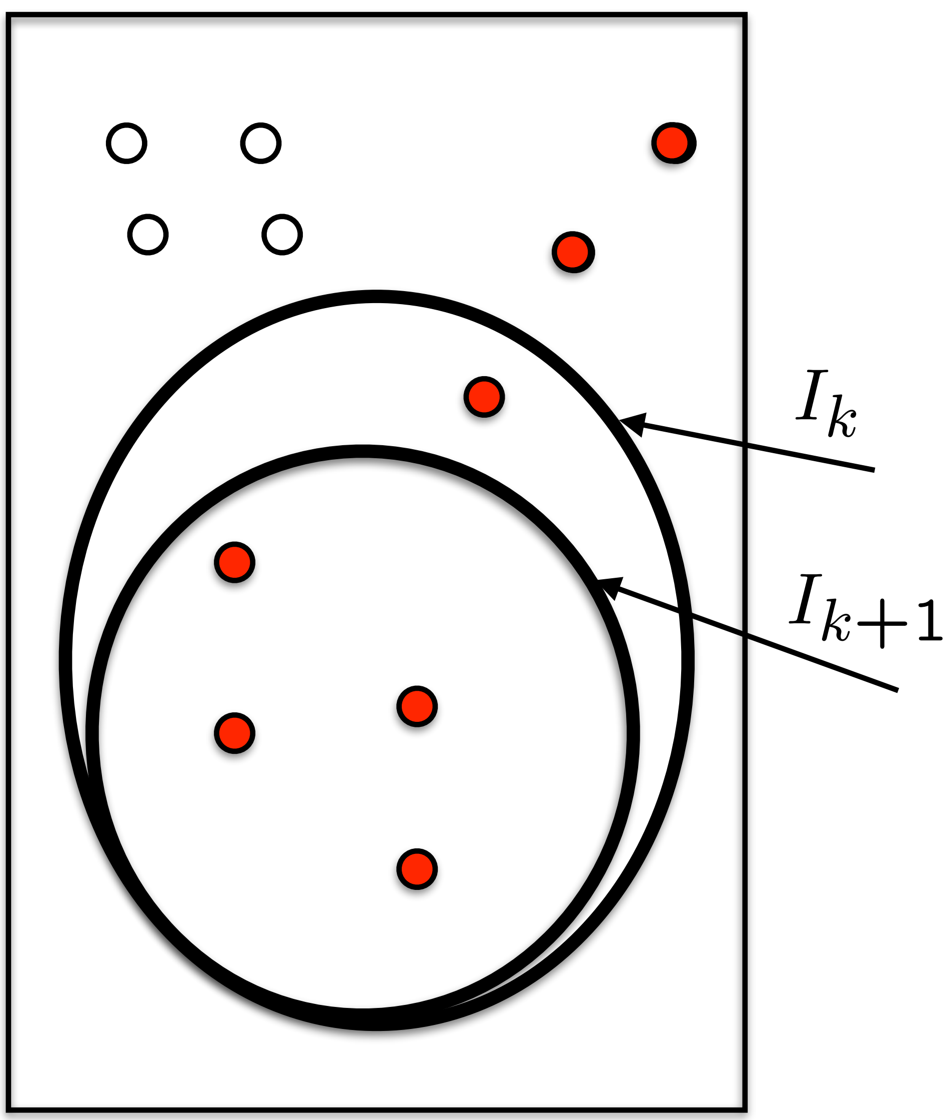

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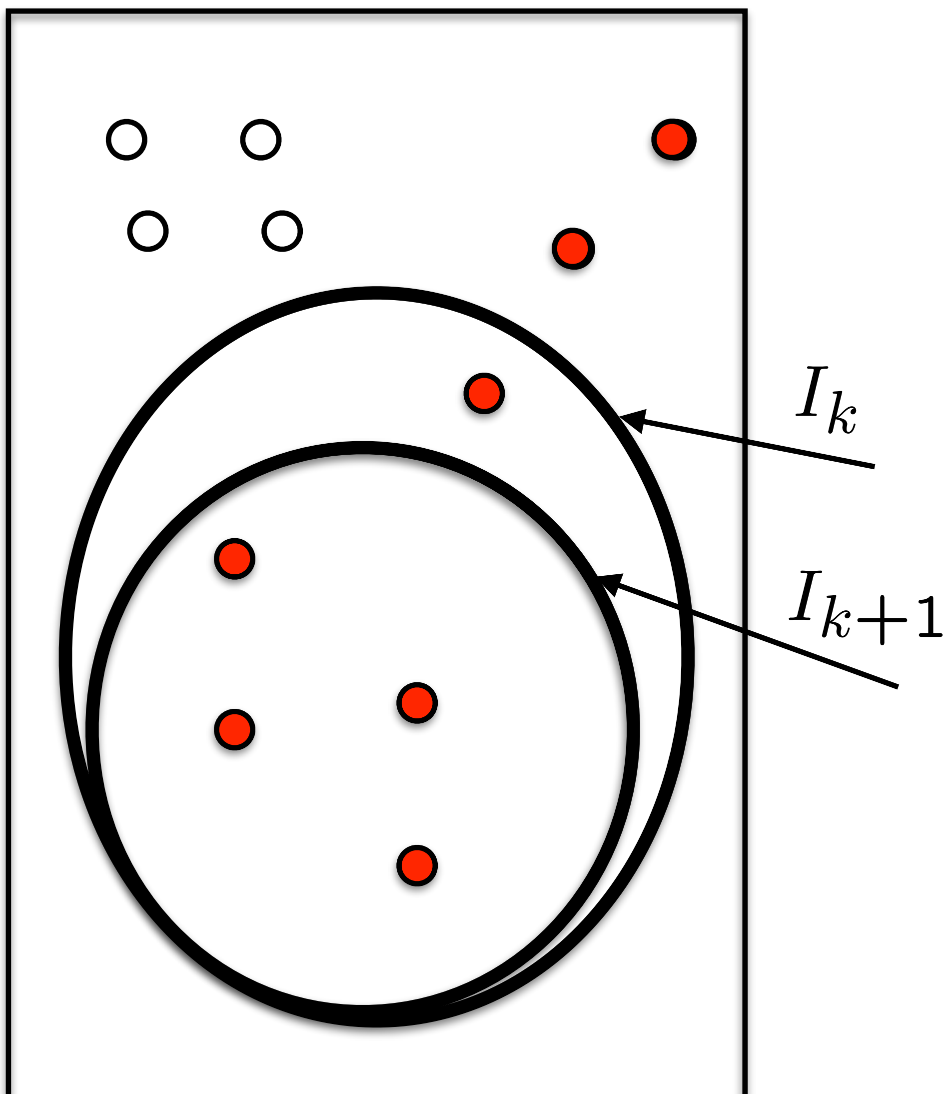

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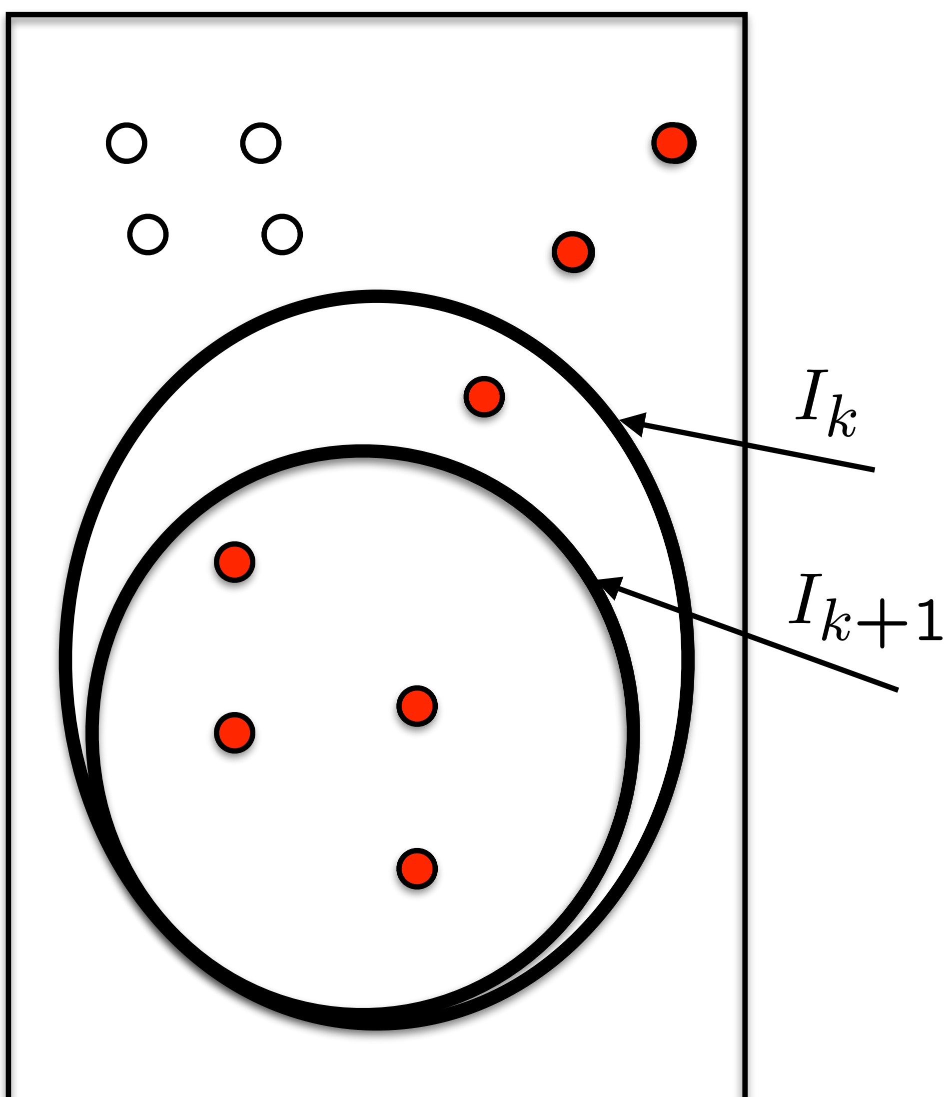

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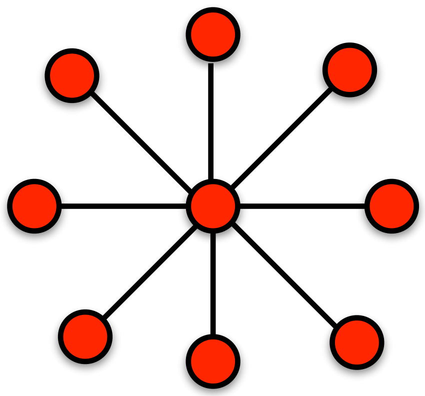
- If failure: let infections happen till $\text{cut}(I_t) \leq B/8$ and restart

