Modeling Convex Subsets of Points – Part 2

Maurice Queyranne

Sauder School of Business at UBC, and
CORE, Université Catholique de Louvain

based on recent and current joint
with several collaborators…

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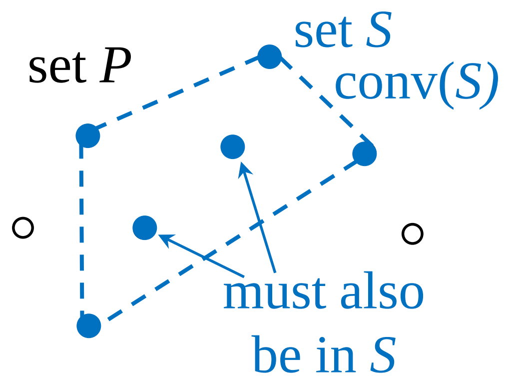
Recall: Convex Subsets of a Given Point Set

- A subset S of a given (finite) set P of n points in $\mathbf{R}^d$ is **convex** (relative to P) iff $S = P \cap \text{conv}(S)$
  equivalently: $P \cap \text{conv}(S) \subseteq S$ (since $P \cap \text{conv}(S) \supseteq S$)
  - i.e., if we select a subset S of points in P then we must also select all points of P that are in their convex hull
  - definition also applies to more general *closure spaces* or *convexity spaces*



- We are interested in formulating the restriction "*the selected set of points must be convex*" in Integer Programming models
  - as hard or soft constraints

Recall: Why Convex Subsets?

- Many (discrete) optimization models seek one or several subsets (of a given set of points, or of elementary regions, a.k.a., “cells”) that should satisfy some “***shape constraint***”
  - often vaguely expressed: the set should “look compact”, its shape should no be “too odd”, etc.
- Convexity is one way of precisely formulating such shape constraints, which is appropriate (or approximately so) in some applications…

Recall: Our Research Agenda

We seek to define notions of ("convex") shapes that are

- relevant to applications, and
- computationally tractable:
  - the Optimization Problem is efficiently solvable (or approximable)
  - the shape requirements can be enforced (or approximated) by a concise system of linear inequalities in natural and/or extended variables

Lectures Overview

Part 1: Computational complexity and algorithms

1. The Maximum Weight Convex Subset problem
2. Dimension 3 and higher: hardness results
3. One-dimension: a well understood case

Part 2: Modeling 2D and related convexities

1. 2D (points in the plane): DP algorithm for the optimization problem
2. 2D convex-shape constraints: IP modeling
3. Other notions of convexity
   a) Poset convexity
   b) Geodesic convexities and related notions

1. Points in the Plane

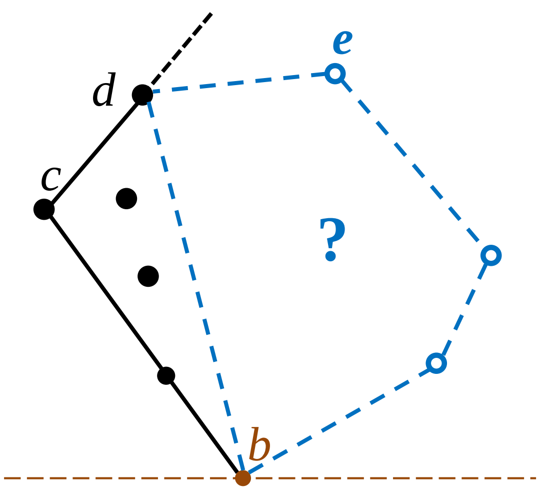
Optimization problem solved by Dynamic Programming

- Basic idea in Eppstein et al. (1992): consider all possible choices of bottom-most selected point $b \in P$
- **Vertex version**: vertices as DP states

$F(b) = \max_{b \in P} \{ \{w_b\} \cup \{w(\mathrm{conv}\{b,c\}) : c \text{ not below } b\}$
$\cup \{ f(b,c,d) : b \text{ is bottom-most point, } c \text{ and } d \text{ are the cw-next two vertices of a convex polytope}\} \}$

where $\boldsymbol{f(b,c,d)}$ is the maximum weight of such a polytope:

$f(b,c,d) = w(\mathrm{conv}\{b,c,d\})$
$+ \max\{ 0, \max_e \{ f(b,d,e) - w(\mathrm{conv}\{b,d\}) :$
e on same side of $\underline{cd}$ as b and
on other side of $\underline{bd}$ than c $\} \}$



