A Framework for Applying First-Order Methods to General Convex Conic Optimization Problems

Part 2

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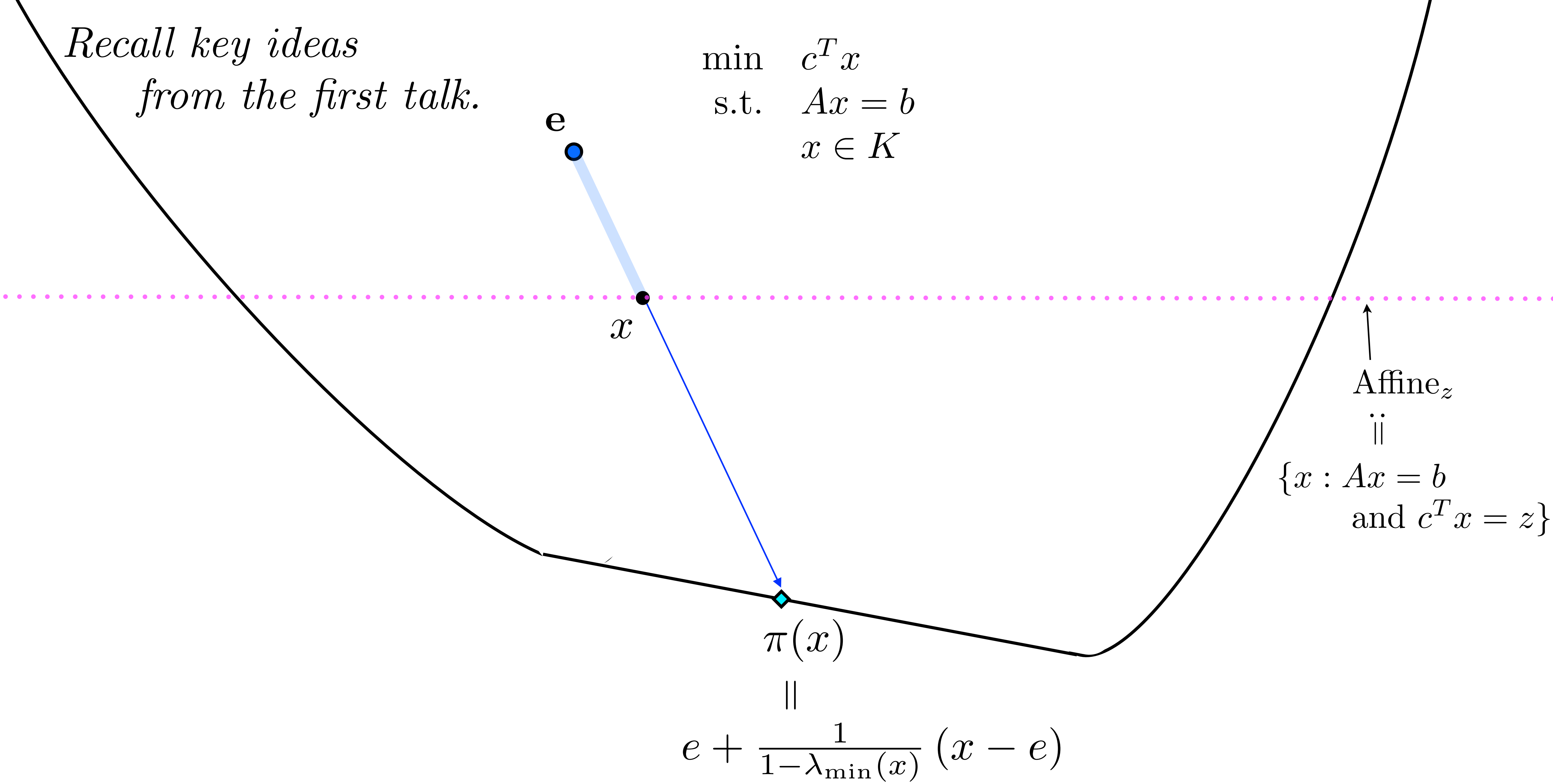
LNMB Lunteren Conference, 2016

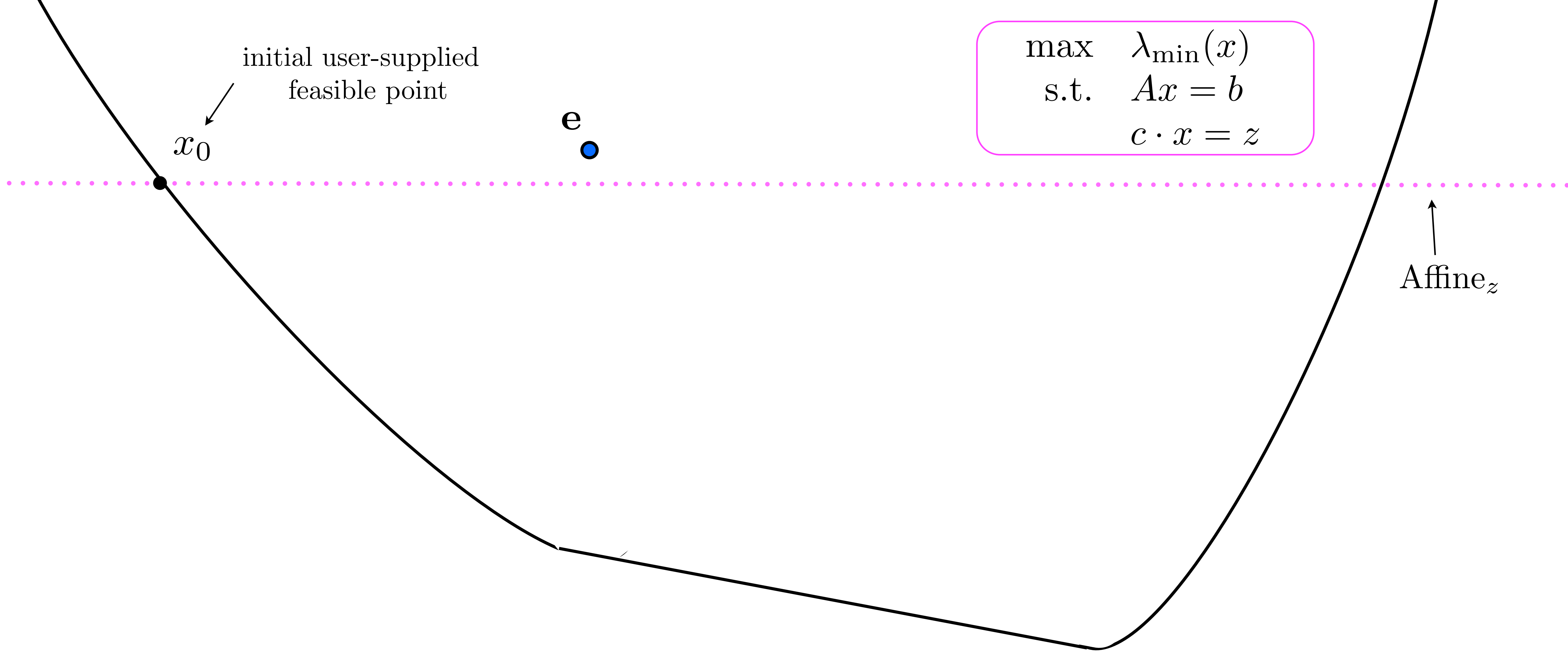
Recall key ideas from the first talk.

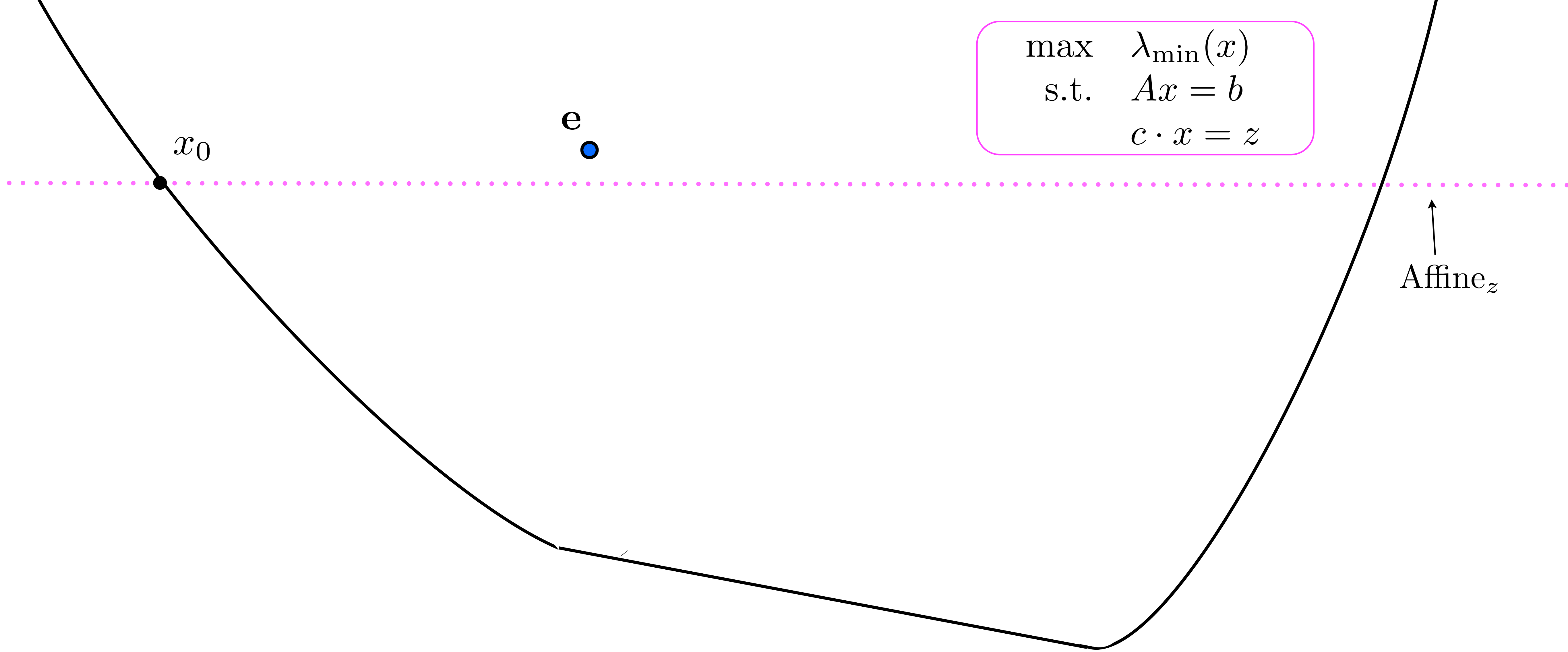
$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \in K \end{array}$$