Simulations, on a star graph



$n = 50 \qquad W \approx 25$

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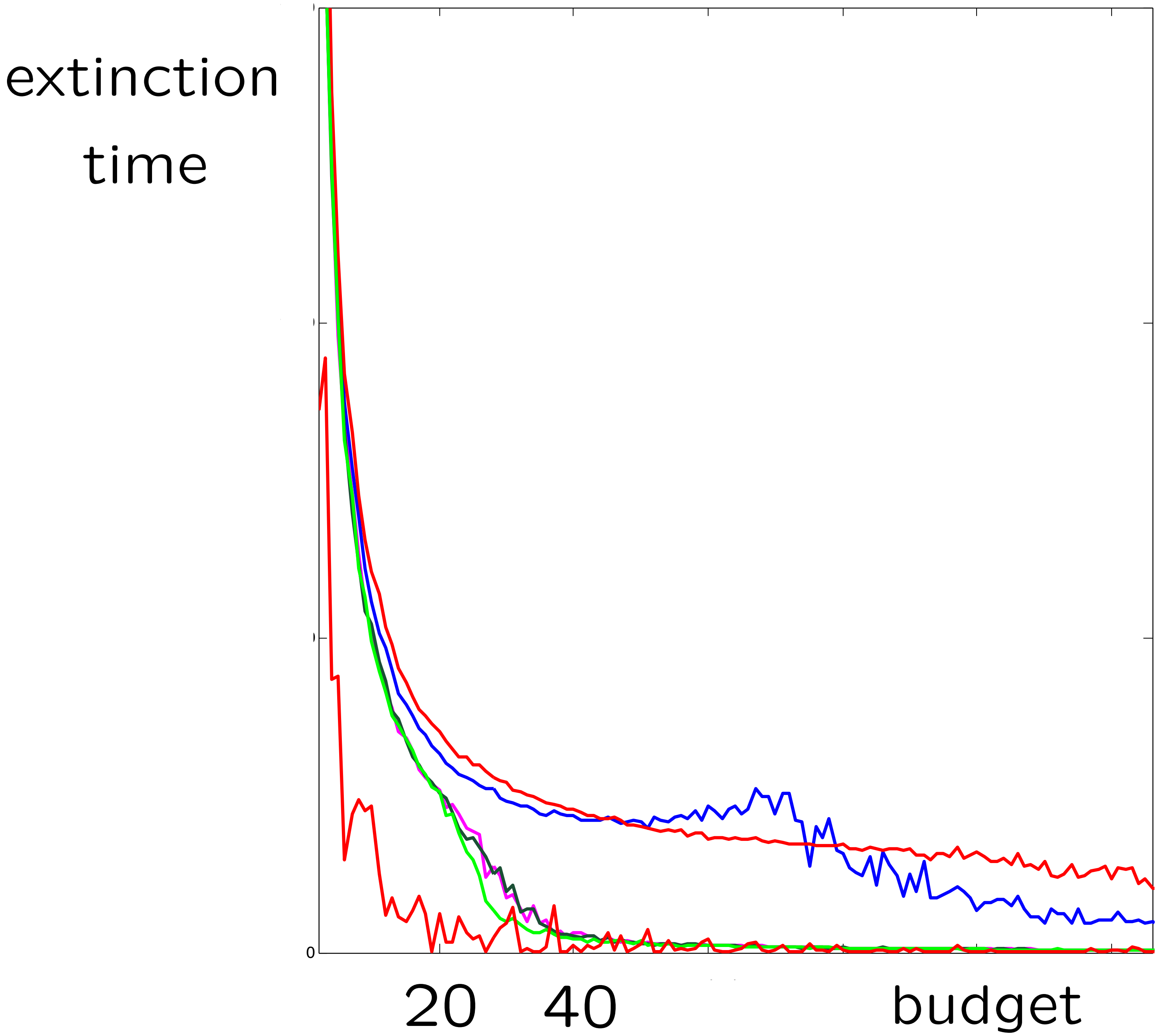

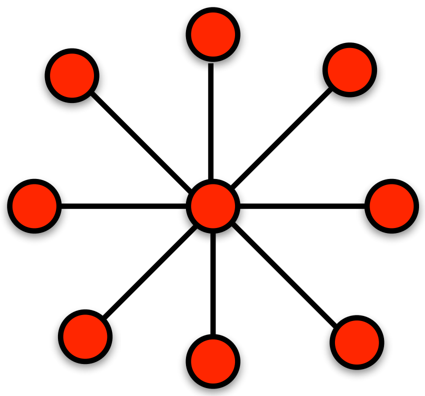

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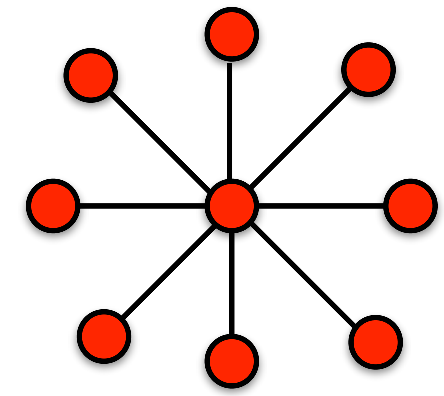

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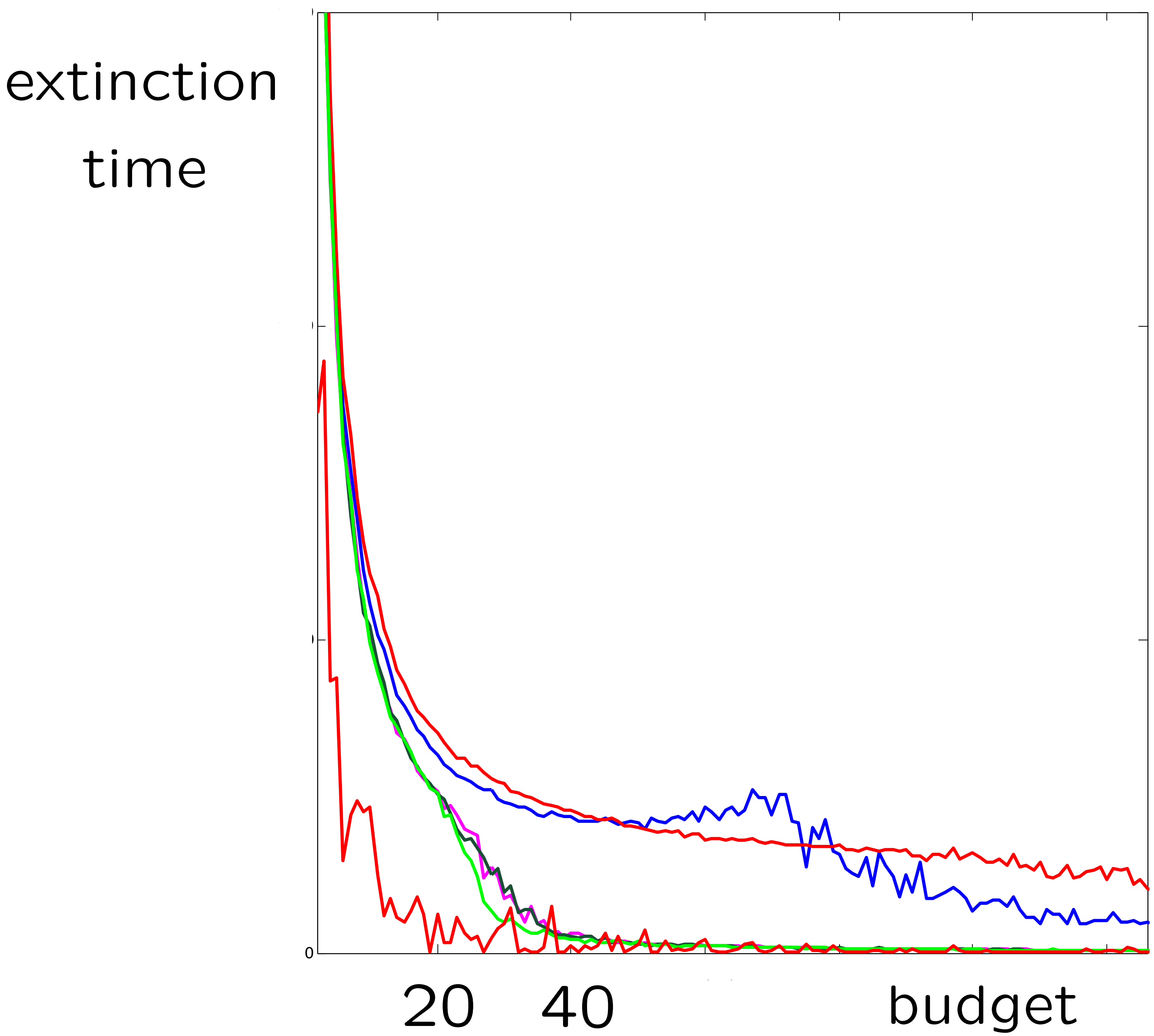