- $O(n^3)$ time and $O(n^2)$ space for each b

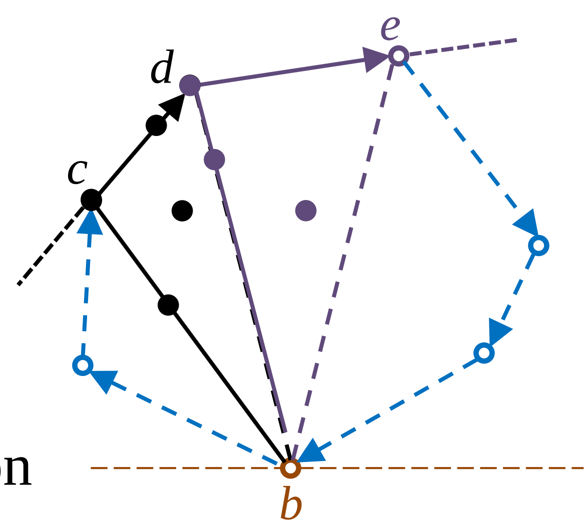
$\Rightarrow O(n^4)$ time and $O(n^2)$ space altogether

1. Points in the Plane: DP Algorithm (2)

- **Edge version**: edges as DP states
- Find a maximum-profit directed cycle beginning and ending at *b* and such that each pair of successive arcs *cd* and *de* are "compatible", i.e., can be edges of a polytope with vertex *b*:
  - *c*, *d* and *e* not colinear, in clockwise order seen from *b*, and
  - *b* in the convex cone they define with apex *d*

  Edge profit $u(cd) = w(\text{conv}\{b,c,d\}) - w(\text{conv}\{b,d\})$
- A special case of longest path with turn restrictions
- $O(n^2)$ time and $O(n^2)$ space for each *b*

  $\Rightarrow O(n^3)$ time and $O(n^2)$ space altogether
  - [Bautista-Santiago et al., 2011, based on Eppstein et al., 1992]

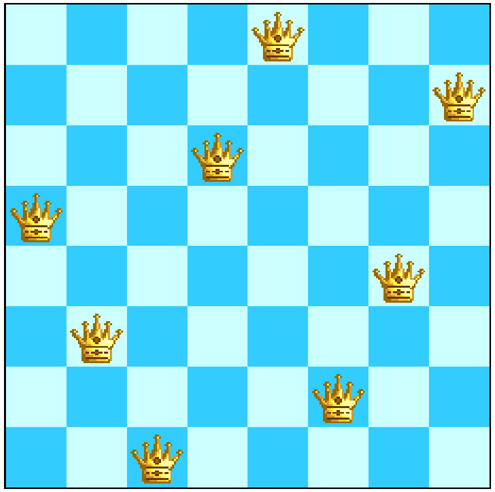


- Generalizes to any edge-decomposable objective, satisfying a monotonicity condition

2. 2D convex-shapes : IP modeling

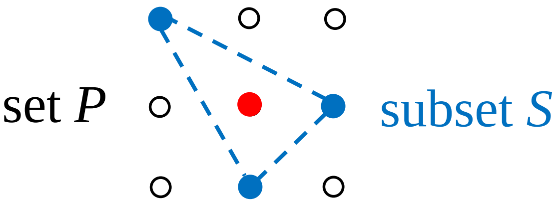
We can formulate 1D-convexity (contiguity) constraints along each coordinate direction, and other directions

- hopefully to obtain approximately convex subsets



but

- the resulting (extended) formulations are not integral
- the solution might not even be "connected"
  - even if we include *all* directions



2. 2D convex-shapes : IP modeling (2)

Constantin Carathéodory
(1873-1950)

Carathéodory's Theorem:

$$p \in \text{conv}\, S \quad \text{iff}$$
$$p \in \text{conv}\, T \;\text{ for some } T \subseteq S \text{ with } |T| \leq d+1$$

$\rightarrow$ in $\mathbf{R}^2$: $\mathbf{O(n^4)}$ constraints

$[\ p_h, p_j, p_k \in S$ and $p_i \in \text{conv}\{p_h, p_j, p_k\}\ ] \Rightarrow p_i \in S$

- e.g., $y_i \geq y_h + y_j + y_k - 2$ for all such h, i, j, k

suffice (in binary variables)

- some of these constraints are redundant
  - Recall: in one dimension, we reduced from O(n^3) to O(n^2) constraints, so:

- **Formulation Question**: to find a sufficient set of fewer, say, O(n^3), constraints (in the natural binary variables)

2. 2D convex-shapes : IP modeling (3)

- Recall: in 1D we had a complete characterization of the convex hull of characteristic vectors of all convex (i.e., contiguous) subsets, using "alternating constraints", and a linear-time separation algorithm
- **Polyhedral Question**: to find a system of linear inequalities defining the convex hull of characteristic vectors of all convex subsets of a given set $P \subseteq \mathbf{R}^2$
  - some progress on restricted problem with given bottom-most point $b \in S$
- **Algorithmic Question**: to find a combinatorial algorithm for the Separation Problem for this convex hull
  - polytime solvable by the Ellipsoid method

2. 2D convex-shapes : IP modeling (4)