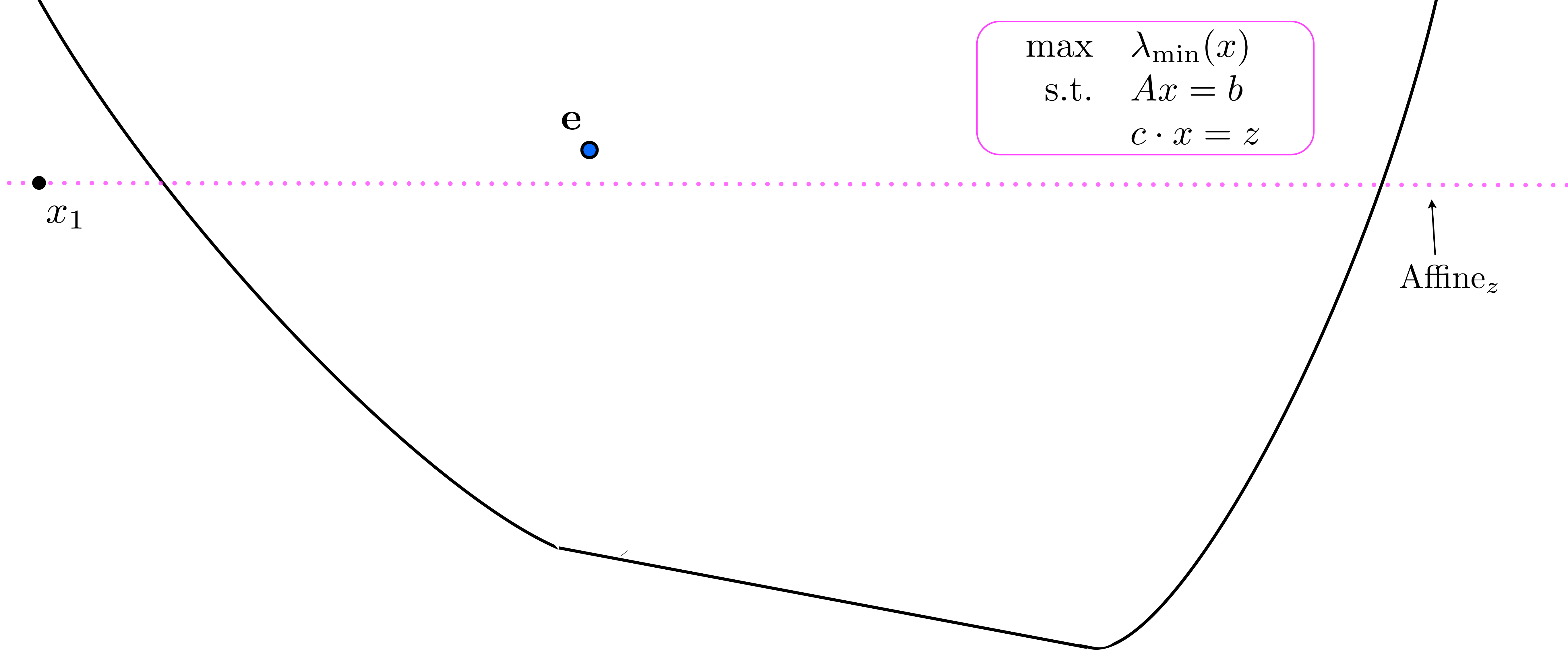
$\boxed{\ldots \text{where } \lambda_{\min}(x) \text{ is the scalar } \lambda \text{ satisfying } \; x - \lambda e \in \text{boundary}(K)}$

$$\begin{array}{ll} \min & c \cdot x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{K} \end{array} \quad \equiv \quad \begin{array}{ll} \max & \lambda_{\min}(x) \\ \text{s.t.} & Ax = b \\ & c \cdot x = z \end{array}$$

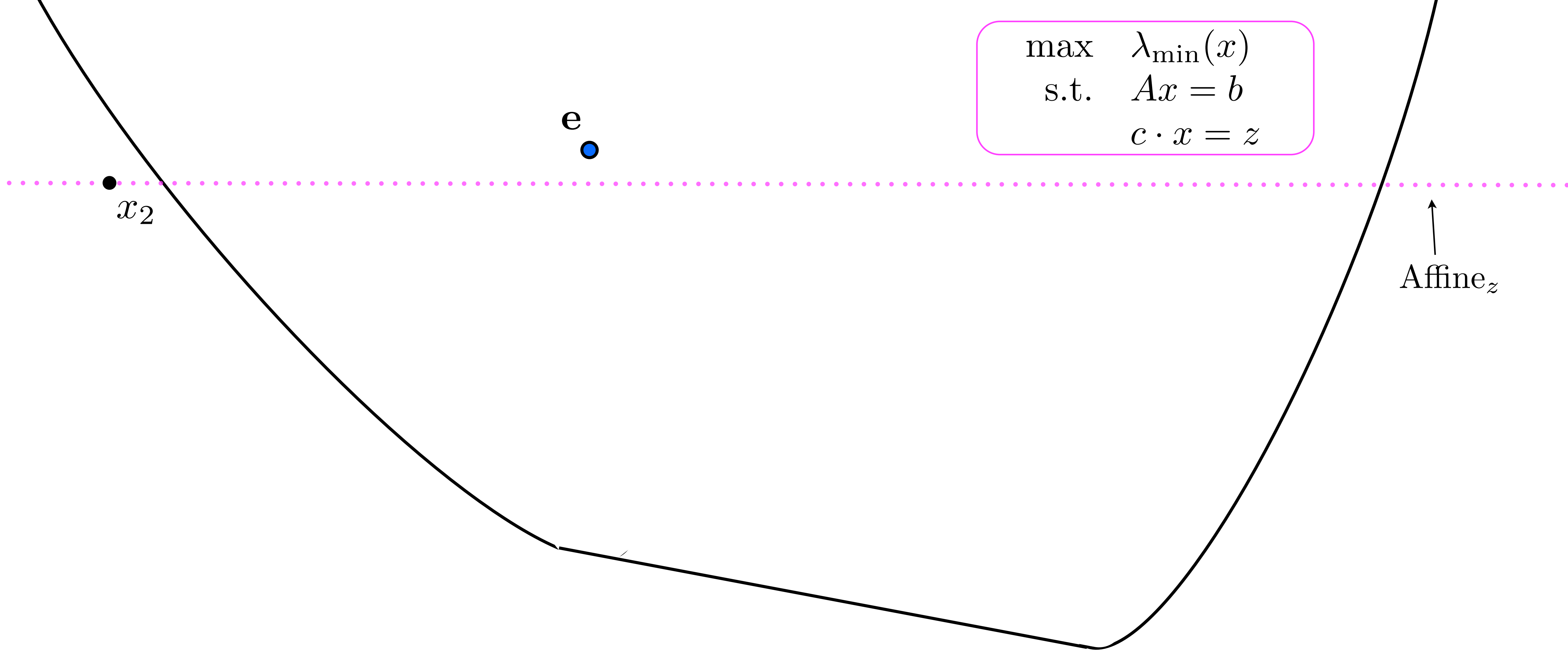

initial user-supplied feasible point
x_0
$\mathbf{e}$
$\max \quad \lambda_{\min}(x)$
$\text{s.t.} \quad Ax = b$
$c \cdot x = z$
Affine_z



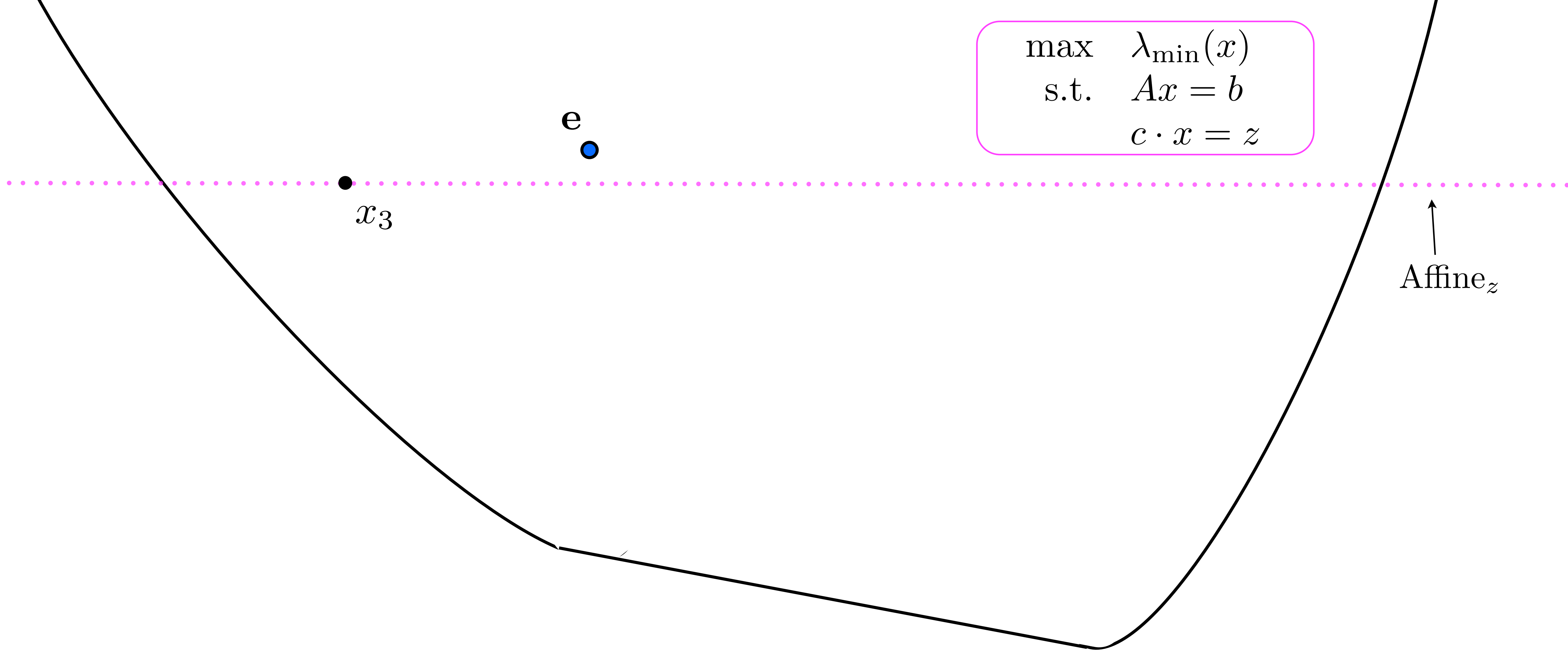
max $\lambda_{\min}(x)$
s.t. $Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_0
Affine_z



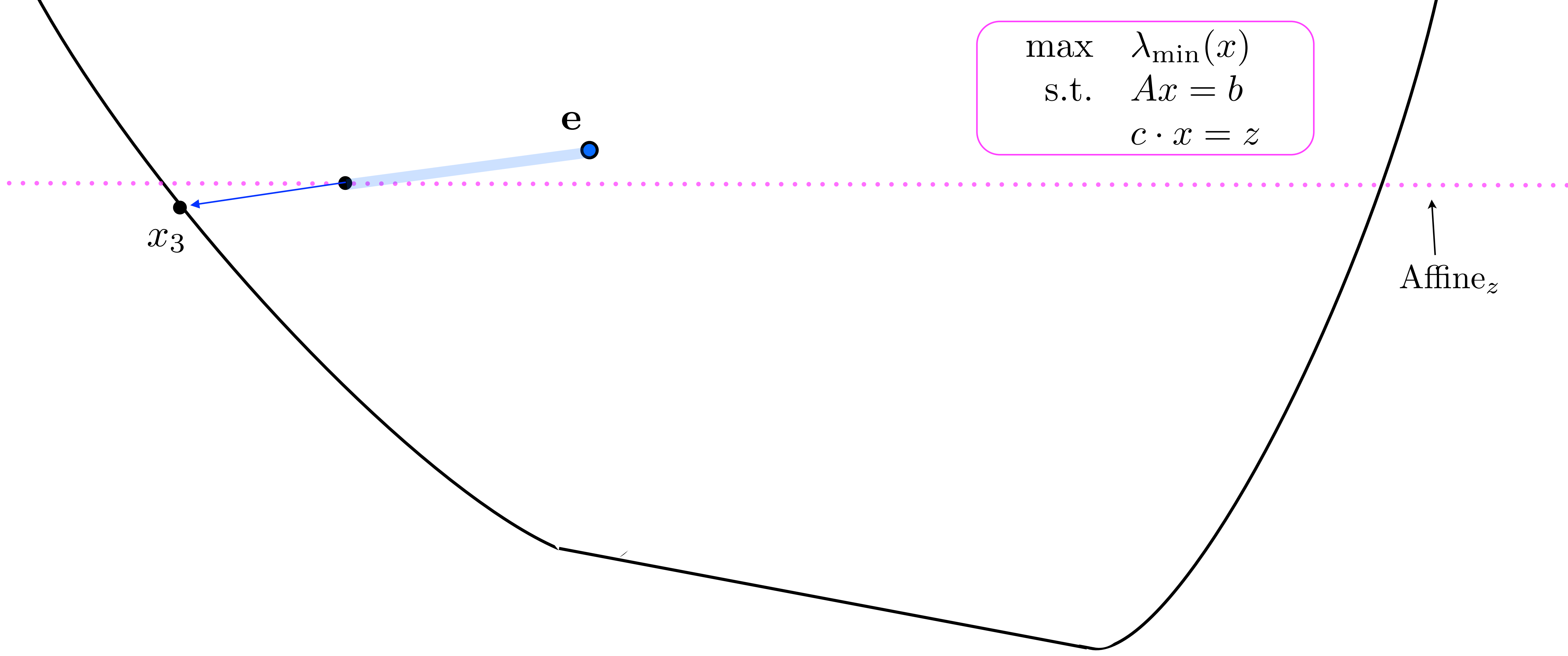
$\max \quad \lambda_{\min}(x)$
$\text{s.t.} \quad Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_1
Affine_z