$\rho_i = R/n$

$\rho_i = R\frac{\deg(i)}{\sum_{i\in V}\deg(i)}$

$\rho_i = \frac{R}{\sum X_j(t)},\ X_i(t) = 1$

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ρ_i– CW-optimal

Can we do better? Lower bounds on expected extinction time

- **Theorem:** Assume: $W \geq c_w n$ $\qquad$ $B \leq c_b n \leq W$

  If c_b small enough $\quad [c_b \leq f(c_w, \Delta)]$,

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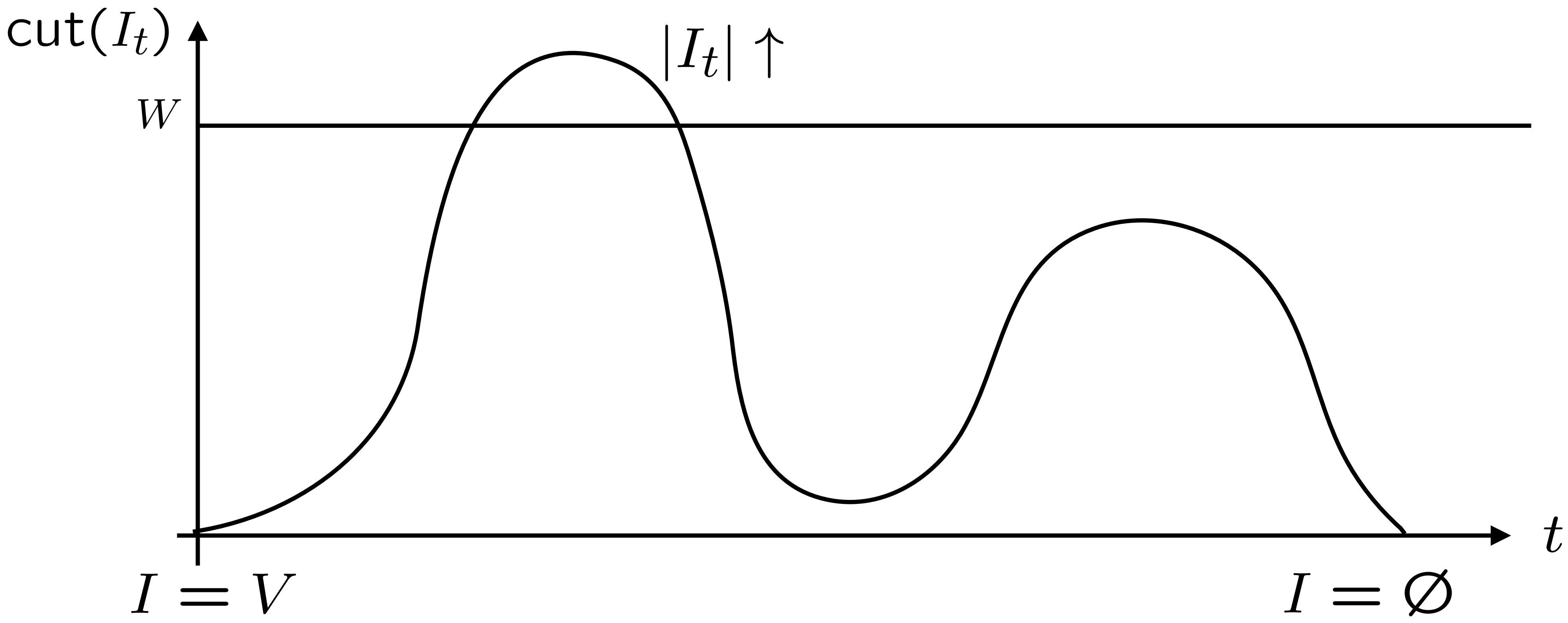

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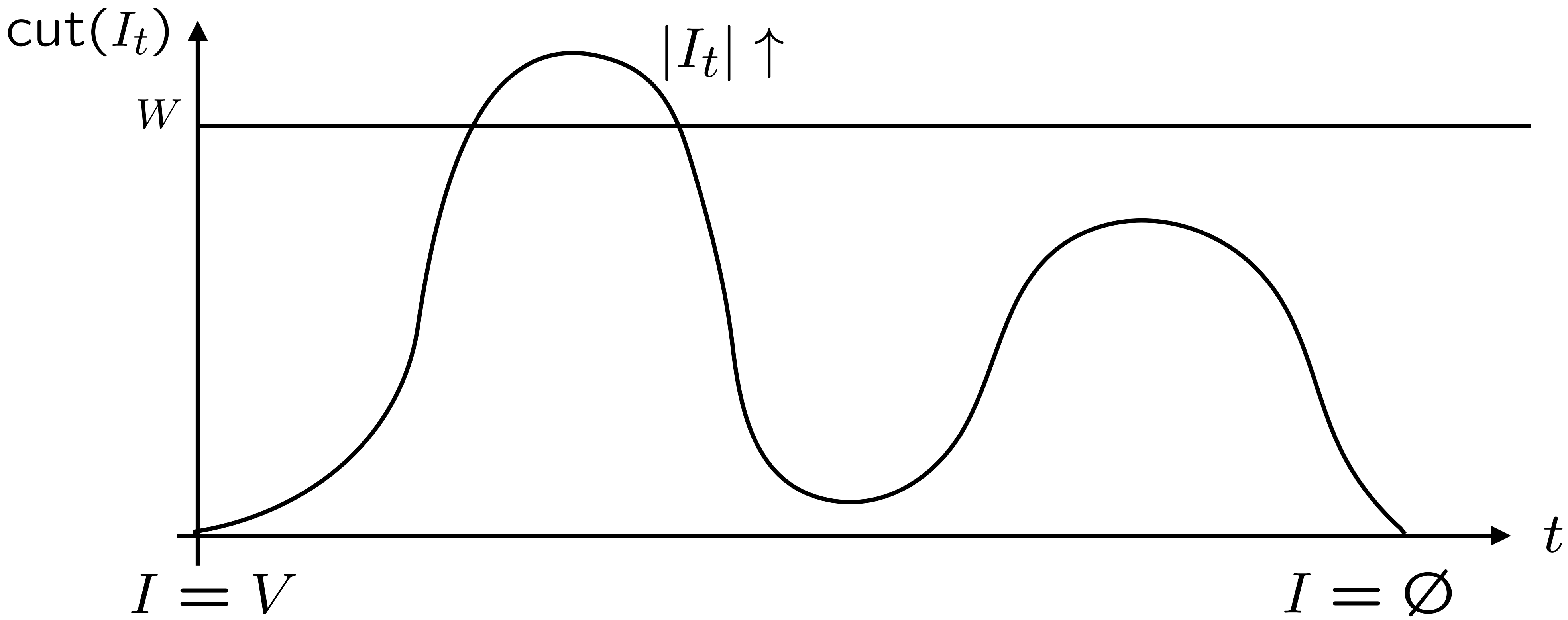

- Not enough. Must show upward drift for substantial amount of time

Resistance $\gamma(I)$

- Difficulty, starting from I

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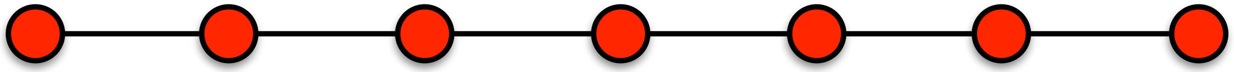

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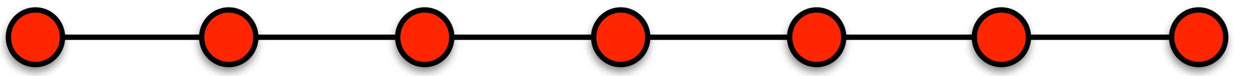

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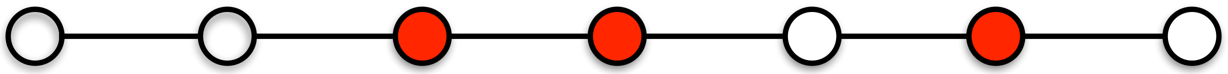

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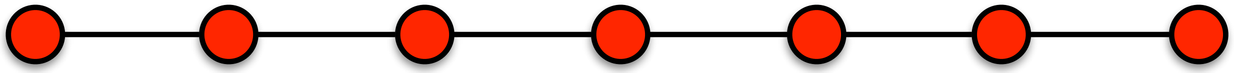

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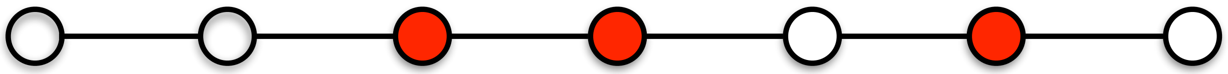

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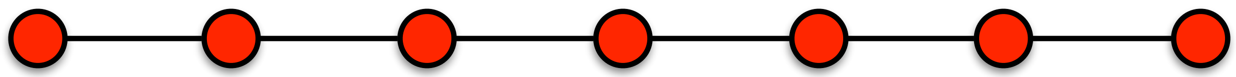

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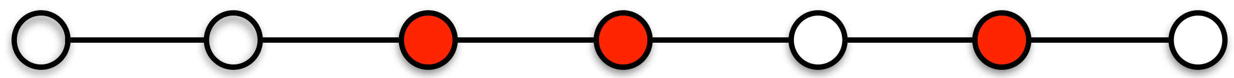

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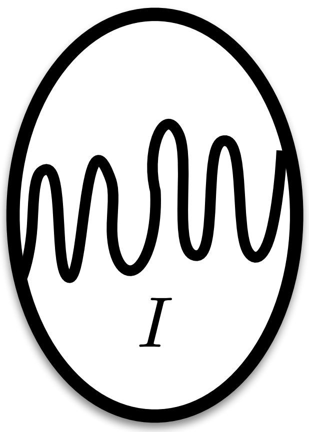

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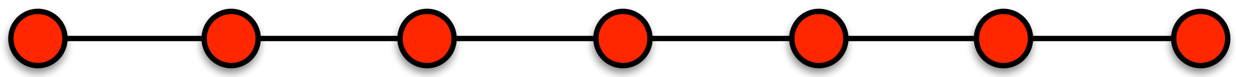

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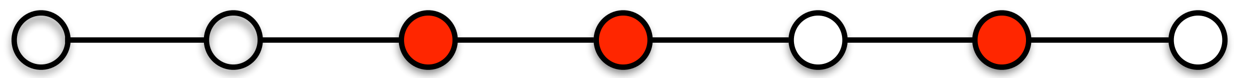

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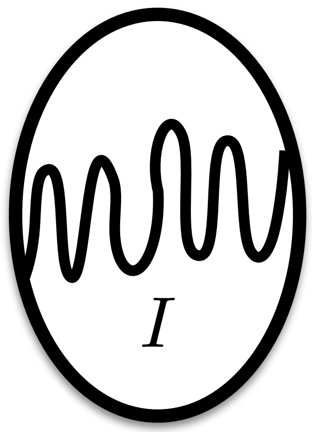
add/infect nodes



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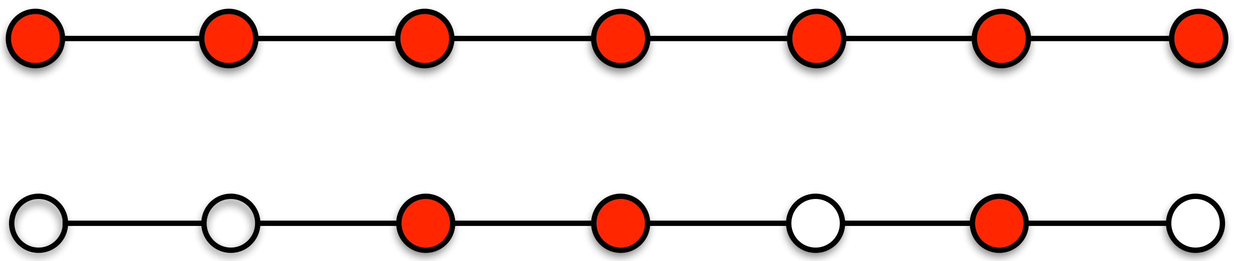

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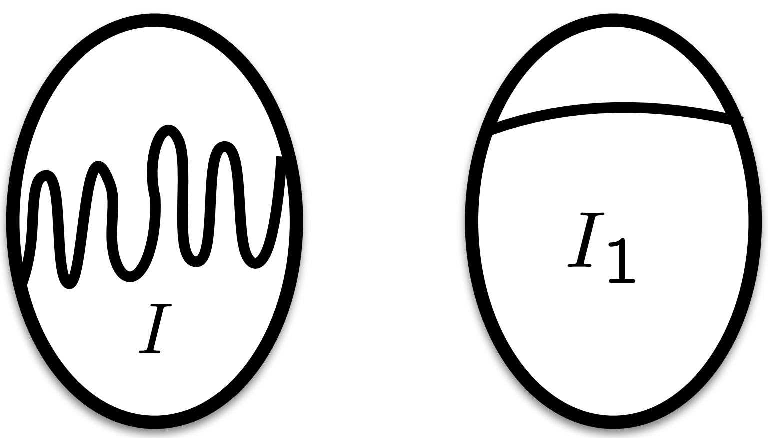

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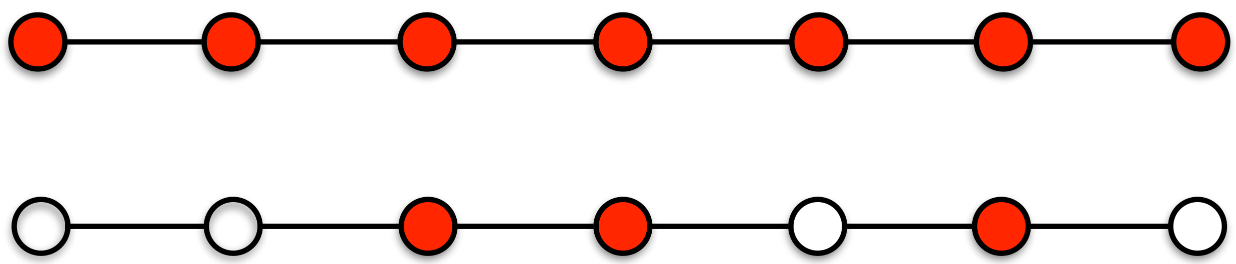

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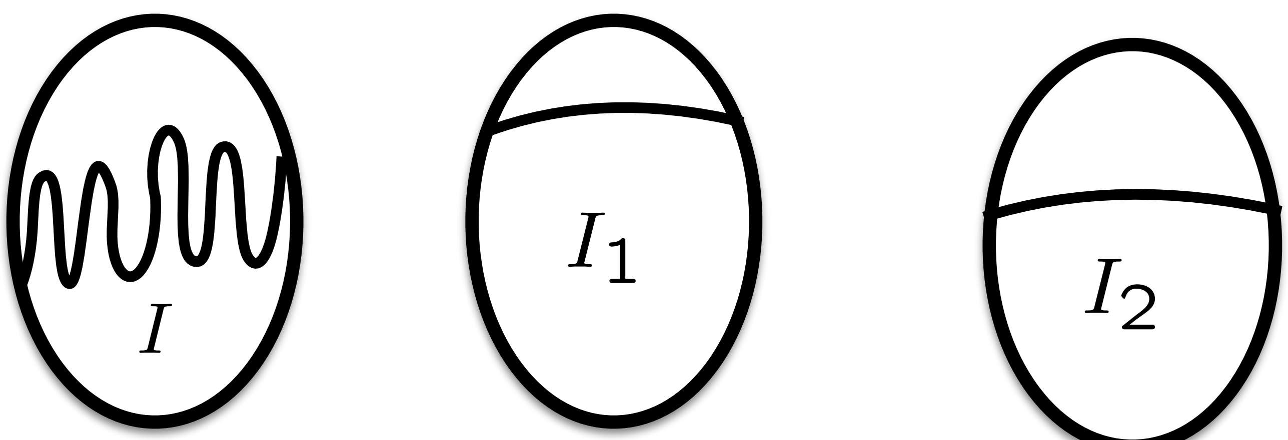

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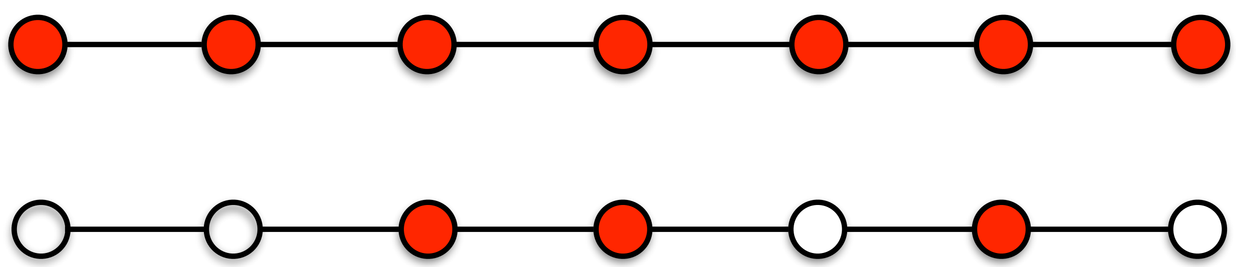

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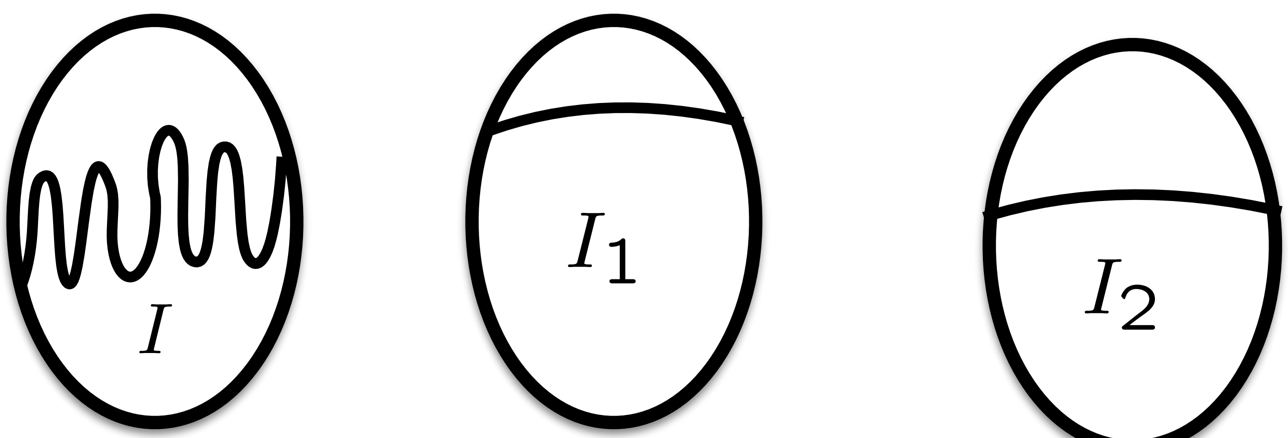

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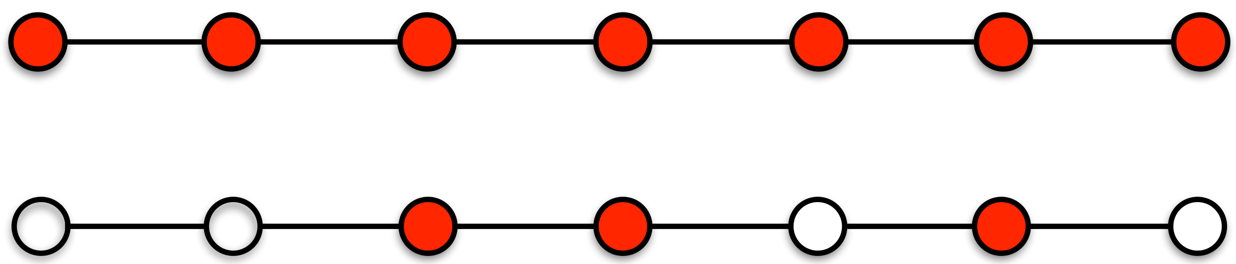

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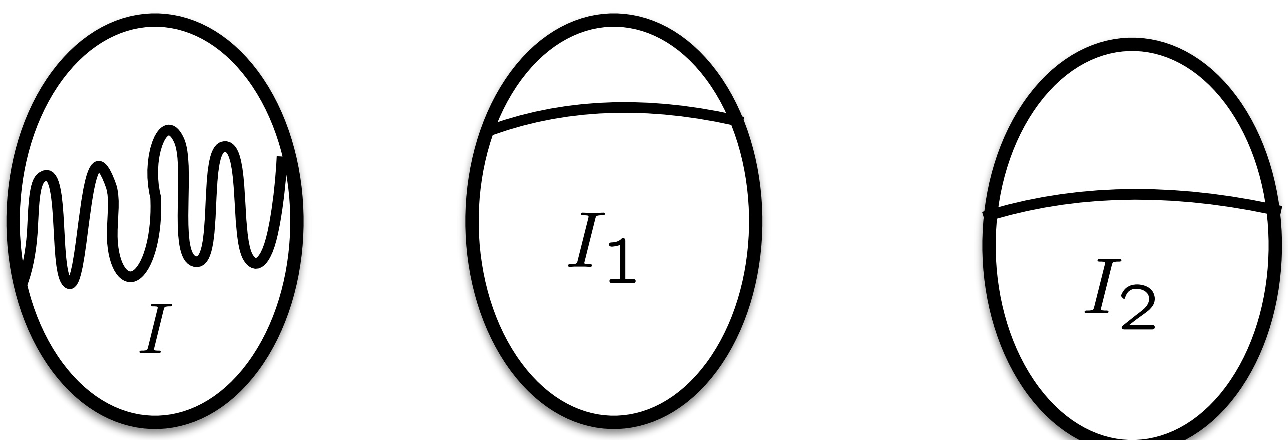