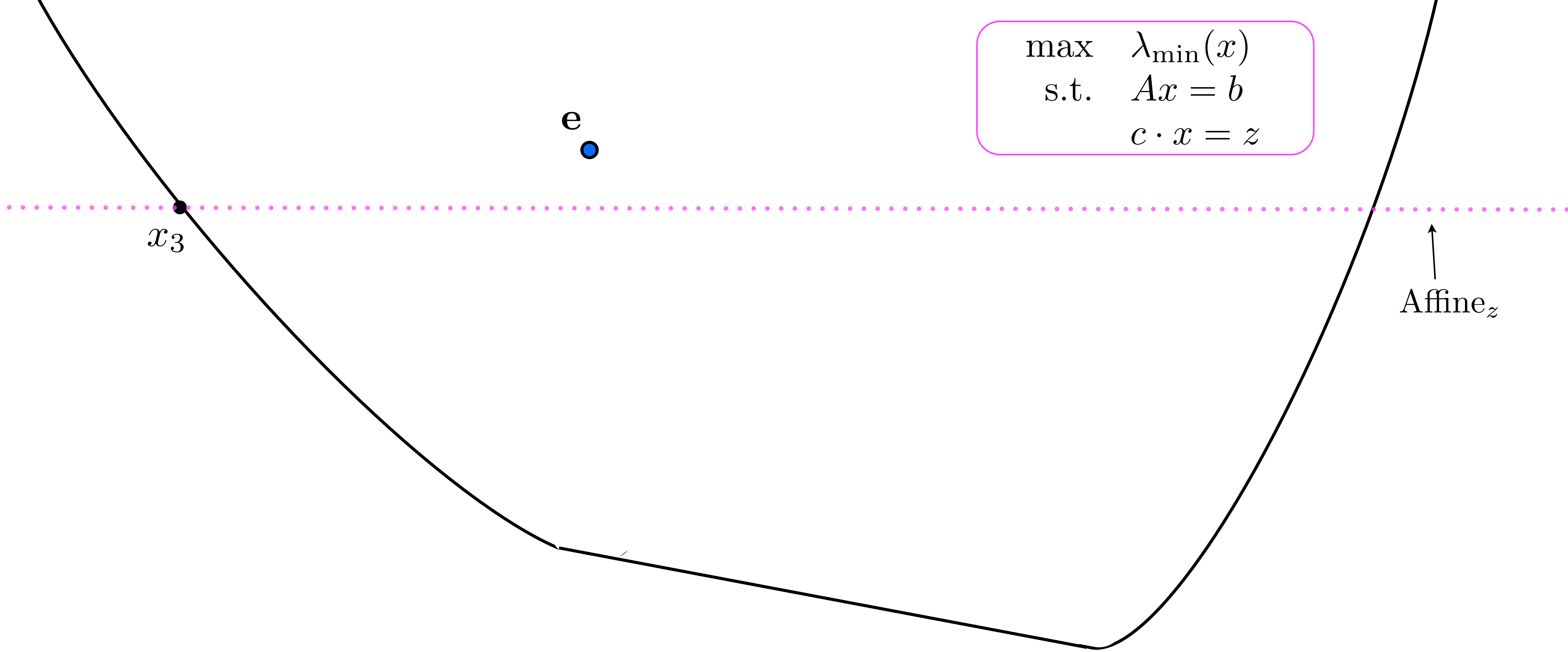
$\max \quad \lambda_{\min}(x)$
$\text{s.t.} \quad Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_2
Affine_z



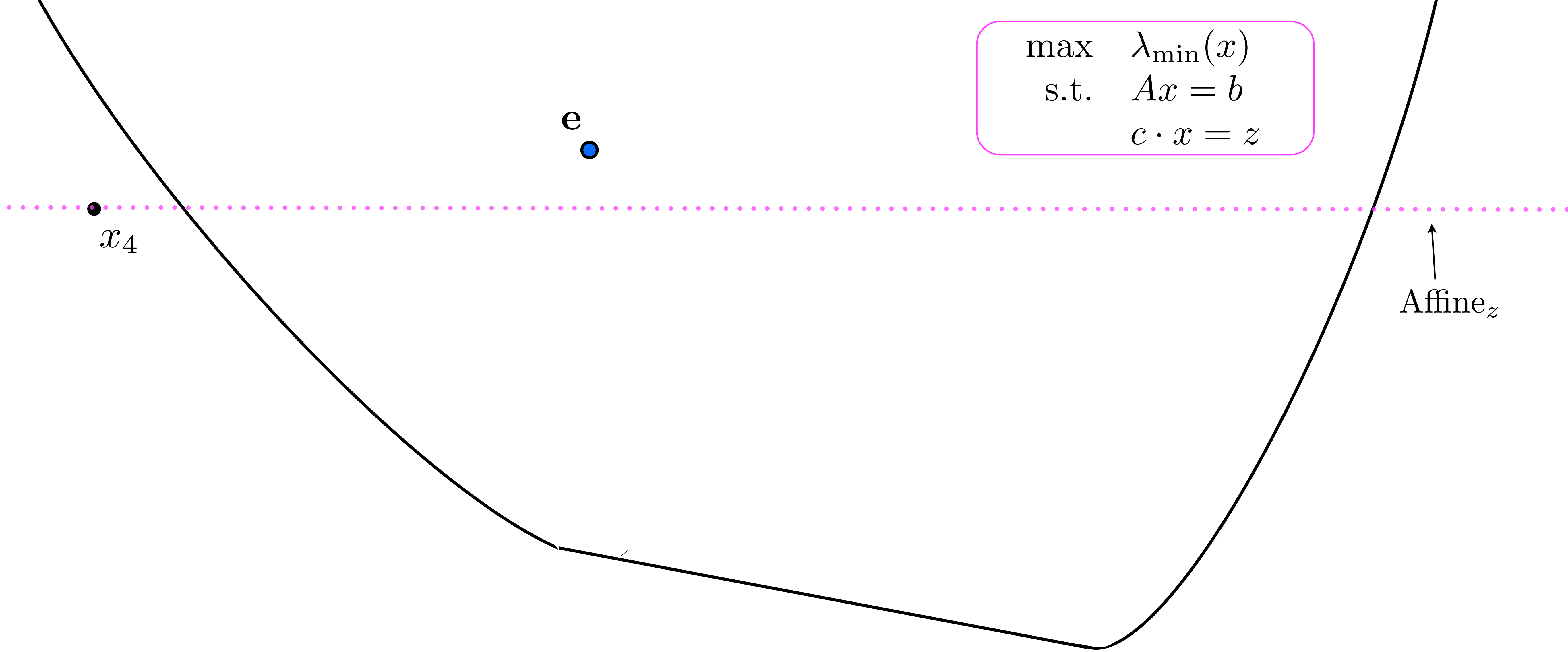
max $\lambda_{\min}(x)$
s.t. $Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_3
Affine$_z$



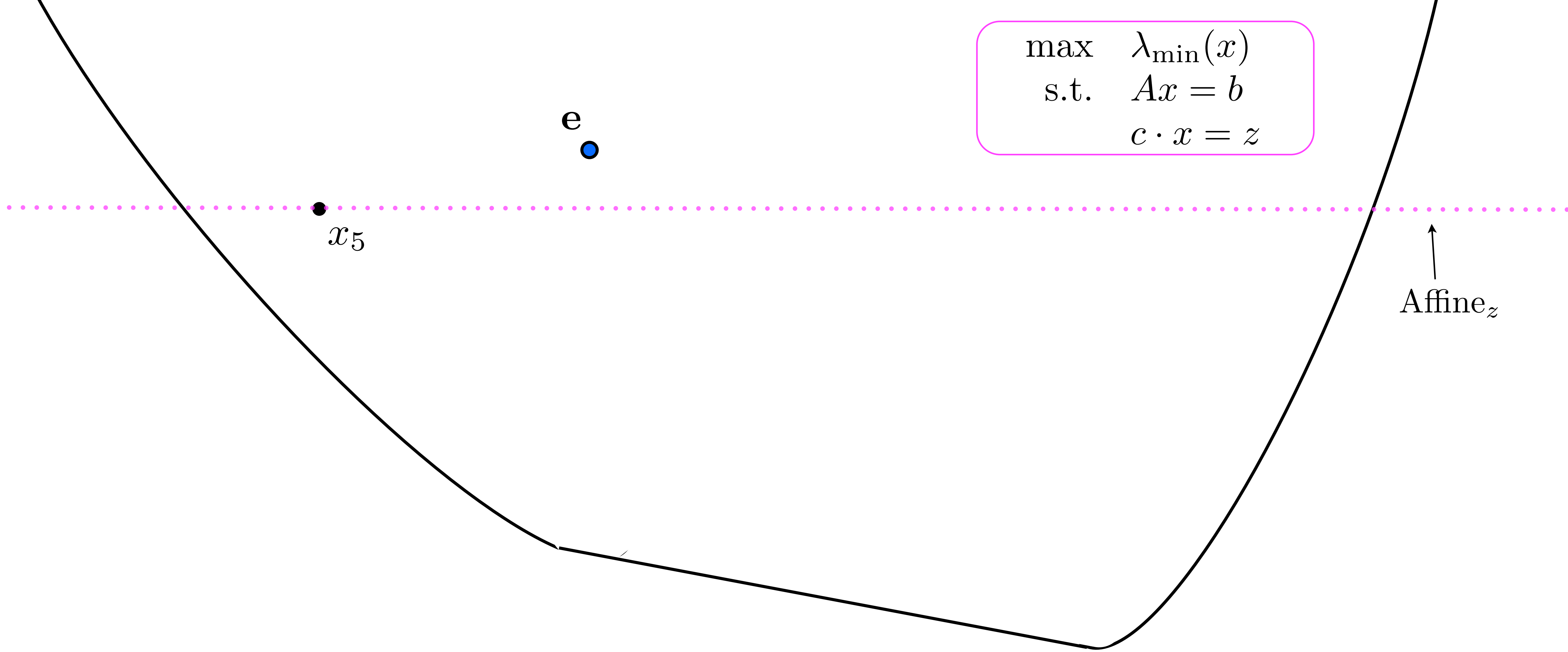
max $\lambda_{\min}(x)$
s.t. $Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_3
Affine$_z$



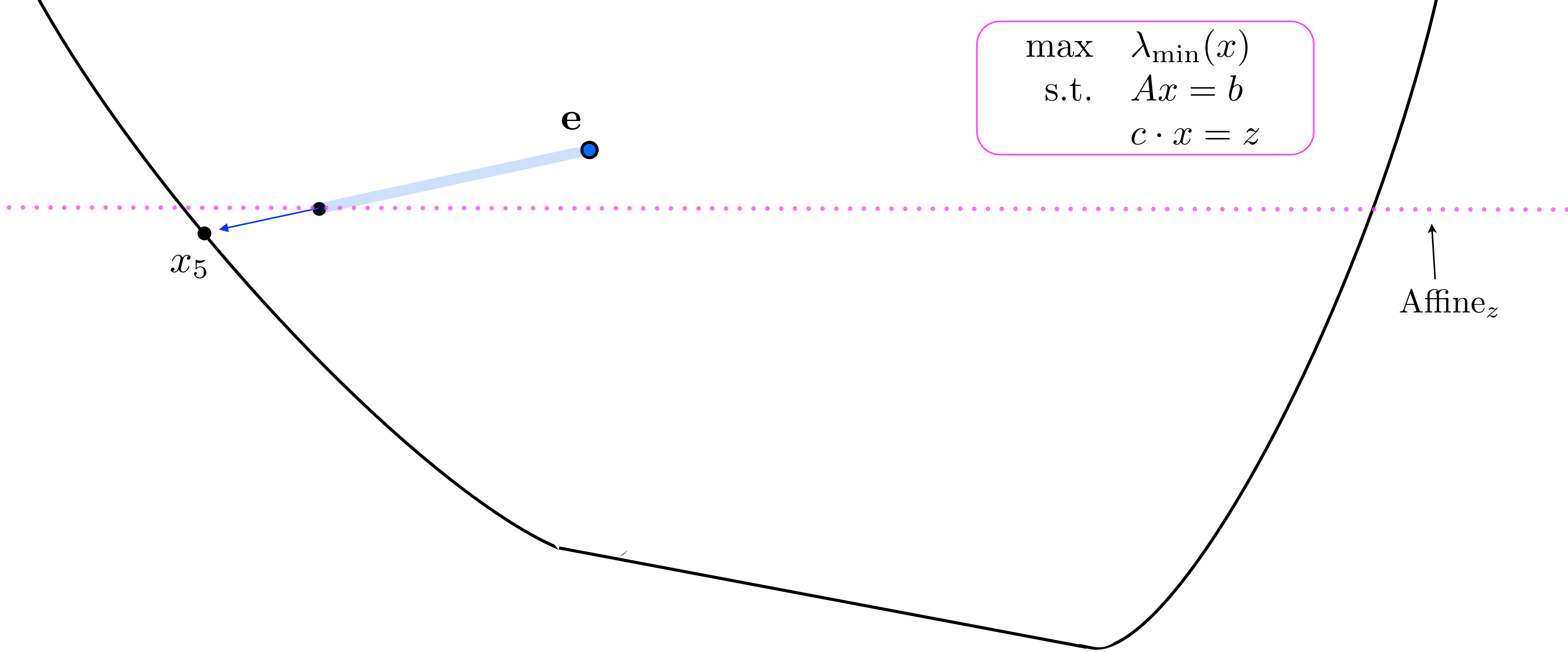
max $\lambda_{\min}(x)$
s.t. $Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_3
Affine$_z$