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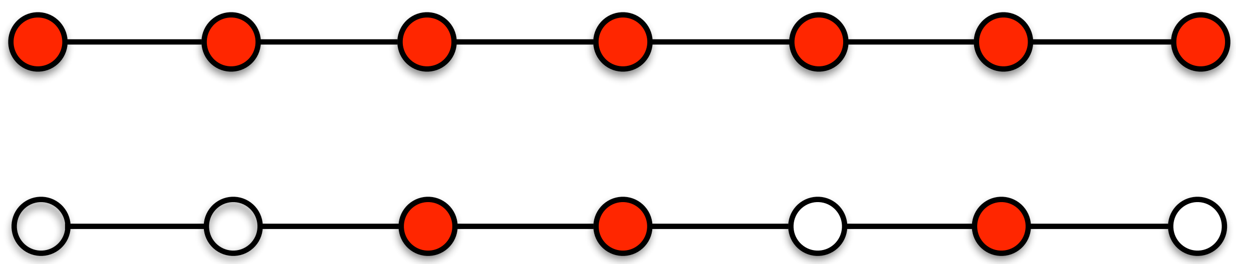

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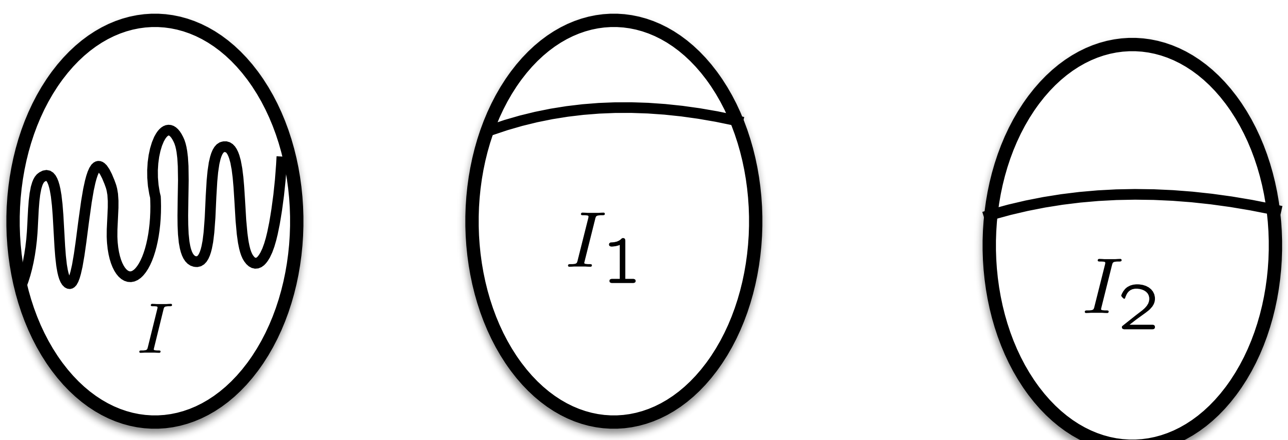

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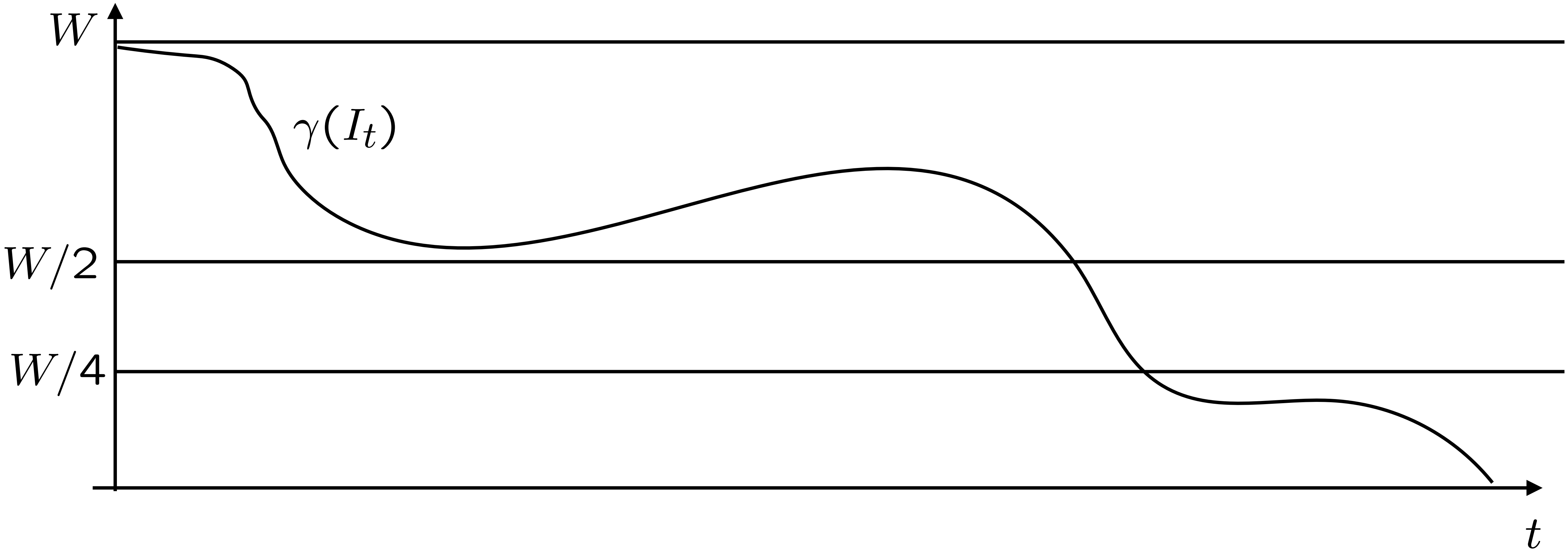

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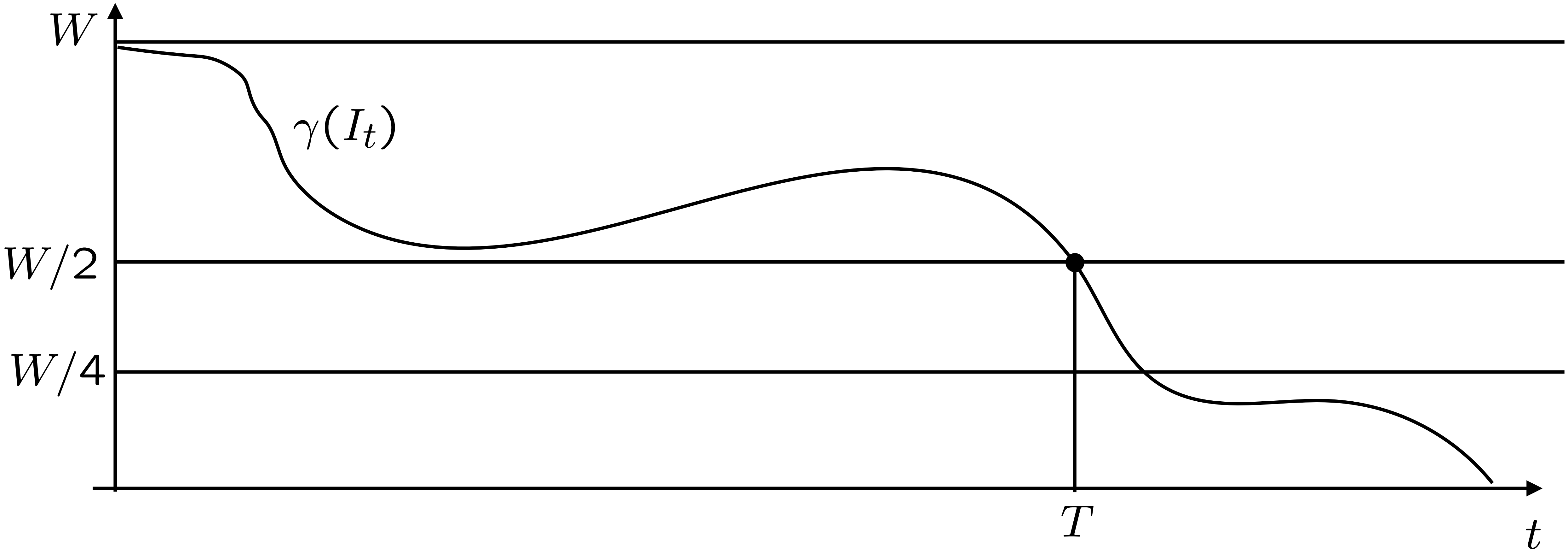

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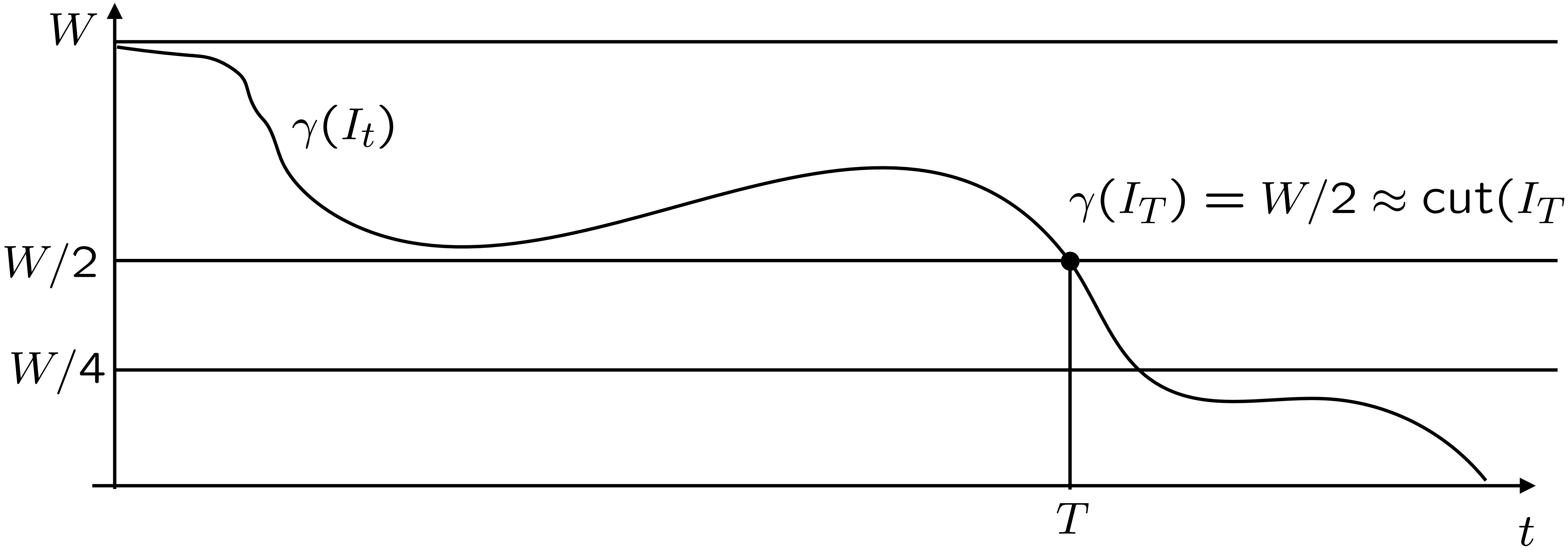