$\max \quad \lambda_{\min}(x)$
s.t. $\quad Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_4
Affine_z



max $\lambda_{\min}(x)$
s.t. $Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_5
Affine$_z$


$\max \quad \lambda_{\min}(x)$
$\text{s.t.} \quad Ax = b$
$c \cdot x = z$
$\mathbf{e}$
x_5
Affine_z

Applying a supgradient method results in a sequence $x_0, x_1, \ldots$ for which $\ldots$

Thm:

Lipschitz constant $\leq 1/r_e$

$$\ell \;\geq\; 8\,(M\,\mathrm{Diam})^2 \cdot \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log_{4/3}\left(\frac{c \cdot e - z^*}{c \cdot e - c \cdot x_0} \right) + 1 \right)$$

$$\Rightarrow \quad \min_{k \leq \ell} \frac{c \cdot \pi(x_k) - z^*}{c \cdot e - z^*} \;\leq\; \epsilon$$

Let's see what results by applying the framework
to general convex optimization problems
by putting those problems into conic form.

First we consider minimizing a convex function subject to no constraints,
but we make some assumptions on the function
so that we can clarify how the new approach
differs from applying subgradient methods directly ...

Assume:

- f is lower semicontinuous and has a minimizer

$$\min \quad f(x)$$

$$\min \quad f(x) \quad \equiv \quad \begin{array}{rl} \min_{x,t} & t \\ \text{s.t.} & (x,t) \in \operatorname{epi}(f) \end{array} \; := \{(x,t) : f(x) \leq t\}$$

closed convex set

$$\begin{array}{rl} \min_{x,t,t'} & t \\ \text{s.t.} & t' = 1 \\ & (x,t,t') \in \mathcal{K} \end{array} \quad \equiv \quad \begin{array}{rl} \min_{x,t} & t \\ \text{s.t.} & (x,t) \in \mathrm{epi}(f) \;\; := \{(x,t) : f(x) \le t\} \end{array}$$

closed convex set

where $\mathcal{K}$ is the closed cone for which

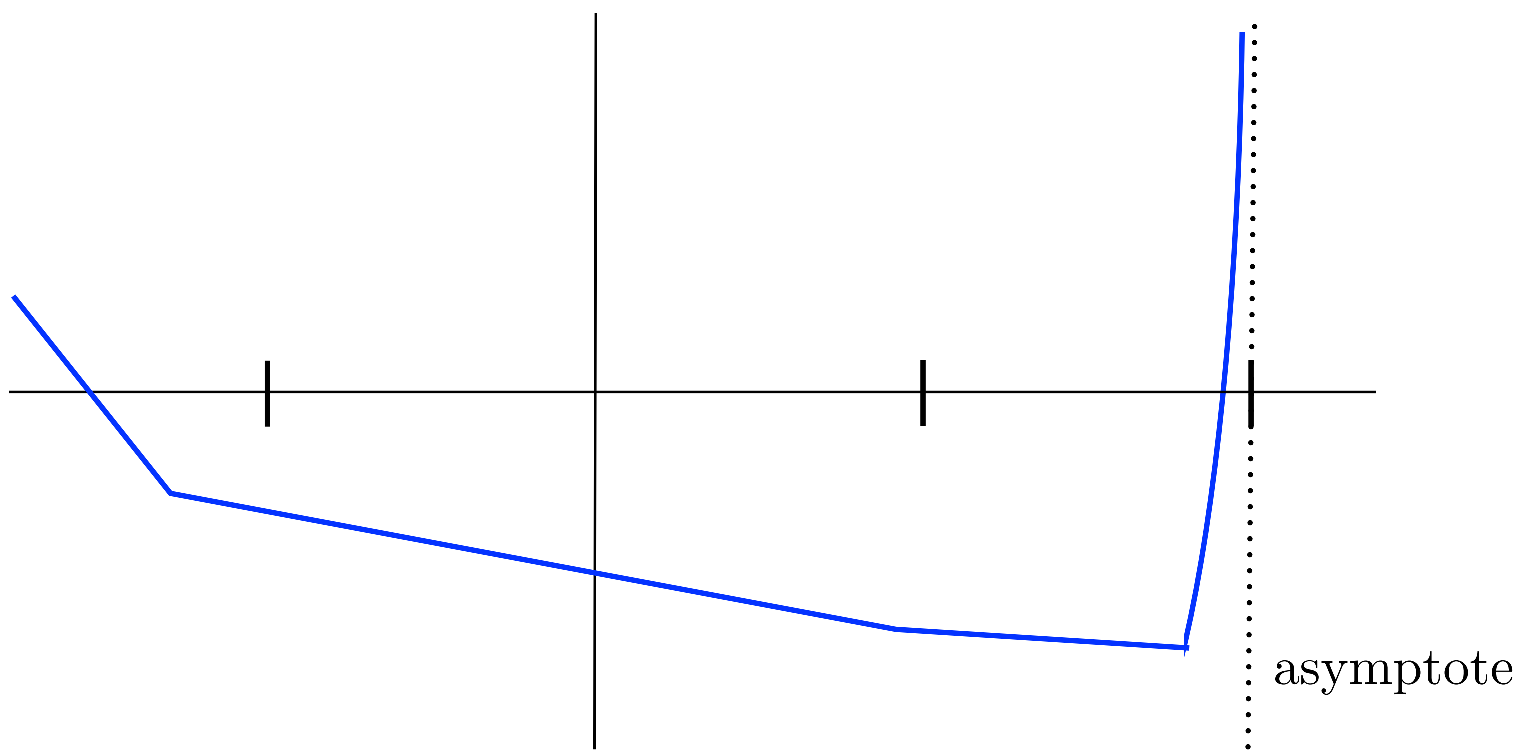
$$(x,t,1) \in \mathcal{K} \Leftrightarrow (x,t) \in \mathrm{epi}(f)$$

The problem on the left is the conic formulation to which we apply our approach.

$$\min \quad f(x)$$

Assume:

- f is lower semicontinuous and has a minimizer
- $\{x : f(x) < \infty\}$ is open
- $\|x\| < 1 \;\Rightarrow\; f(x) < 0$



graph of such a function f

$$\min \quad f(x)$$

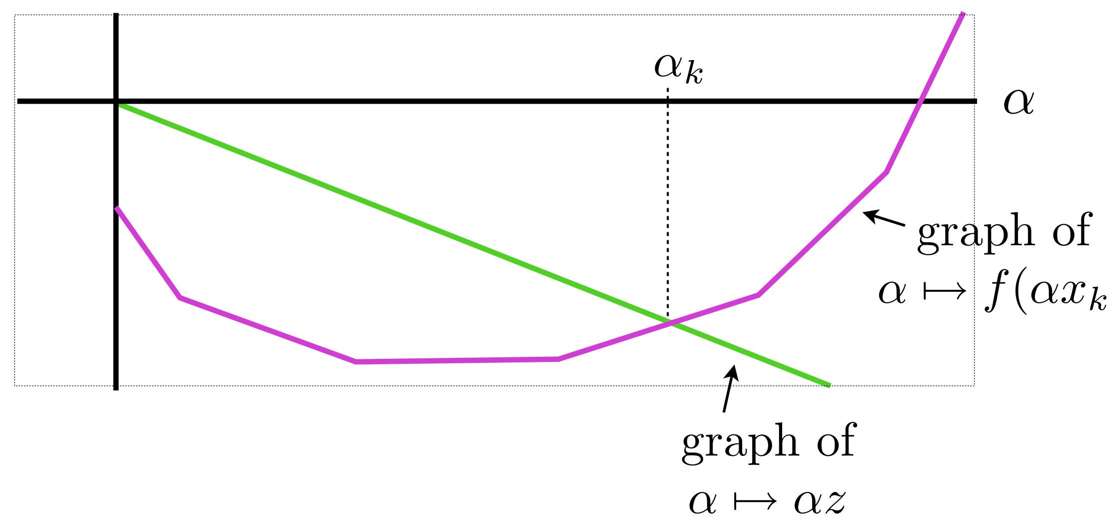
Assume:

- f is lower semicontinuous and has a minimizer
- $\{x : f(x) < \infty\}$ is open
- $\|x\| < 1 \ \Rightarrow \ f(x) < 0$

Initialize: $\quad x_0 = \vec{0}, \quad z = f(\vec{0})$

Iterate:

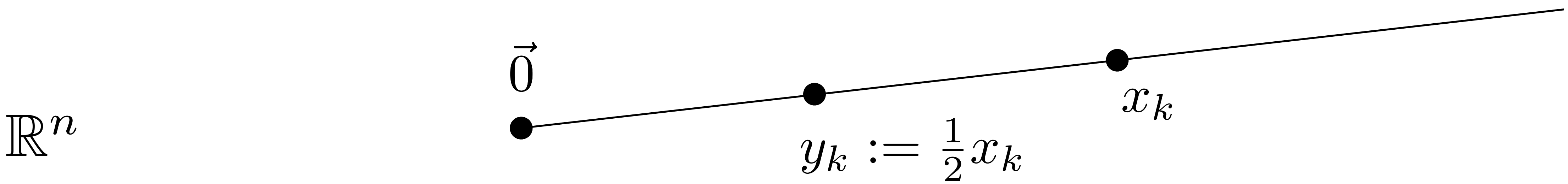
(1) Compute the positive scalar α_k satisfying $f(\alpha_k x_k) = \alpha_k z$.



(2) Let $y_k := \alpha_k x_k$.

$z < 0$ – an upper bound on the optimal value of $\min f(x)$

(the value z is occasionally updated)

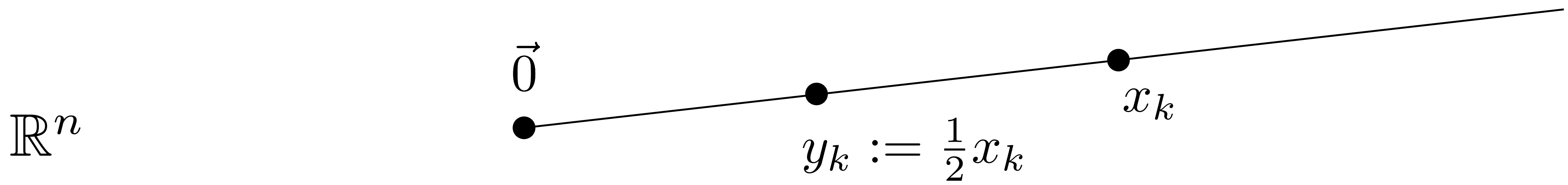


Determine the positive scalar α_k for which $f(\alpha_k x_k) = \alpha_k z$,

and then define $y_k = \alpha_k x_k$.