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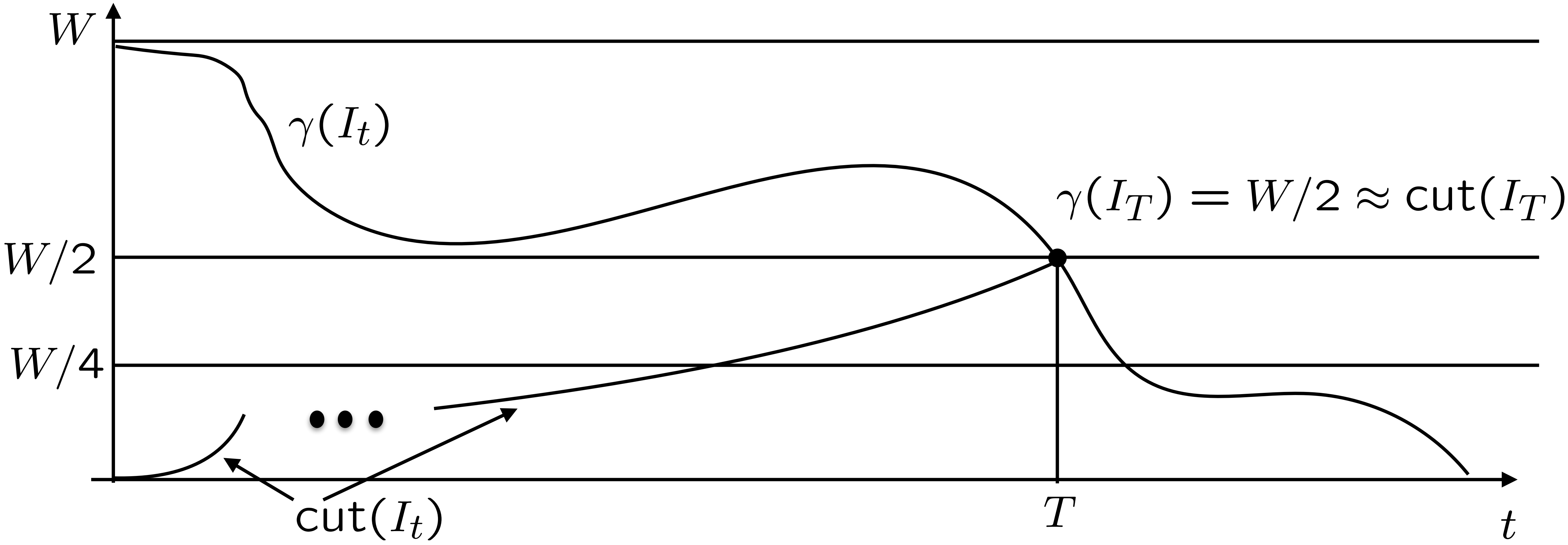

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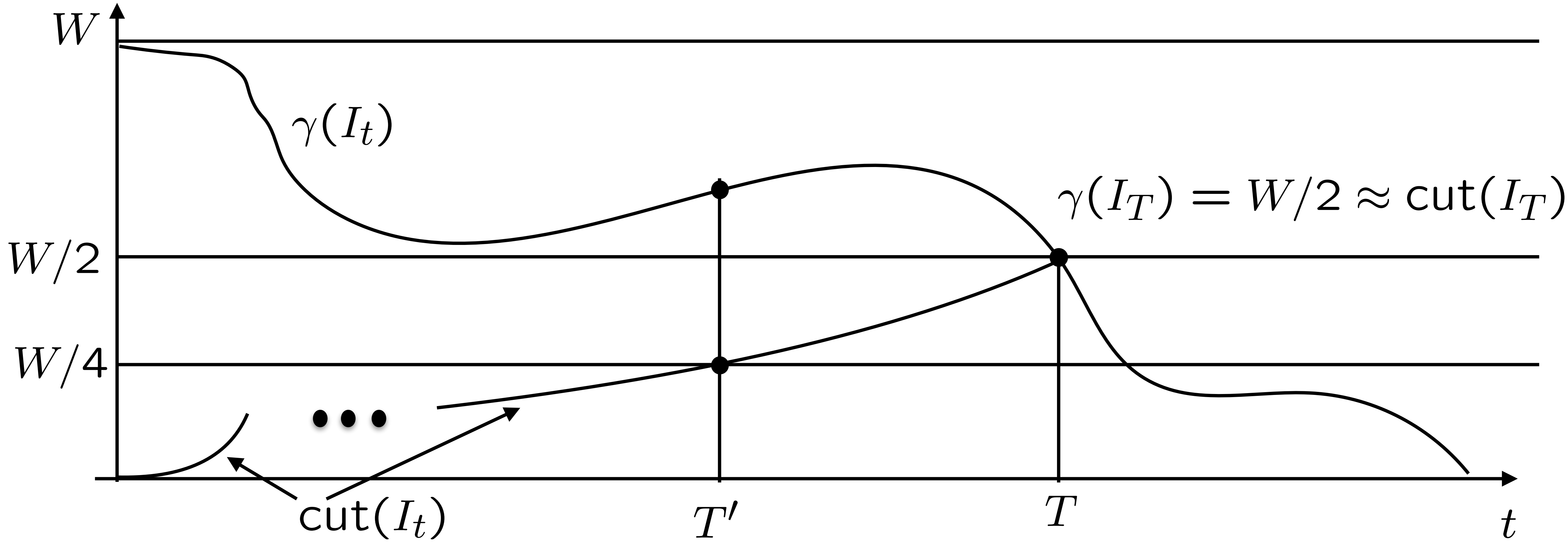

- **Claim:** During $[T', T]$ have many recoveries (w.h.p.)

Lower bound proof sketch

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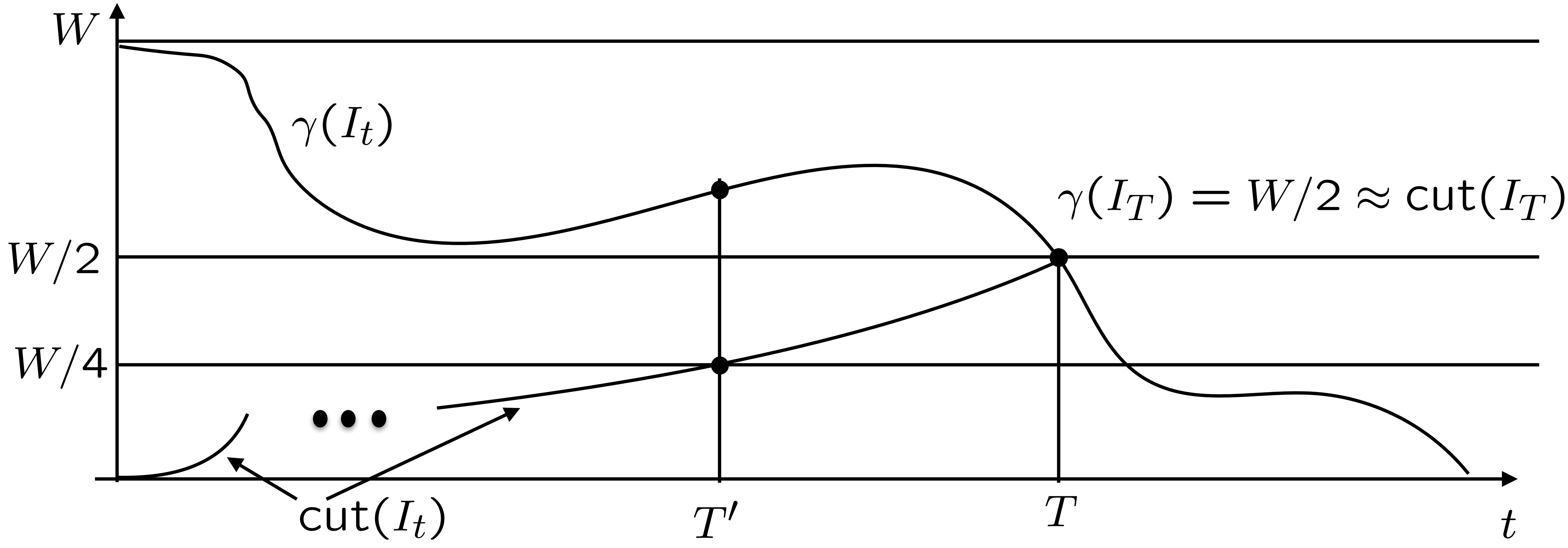

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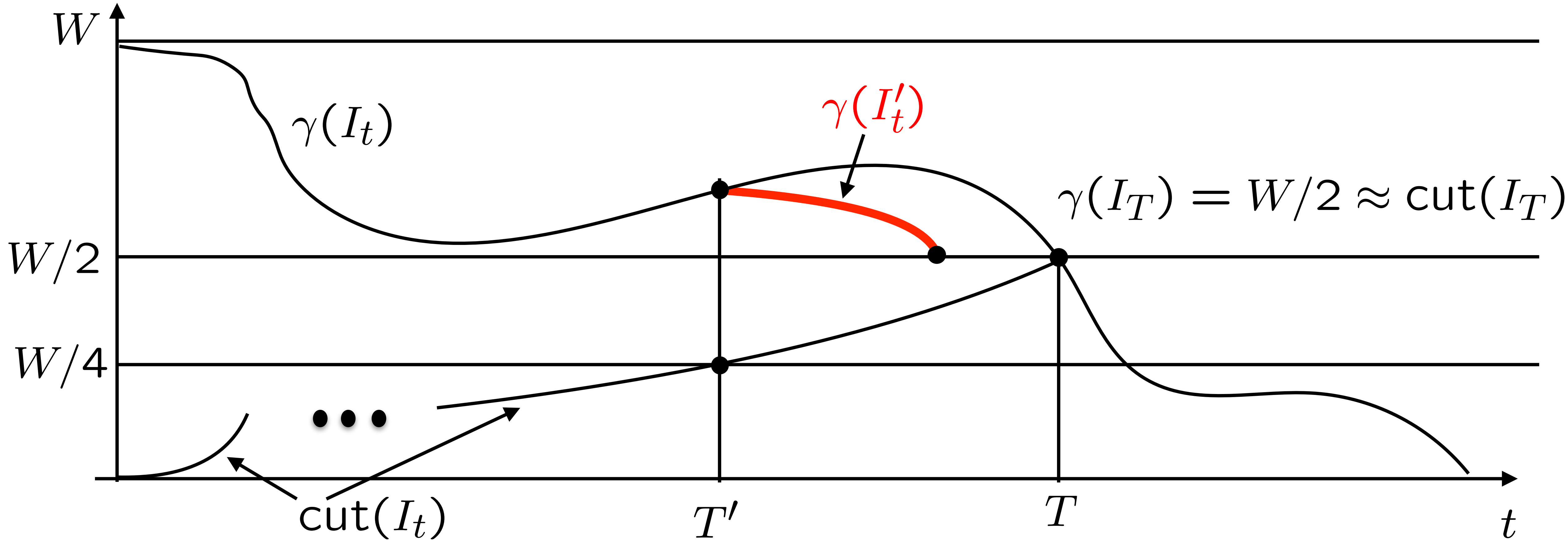