$z < 0$ – an upper bound on the optimal value of $\min f(x)$

(the value z is occasionally updated)



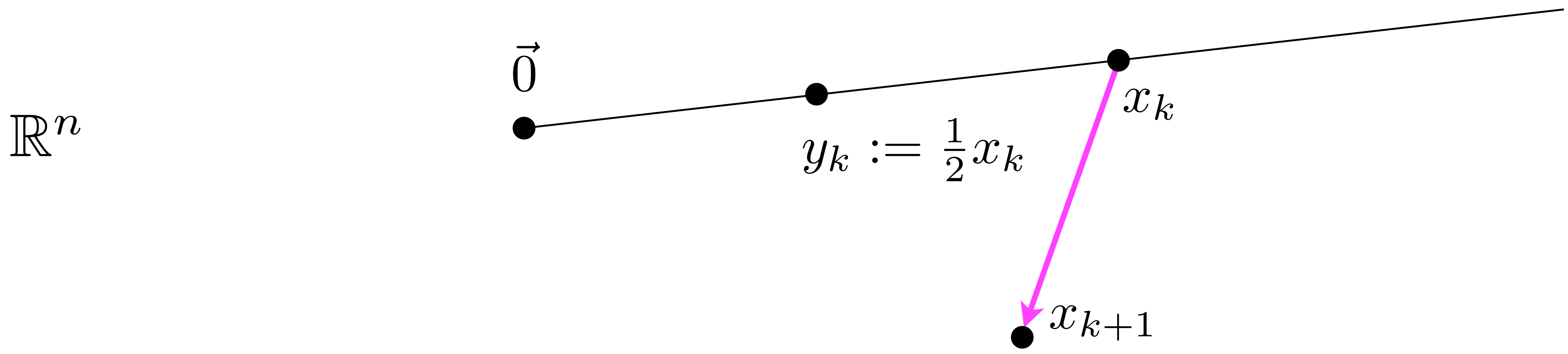
subgradient at y_k

If $\alpha_k < 4/3$, then let $x_{k+1} = x_k + \frac{\epsilon}{2\|g\|^2}\, g$

where $g = \frac{1}{f(y_k) + \langle \nabla f(y_k), \vec{0} - y_k \rangle} \nabla f(y_k)$

$z < 0$ – an upper bound on the optimal value of $\min f(x)$

(the value z is occasionally updated)



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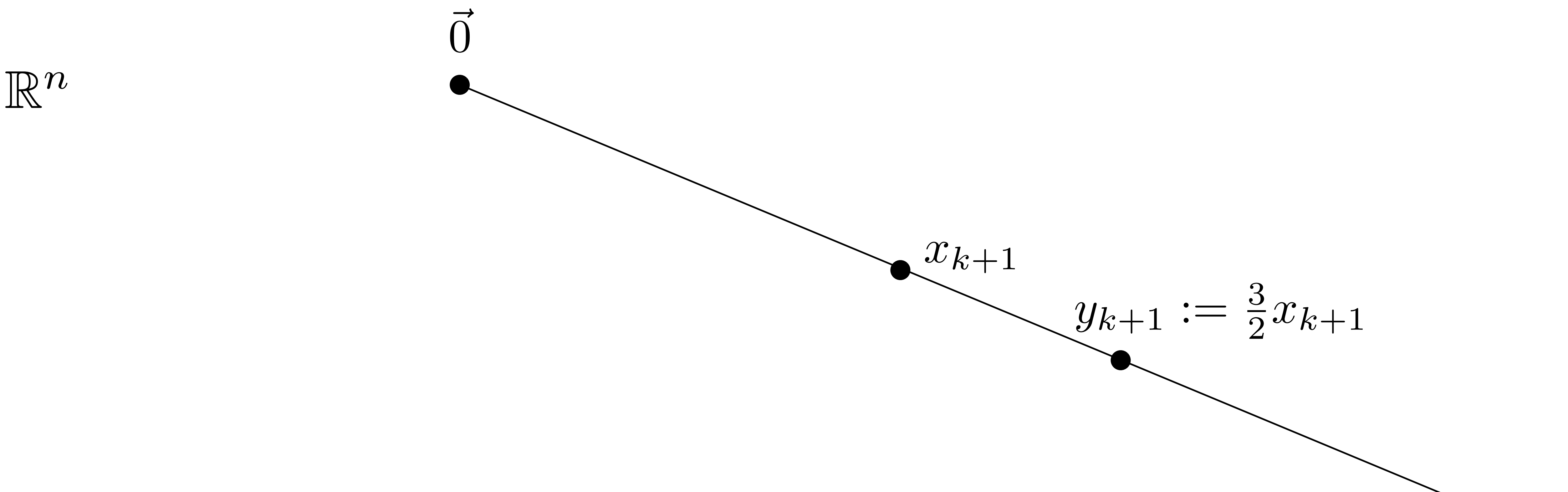
subgradient at y_k

where $g = \frac{1}{f(y_k) + \langle \nabla f(y_k), \vec{0} - y_k \rangle} \nabla f(y_k)$

The subgradient is for y_k but the step is taken from x_k!

$z < 0$ – an upper bound on the optimal value of $\min f(x)$

(the value z is occasionally updated)

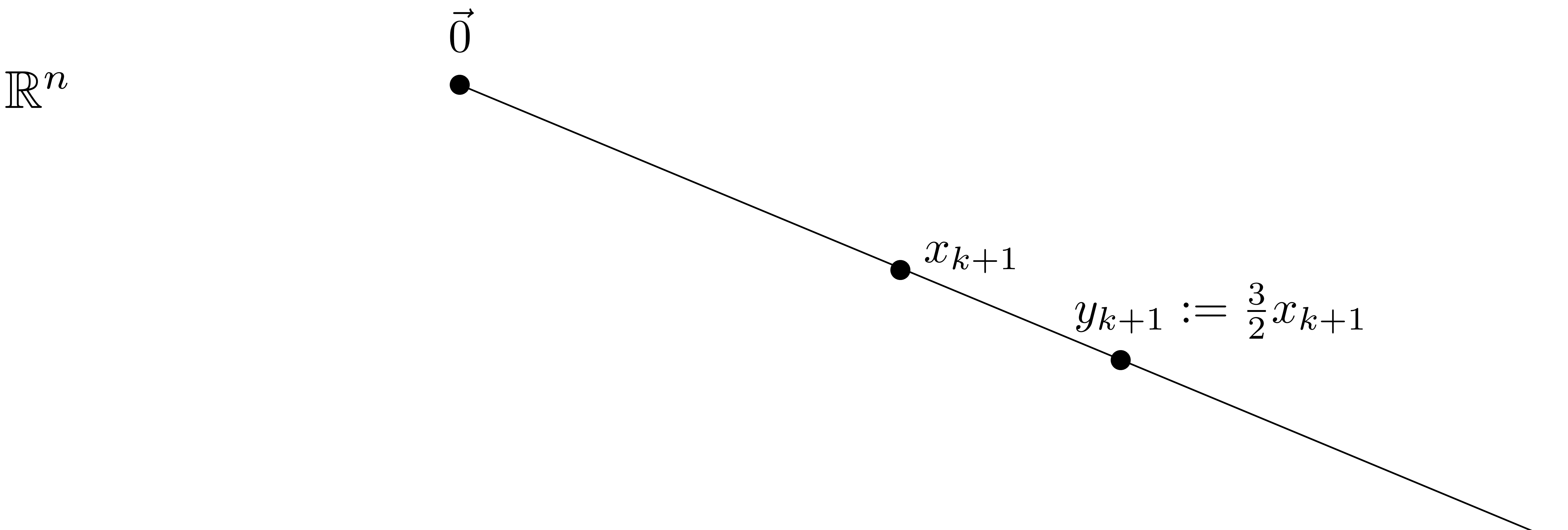


Determine the positive scalar α_{k+1} for which $f(\alpha_{k+1} x_{k+1}) = \alpha_{k+1} z$,

and then define $y_{k+1} = \alpha_{k+1} x_{k+1}$.

$z < 0$ – an upper bound on the optimal value of $\min f(x)$

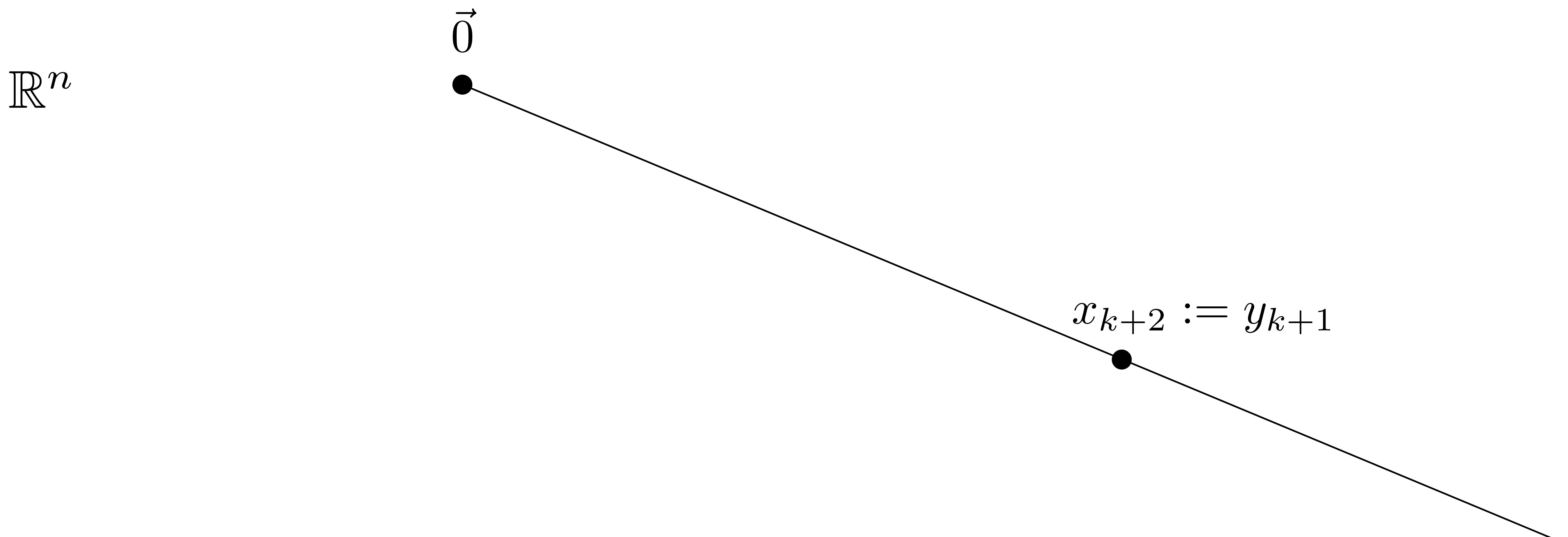
(the value z is occasionally updated)



If $\alpha_{k+1} \geq 4/3$, define $x_{k+2} = y_{k+1}$ and update z: $z \leftarrow f(x_{k+2})$

$z < 0$ – an upper bound on the optimal value of $\min f(x)$

(the value z is occasionally updated)



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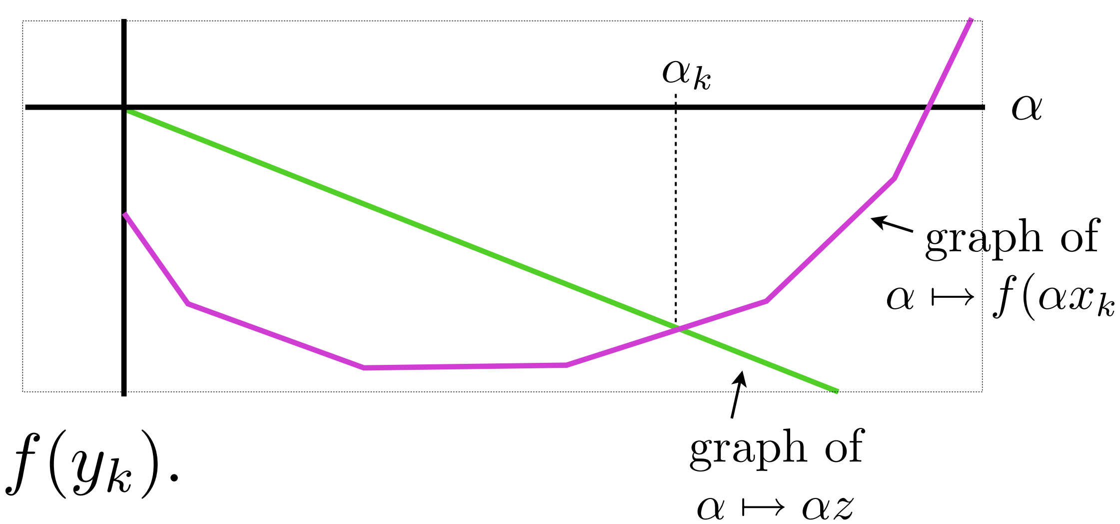

(2) Let $y_k := \alpha_k x_k$.