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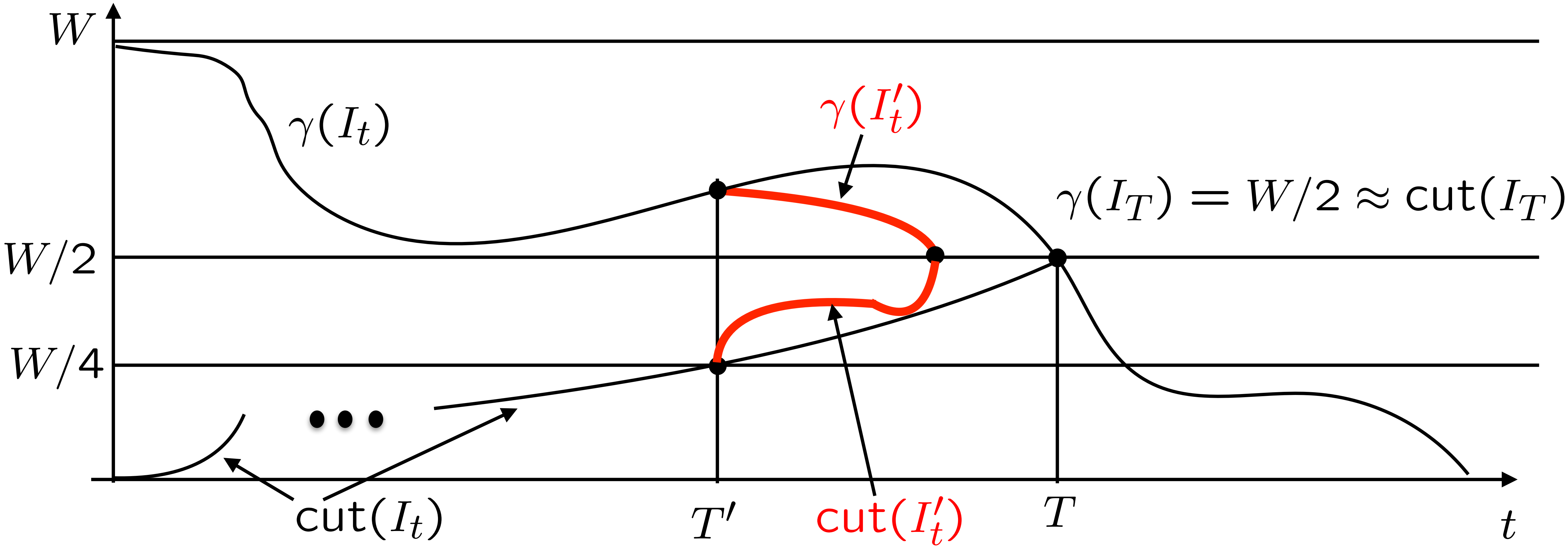

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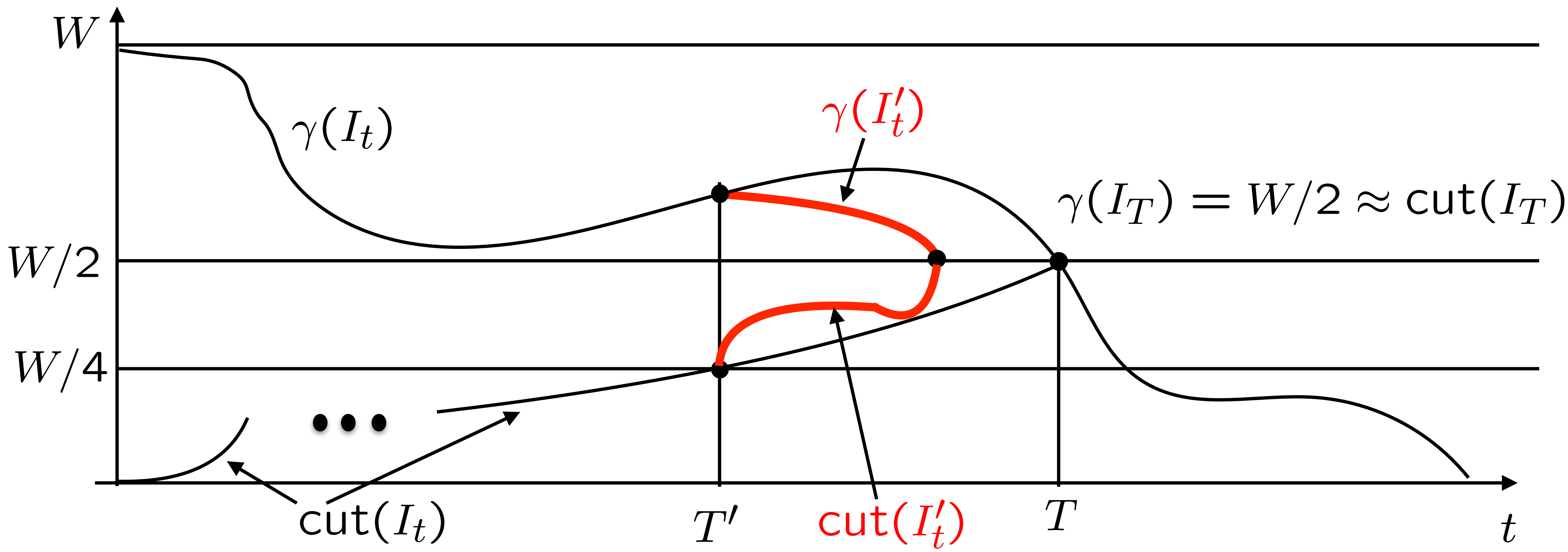

- **Claim:** During $[T', T]$ have many recoveries (w.h.p.)
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- $\mathrm{cut}(I'_t)$ increases by $W/4$

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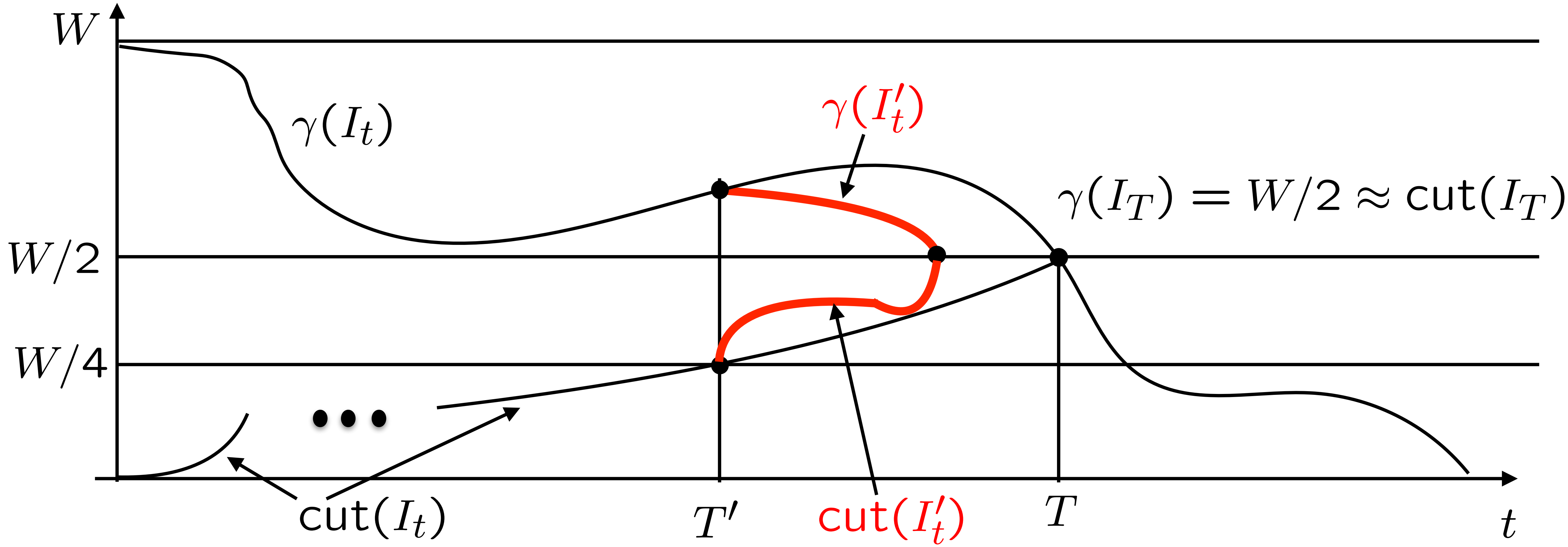