(3) If $\alpha_k \geq 4/3$, let $x_{k+1} = y_k$ and $z \leftarrow f(y_k)$.

Else let $\quad x_{k+1} = x_k + \dfrac{\epsilon}{2\|g_k\|^2}\, g_k$

$$\text{where} \quad g_k = \frac{1}{\underbrace{\nabla f(y_k) + \langle \nabla f(y_k), \vec{0} - y_k \rangle}_{\leq f(\vec{0}) < 0}} \underbrace{\nabla f(y_k)}_{\text{subgradient at } y_k} .$$

$$\min \quad f(x)$$

Assume:

- f is lower semicontinuous and has a minimizer
- $\{x : f(x) < \infty\}$ is open
- $\|x\| < 1 \;\Rightarrow\; f(x) < 0$

Cor:

optimal value < 0

The algorithm computes y_k satisfying $\dfrac{f(y_k) - f^*}{0 - f^*} \leq \epsilon$

$$\text{where} \quad k \leq 8\,D^2 \left(\frac{1}{\epsilon^2} + \frac{1}{\epsilon} \log_{4/3}\big((D+1)(1-\epsilon)\big) + 1 \right)$$

$$\text{defining} \quad D = \text{diameter}\big(\{x : f(x) \leq f(\vec{0})\}\big) \; .$$

Differs from traditional subgradient literature in that f is not required to be Lipschitz continuous!!!

More generally . . .

extended valued, convex, lower semicontinuous

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & x \in \text{Feas} \end{array}$$

$$\text{Feas} = \{x \in S : Ax = b\}$$

closed and convex

Assume:

- $\bar{x}$ satisfies $A\bar{x} = b$ and $\bar{x} \in \text{interior}\big(S \cap \text{effective\_domain}(f)\big)$
- Euclidean norm satisfies $\underbrace{\{x \in B(\bar{x}, 1) : Ax = b\}}_{\substack{\text{let } \widehat{f} \text{ be a scalar upper bound} \\ \text{on } f(x) \text{ for all } x \text{ in this set}}} \subseteq S \cap \text{effective\_domain(f)}$
- $D = \text{diameter}\big(\{x \in \text{Feas} : f(x) \leq f(\bar{x})\}\big)$

Then can compute feasible x satisfying $\dfrac{f(x) - f^*}{\widehat{f} - f^*} \leq \epsilon$ (optimal value: f^*)

within $\mathcal{O}\left(D^2 \cdot \left(\dfrac{1}{\epsilon^2} + \dfrac{1}{\epsilon} \log D \right) \right)$ iterations.

(see arXiv posting for details)

• • •

Now we turn to a second topic,
the "smoothing" of a convex conic optimization problem,
thus allowing accelerated gradient methods to be applied,
resulting in better complexity bounds.

In order to provide an explicit smoothing,
we need the optimization problem to have algebraic structure,
and thus we restrict attention to "hyperbolic programs",
still a very general class of conic optimization problems.

Yu. Nesterov*

Smooth minimization of non-smooth functions

Received: February 4, 2003 / Accepted: July 8, 2004
Published online: December 29, 2004 –

Abstract. In this paper we propose a new approach for constructing efficient schemes for non-smooth convex optimization. It is based on a special smoothing technique, which can be applied to functions with explicit max-structure. Our approach can be considered as an alternative to black-box minimization. From the viewpoint of efficiency estimates, we manage to improve the traditional bounds on the number of iterations of the gradient schemes from $O\left(\frac{1}{\epsilon^2}\right)$ to $O\left(\frac{1}{\epsilon}\right)$, keeping basically the complexity of each iteration unchanged.

Our motivation was to develop an approach similar to the one of Nesterov,
but which applies to optimization problems with complicated feasible regions
rather than just "simple" ones.

We depend heavily on various works of Nesterov,
as well as results from the literature on "hyperbolic polynomials."

(Our most recent arXiv posting has all of the details.)

Yu. Nesterov*

Smooth minimization of non-smooth functions

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Abstract. In this paper we propose a new approach for constructing efficient schemes for non-smooth convex optimization. It is based on a special smoothing technique, which can be applied to functions with explicit max-structure. Our approach can be considered as an alternative to black-box minimization. From the viewpoint of efficiency estimates, we manage to improve the traditional bounds on the number of iterations of the gradient schemes from $O\left(\frac{1}{\epsilon^2}\right)$ to $O\left(\frac{1}{\epsilon}\right)$, keeping basically the complexity of each iteration unchanged.

Yurii Nesterov

Smoothing technique and its applications in semidefinite optimization*

Received: 20 January 2005 / Accepted: 23 February 2006 /
Published online: 27 April 2006

Smoothing

Following Nesterov, rely on the smooth concave function

$$f_\mu(X) := -\mu \ln \sum_j \exp(-\lambda_j(X)/\mu) \qquad \text{(for fixed } \mu > 0\text{)}$$

Easy to see: $\lambda_{\min}(X) - \mu \ln n \;\le\; f_\mu(X) \;\le\; \lambda_{\min}(X)$

Not so obvious, but which Nesterov showed:

$$\|\nabla f_\mu(X) - \nabla f_\mu(Y)\|_\infty^* \;\le\; \tfrac{1}{\mu}\,\|X - Y\|_\infty$$

that is, $X \mapsto \nabla f_\mu(X)$ has Lipschitz constant $L = 1/\mu$

$$\nabla f_\mu(X) = \frac{1}{\sum_j \exp(-\lambda_j(X)/\mu)}\, Q \begin{bmatrix} \exp(-\lambda_1(X)/\mu) & & \\ & \ddots & \\ & & \exp(-\lambda_n(X)/\mu) \end{bmatrix} Q^T$$

where $X = Q \begin{bmatrix} \lambda_1(X) & & \\ & \ddots & \\ & & \lambda_n(X) \end{bmatrix} Q^T$ is an eigendecomposition of X
expensive!

A relevant line of work thus begins with . . .

d'Aspremont, "Smooth optimization with approximate gradient"
SIAM J Opt (2008)

Smoothing

Following Nesterov, rely on the smooth concave function

$$f_\mu(X) := -\mu \ln \sum_j \exp(-\lambda_j(X)/\mu) \qquad \text{(for fixed } \mu > 0\text{)}$$

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that is, $X \mapsto \nabla f_\mu(X)$ has Lipschitz constant $L = 1/\mu$

$$\nabla f_\mu(X) = \frac{1}{\sum_j \exp(-\lambda_j(X)/\mu)}\, Q \begin{bmatrix} \exp(-\lambda_1(X)/\mu) & & \\ & \ddots & \\ & & \exp(-\lambda_n(X)/\mu) \end{bmatrix} Q^T$$

where $X = Q \begin{bmatrix} \lambda_1(X) & & \\ & \ddots & \\ & & \lambda_n(X) \end{bmatrix} Q^T$ is an eigendecomposition of X

For linear programming: $\nabla f_\mu(x) = \frac{1}{\sum_j \exp(-x_j/\mu)} \begin{bmatrix} \exp(-x_1/\mu) \\ \vdots \\ \exp(-x_n/\mu) \end{bmatrix}$

$$
\begin{array}{ll} \min & \langle C, X\rangle \\ \text{s.t.} & \mathcal{A}(X) = b \\ & X \succeq 0 \end{array}
\quad \equiv \quad
\begin{array}{ll} \max & \lambda_{\min}(X) \\ \text{s.t.} & \mathcal{A}(X) = b \\ & \langle C, X\rangle = z \end{array}
\quad \approx \quad
\begin{array}{ll} \max & f_\mu(X) \\ \text{s.t.} & \mathcal{A}(X) = b \\ & \langle C, X\rangle = z \end{array}
$$

Same goal as before: Compute feasible X satisfying $\dfrac{\langle C, X\rangle - z^*}{\langle C, I\rangle - z^*} \leq \epsilon$

Choosing $\mu = \epsilon/(6 \ln n)$
and relying on Nesterov's original accelerated gradient method ...

Thm:

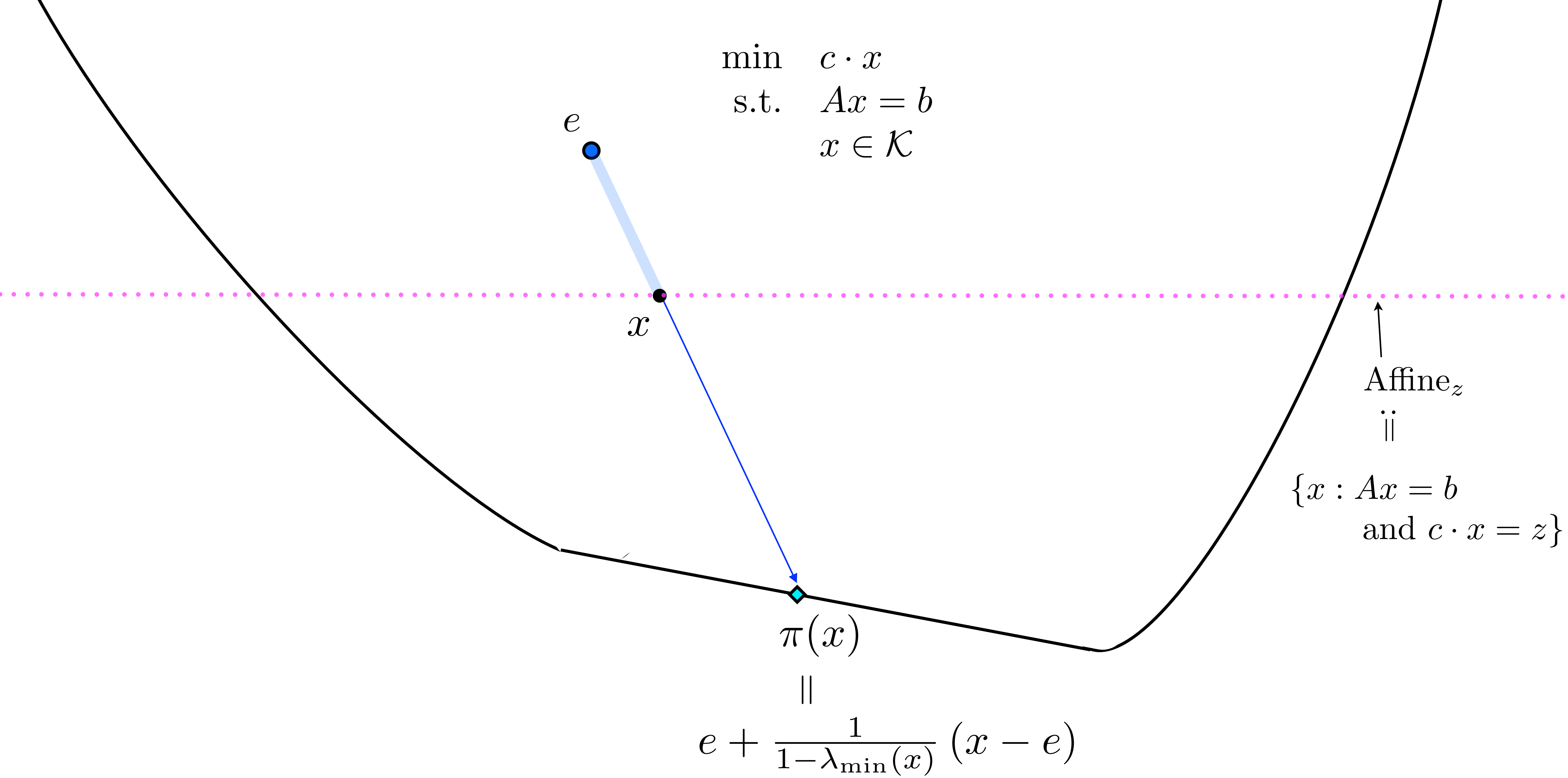
$$
k \geq \underbrace{\kappa}_{\text{a universal constant}} \cdot \left(\sqrt{\ln n} \cdot \text{Diam} \cdot \left(\frac{1}{\epsilon} + \log \frac{\langle C, I\rangle - z^*}{\langle C, I\rangle - \langle C, X_0\rangle} \right) \right)
$$

$$
\Rightarrow \quad \frac{\langle C, \pi(X_k)\rangle - z^*}{\langle C, I\rangle - z^*} \leq \epsilon
$$

Especially-notable earlier work with similar iteration bounds:

Lu, Nemirovski and Monteiro, "Large-scale semidefinite programming via a saddle point Mirror-Prox algorithm" *Math Prog* (2007)

Lan, Lu and Monteiro, "Primal-dual first-order methods with $O(1/\epsilon)$ iteration-complexity for cone programming" *Math Prog* (2011)



$\ldots$ where $\lambda_{\min}(x)$ is the scalar λ satisfying $\quad x - \lambda\, e \in \text{boundary}(\mathcal{K})$

$$\begin{array}{rl} \min & c \cdot x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{K} \end{array}$$

Euclidean space

Defn: $\mathcal{K} \subseteq \mathcal{E}$ is a "hyperbolicity cone"
if there is a homogeneous polynomial p satisfying:

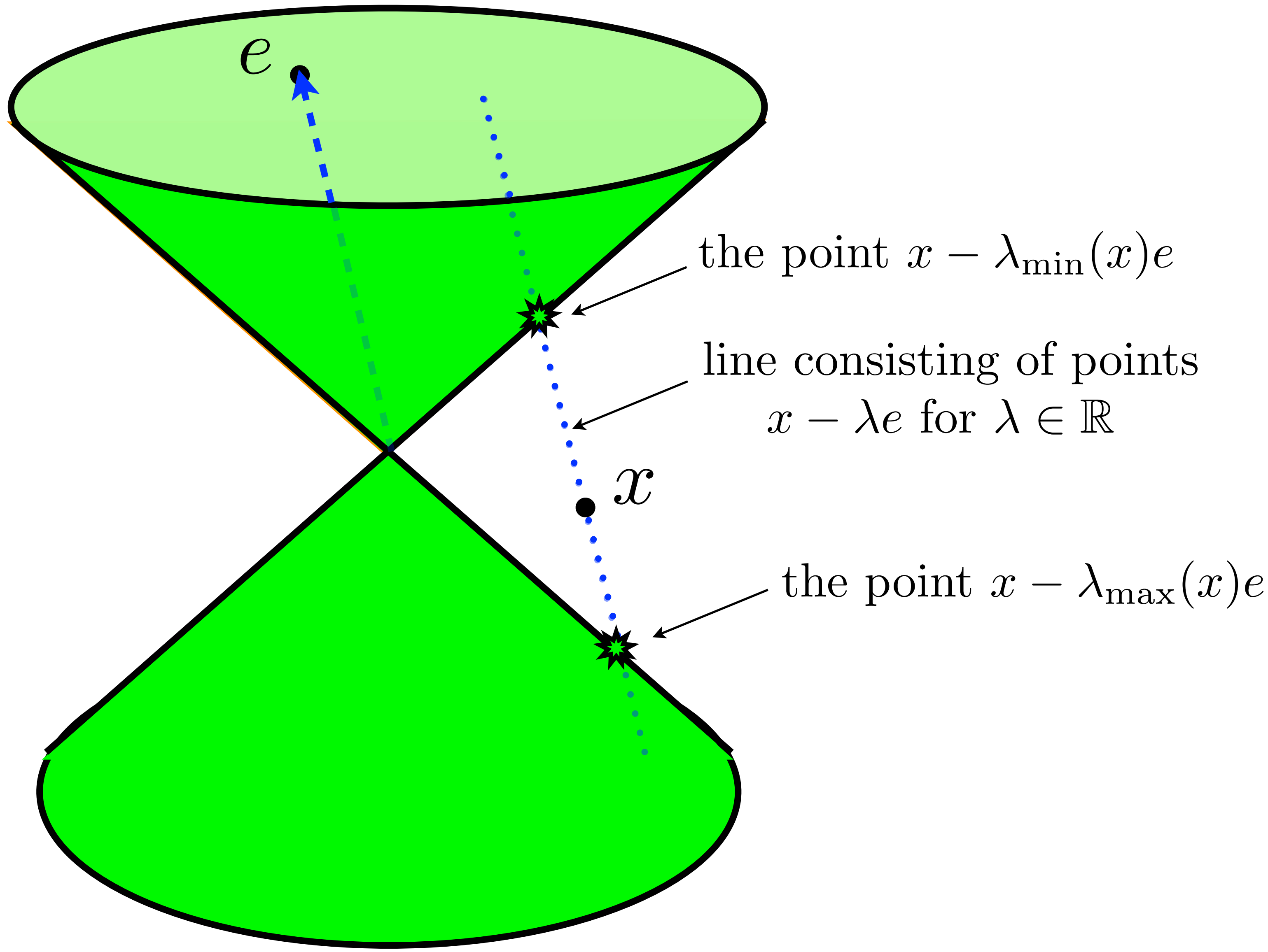
- $p(x) \neq 0$ for all $x \in \mathrm{int}(\mathcal{K})$
- $p(x) = 0$ for all $x \in \mathrm{bdy}(\mathcal{K})$
- $\exists\, e \in \mathrm{int}(\mathcal{K})$ such that for all $x \in \mathcal{E}$,
  the univariate polynomial $\lambda \mapsto p(x - \lambda e)$ has only real roots.

Example: $\mathcal{K} = \mathbb{S}^{n \times n}_{+}$, $p(X) = \det(X)$, $e = I$, $\lambda \mapsto \det(X - \lambda I)$

Also: non-negative orthant, second-order cones, and many others.

$$\mathcal{K} = \{(x_1, x_2, x_3) : x_3 \geq \sqrt{x_1^2 + x_2^2}\,\}$$

$$p(x_1, x_2, x_3) = x_3^2 - x_1^2 - x_2^2$$



Another example (this one not obvious):

Let $E_k : \mathbb{R}^n \to \mathbb{R}$ be the elementary symmetric polynomial of degree k,

that is, $E_k(x) = \sum_{j_1 < j_2 < \ldots < j_k} x_{j_1} x_{j_2} \cdots x_{j_k}$

Then for $k = 1, \ldots, n$,

$$\mathcal{K}^{(k)} := \{x : E_i(x) \geq 0 \text{ for all } i = 1, \ldots, k\}$$

is a hyperbolicity cone with polynomial $p(x) = E_k(x)$

These cones are nested:

$$\mathbb{R}^n_+ = \mathcal{K}^{(n)} \subset \mathcal{K}^{(n-1)} \subset \ldots \subset \mathcal{K}^{(2)} \subset \mathcal{K}^{(1)}$$

Each hyperbolicity cone has such a nested set of "derivative cones" which themselves are hyperbolicity cones.

$$\begin{array}{rl} \min & c \cdot x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{K} \end{array}$$

Defn: $\mathcal{K} \subseteq \overset{\text{Euclidean space}}{\mathcal{E}}$ is a "hyperbolicity cone"

if there is a homogeneous polynomial p satisfying:

- $p(x) \neq 0$ for all $x \in \operatorname{int}(\mathcal{K})$
- $p(x) = 0$ for all $x \in \operatorname{bdy}(\mathcal{K})$
- $\exists\, e \in \operatorname{int}(\mathcal{K})$ such that for all $x \in \mathcal{E}$,

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Example: $\mathcal{K} = \mathbb{S}_+^{n \times n}$, $p(X) = \det(X)$, $e = I$, $\lambda \mapsto \det(X - \lambda I)$

Also: non-negative orthant, second-order cones, and many others.

Theorem (Gårding, 1959): For ***every*** $e \in \operatorname{int}(\mathcal{K})$ and all $x \in \mathcal{E}$,

the univariate polynomial $\lambda \mapsto p(x - \lambda e)$ has only real roots.

If each of $\mathcal{K}_1$ and $\mathcal{K}_2$ is a hyperbolicity cone, then so is $\mathcal{K}_1 \times \mathcal{K}_2$ and* $\mathcal{K}_1 \cap \mathcal{K}_2$.

If $\mathcal{K}' \subseteq \mathcal{E}'$ is a hyperbolicity cone and $T : \mathcal{E} \to \mathcal{E}'$ is a linear transformation,

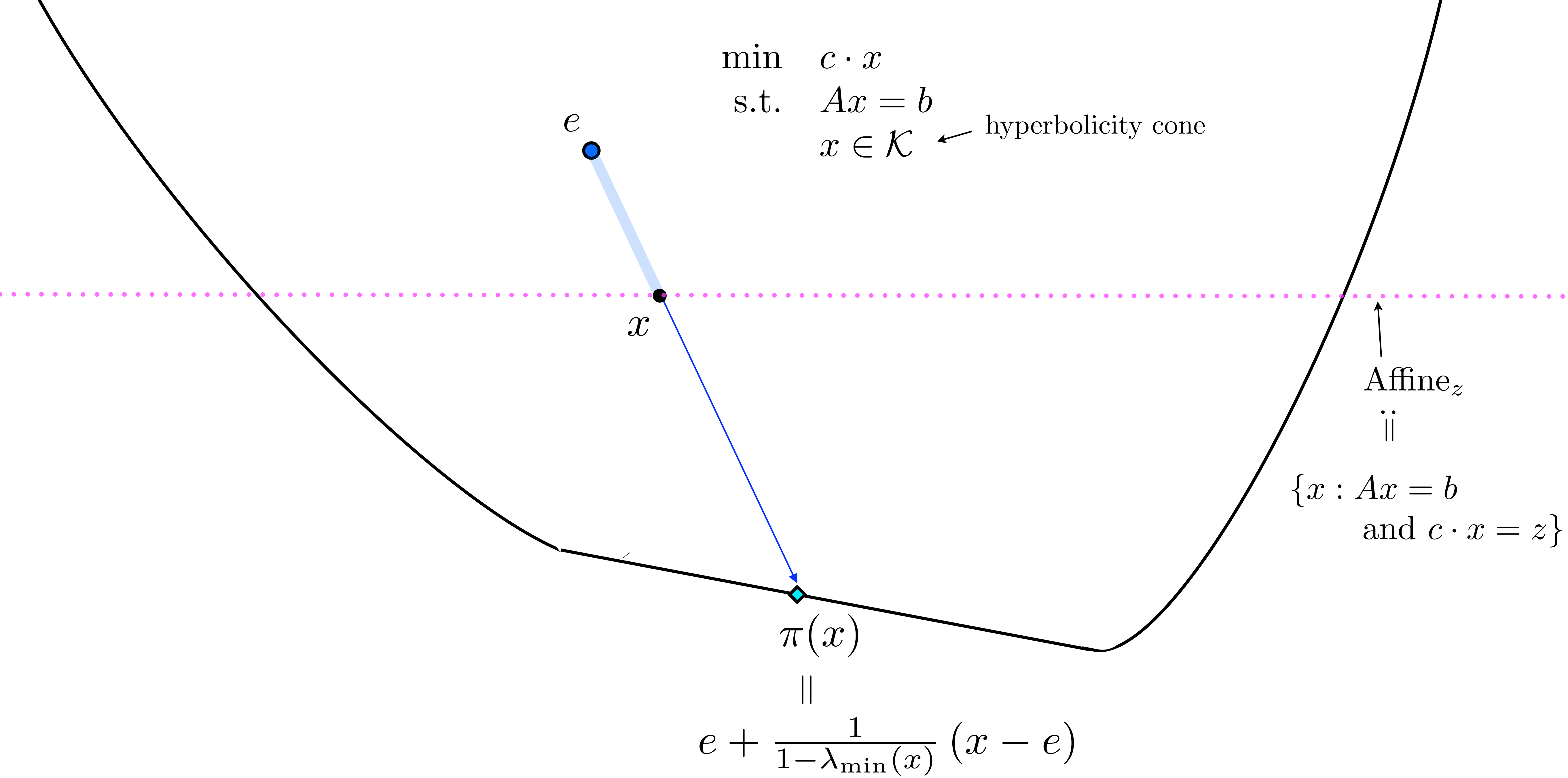
then* $\mathcal{K} := \{x : T(x) \in \mathcal{K}'\}$ is a hyperbolicity cone.