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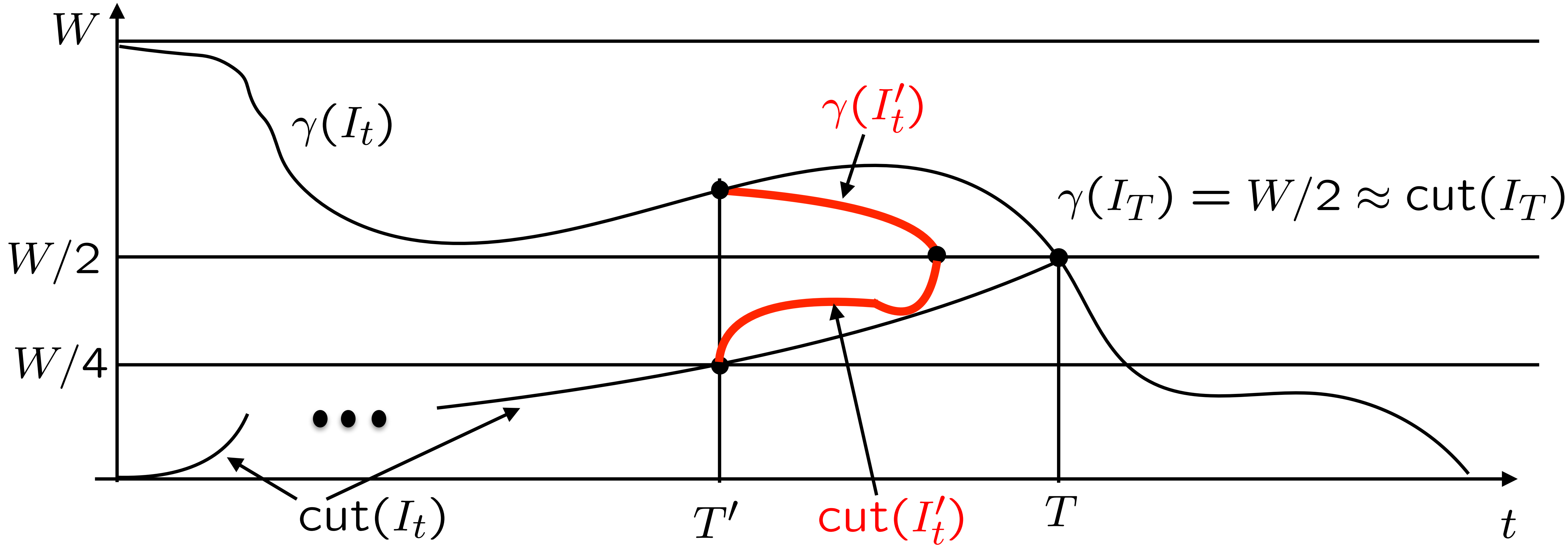

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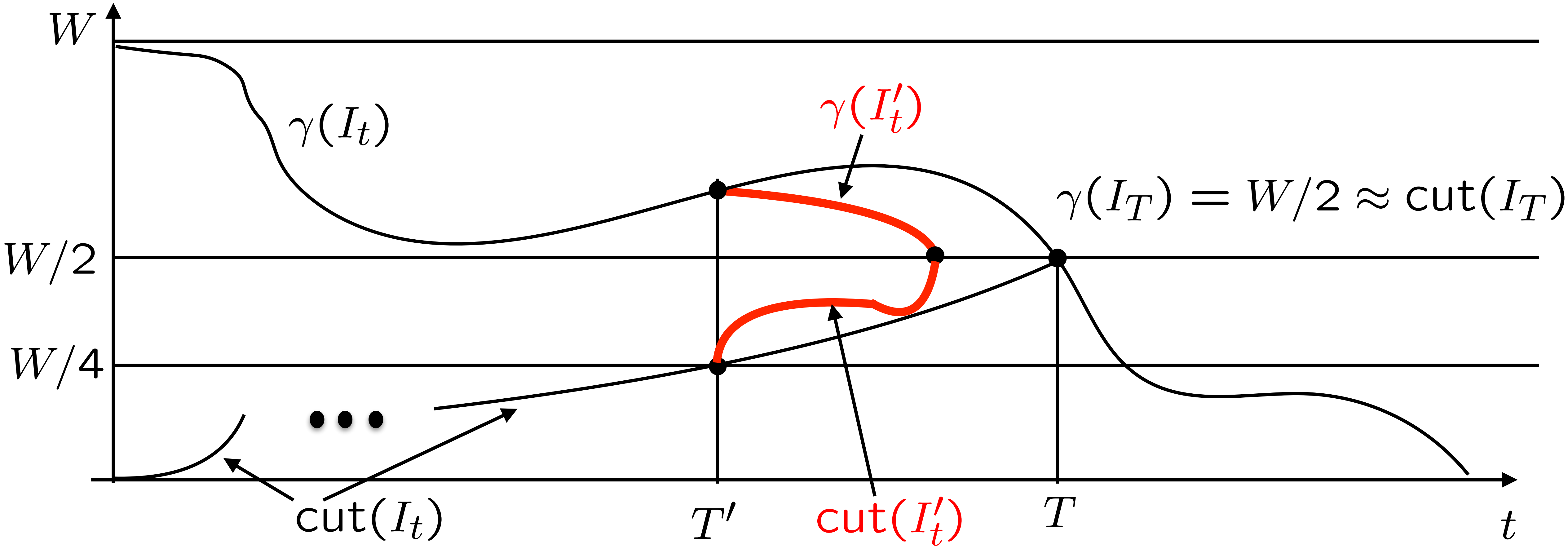

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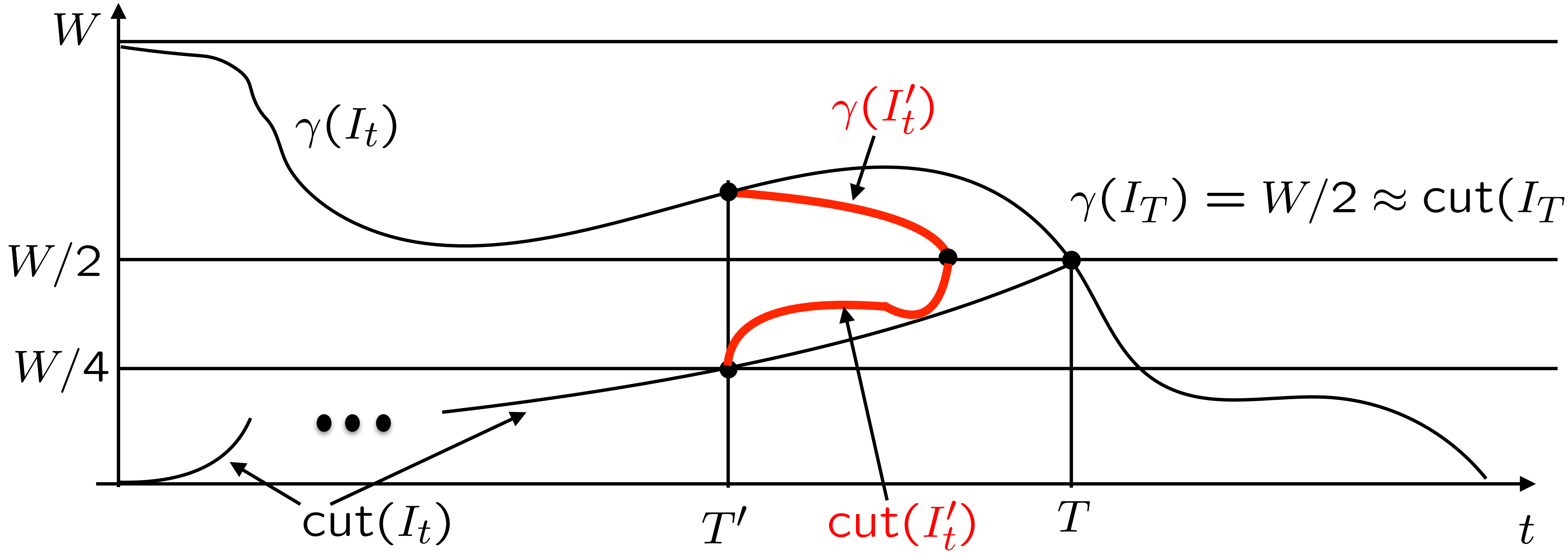

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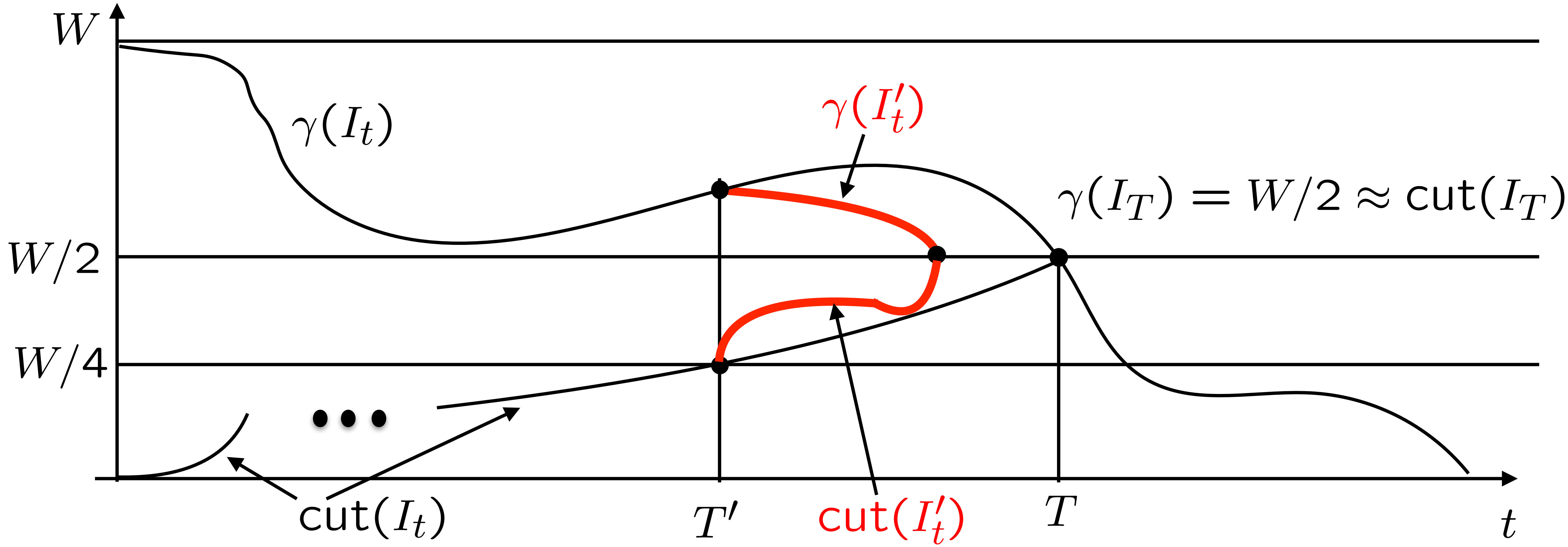

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  $\Rightarrow \geq 2n$ infections (w.h.p.): contradiction
- Probability of such a scenario: exponentially small
- Need exponential time for such a scenario to materialize

Extensions, open problems

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- SIR models (cured cannot get reinfected)
- Imperfect information, etc.