"hyperbolic program"

$$\begin{aligned} \min \quad & c \cdot x \\ \text{s.t.} \quad & Ax = b \\ & x \in \mathcal{K} \end{aligned}$$

hyperbolicity cone

Güler, "Hyperbolic polynomials and interior point methods for convex programming" *Math of Oper Res* (1997)



Now every x has n real "eigenvalues" $\lambda_1(x), \ldots, \lambda_n(x)$,

degree of p

the roots of $\lambda \mapsto p(x - \lambda e)$

$$f_\mu(x) \;:=\; -\mu \,\ln \sum_j \exp(-\lambda_j(x)/\mu) \qquad\qquad (\text{for fixed } \mu > 0)$$

Easy to see: $\quad \lambda_{\min}(x) - \mu \,\ln n \;\leq\; f_\mu(x) \;\leq\; \lambda_{\min}(x)$

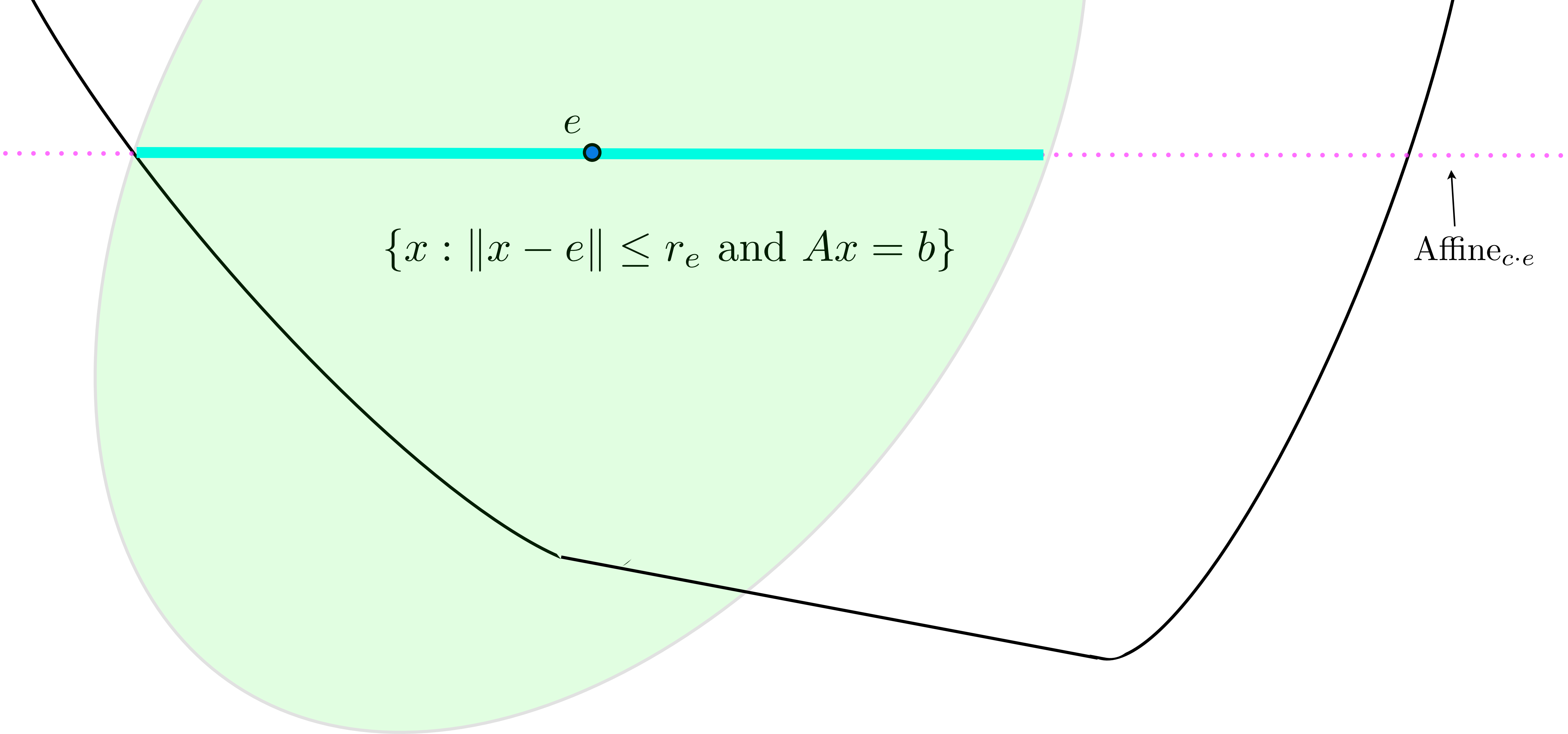
Prop: f_μ is concave and infinitely Fréchet differentiable.

Moreover, $\|\nabla f_\mu(x) - \nabla f_\mu(y)\|_\infty^* \;\leq\; \frac{1}{\mu}\,\|x-y\|_\infty$ for all x, y.

(Mostly a corollary to Nesterov's result and the Helton-Vinnikov Theorem.)

Moreover, the gradients are readily computable, especially if the underlying "hyperbolic polynomial" can be factored as the product of polynomials of low degrees.

(see the arXiv posting for details)



$$\textbf{Cor:} \quad \|P\,\nabla f_\mu(x) - P\,\nabla f_\mu(y)\| \leq \tfrac{1}{r_e^2 \mu}\,\|x - y\| \quad \text{for all } x, y \in \text{Affine}_z \text{ and for every } z$$

$$\begin{array}{ll} \min & c\cdot x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{K} \end{array} \quad \equiv \quad \begin{array}{ll} \max & \lambda_{\min}(x) \\ \text{s.t.} & Ax = b \\ & c\cdot x = z \end{array} \quad \approx \quad \begin{array}{ll} \max & f_\mu(x) \\ \text{s.t.} & Ax = b \\ & c\cdot x = z \end{array}$$

Same goal as before: Compute feasible x satisfying $\dfrac{c\cdot x - z^*}{c\cdot e - z^*} \leq \epsilon$

Choosing $\mu = \epsilon/(6\,\ln n)$
and using a "uniformly optimal" (or "universal")
accelerated gradient method (Lan (2010), Nesterov (2014)) ...

Thm:

The algorithm produces x_k satisfying $\dfrac{c\cdot \pi(x_k) - z^*}{c\cdot e - z^*} \leq \epsilon$

and does so within computing a total number of gradients not exceeding

$$\mathcal{O}\left(\text{Diam}\cdot\sqrt{L}\cdot\left(\frac{1}{\sqrt{\epsilon}} + \log\frac{c\cdot e - z^*}{c\cdot e - c\cdot x_0} + \left|\log\frac{L}{L'}\right|\right)\right)$$

where L is the Lipschitz constant for the gradient map
and where $L' > 0$ is the input guess of L.

$$\boxed{\text{Note: } L \leq \frac{1}{r_e^2\,\mu} = \frac{6\,\ln n}{r_e^2\,\epsilon}}$$

Helton-Vinnikov Theorem

Assume $\mathcal{K}$ is a hyperbolicity cone with polynomial p of degree n.

Fix $e \in \mathrm{int}(\mathcal{K})$

and let $\mathcal{L}$ be a 3-dimensional subspace containing e.

Then there exists a linear transformation $T : \mathcal{L} \to \mathbb{S}^n$

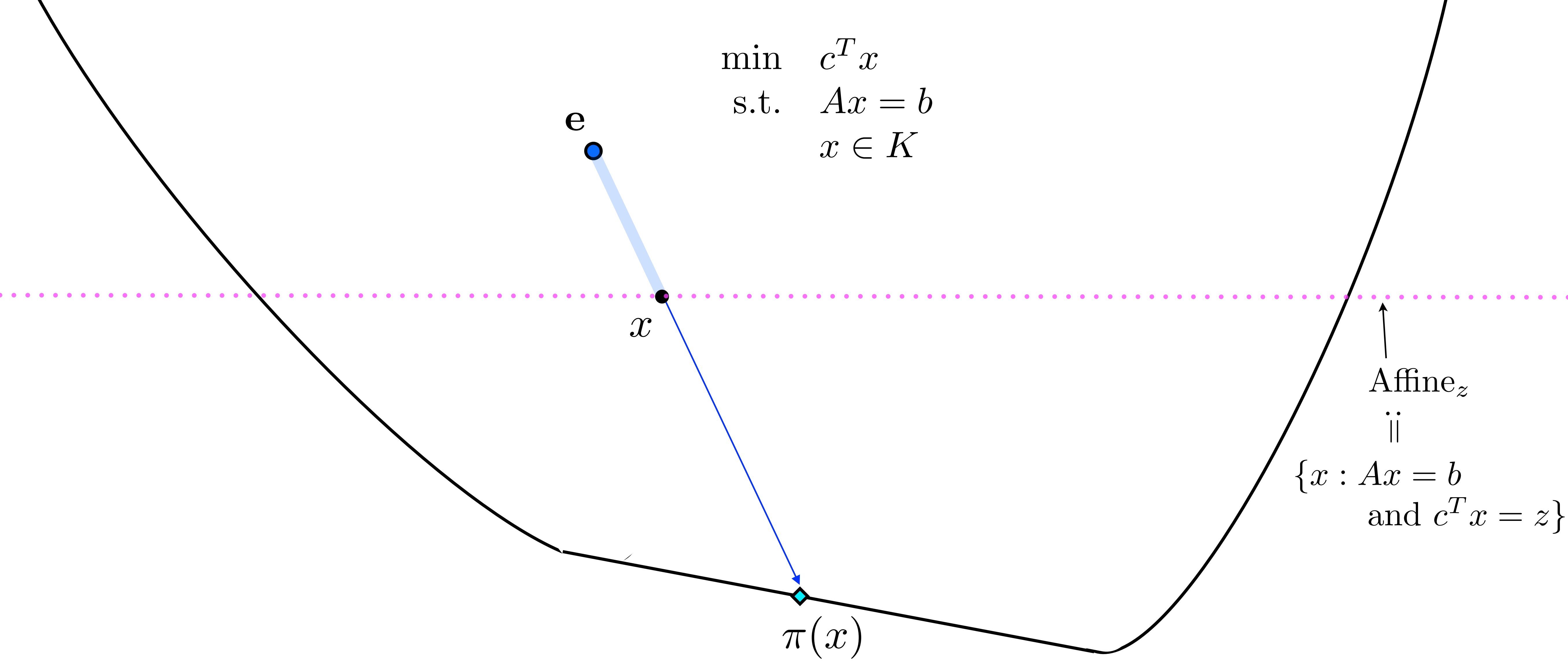
such that $T(e) = I$

and for all $x \in \mathcal{L}$,

$$p(x) = p(e) \det(T(x))$$

"If you can prove something for the PSD cone
by relying only on subspaces which contain I and are of dimension 3,
then you likely can generalize the proof to all hyperbolicity cones
by making use of the H-V Theorem."

But the most important takeaway from these talks
is entirely elementary $\ldots$



$$\begin{array}{rl} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \in K \end{array}$$

$$\mathrm{Affine}_z \;=\; \{x : Ax = b \text{ and } c^T x = z\}$$

$$\pi(x) \;=\; e + \frac{1}{1-\lambda_{\min}(x)}\,(x - e)$$

$\boxed{\ldots \text{where } \lambda_{\min}(x) \text{ is the scalar } \lambda \text{ satisfying } \; x - \lambda e \in \text{boundary}(K)}$

$$\begin{array}{rl} \min & c \cdot x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{K} \end{array} \quad \equiv \quad \begin{array}{rl} \max & \lambda_{\min}(x) \\ \text{s.t.} & Ax = b \\ & c \cdot x = z \end{array}$$

Thanks for listening!