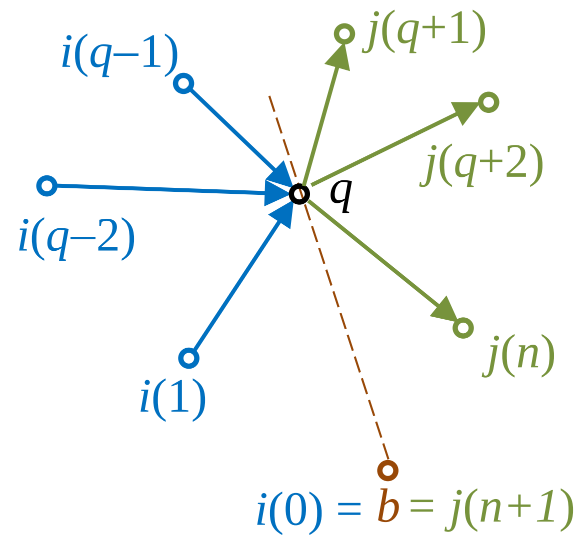
Back to **extended formulations**. Recall:

- in one dimension, we had an ideal extended formulation with $2n$ variables and $2n$ constraints

From the DP algorithm we can derive an extended formulation with $O(n^3)$ variables and constraints

  - *current joint work with Laurence Wolsey (CORE)*

- Based on an inclusion condition: given bottom-most point b,
  - Label points $b = 0, 1, \ldots, n, n+1 = b$ in clockwise order as seen from b
  - At each $q \in P$, the "before" nodes B_q are labeled $i(0), \ldots, i(q-1)$ in cw order as seen **from q**
  - and similarly for the "after" nodes A_q
  - If edges $i(h)\, q$ and $q\, j(k)$ compatible, then so are $i(h')\, q$ and $q\, j(k')$ for $h' \leq h$ and $k \leq k'$



2. 2D convex-shapes : IP modeling (5)

Ideal extended formulation:

- bottom-most indicator variables z_b
- edge variables e_{bpq} = 1 iff b is b-most point and pq an edge of the convex hull of selected points

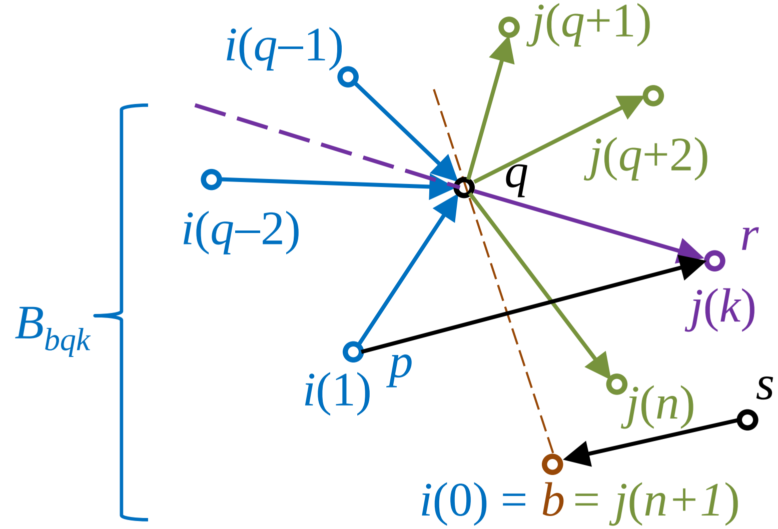
$\Sigma_b \, z_b \leq 1$

$y_{bq} + \Sigma \, \{e_{bpr}$: edge pr cuts edge $bq\} = z_b \quad \forall b, q$

$\Sigma_s \, e_{b,s,(n+1)} = z_b \quad \forall b$

$\Sigma \, \{e_{bpq} : p \in B_{bqk}\} + s_{b,q,(k-1)}$
$= e_{b,q,j(k)} + s_{b,q,k} \qquad \forall b, q, k$

where $s_{b,q,k}$ is a "slack" variable



- O(n^3) variables and constraints

2. 2D convex-shapes : IP modeling (6)

More Formulation Questions:

- to find a more concise extended formulation
  - e.g., with $O(n)$ variables and $O(n^2)$ constraints?
- …and that is ideal?

3. Other Notions of Convexity

A ***closure system*** (or *Moore family*) is a set system $(X, \mathcal{F})$

(1) containing the empty and full sets ($\emptyset \in \mathcal{F}$ and $X \in \mathcal{F}$), and

(2) stable for (arbitrary) intersection (i.e., $\cap \mathcal{G} \in \mathcal{F}$ for all $\mathcal{G} \subseteq \mathcal{F}$)

In a closure system every subset $S \subseteq X$ has a ***closure*** $\mathrm{cl}_{\mathcal{F}} S$

- the smallest set in $\mathcal{F}$ that contains S: $\mathrm{cl}_{\mathcal{F}} S = \cap\{ F \in \mathcal{F} : S \subseteq F \}$

A ***convex structure*** (or *aligned space*) is a closure system $(X, \mathcal{F})$ also

(3) stable for nested unions (i.e., if $\mathcal{G} \subseteq \mathcal{F}$ is totally ordered by inclusion then $\cup \mathcal{G} \in \mathcal{F}$)

In a convex structure the closure $\mathrm{cl}_{\mathcal{F}} S$ is called the ***convex hull*** $\mathrm{co}_{\mathcal{F}} S$

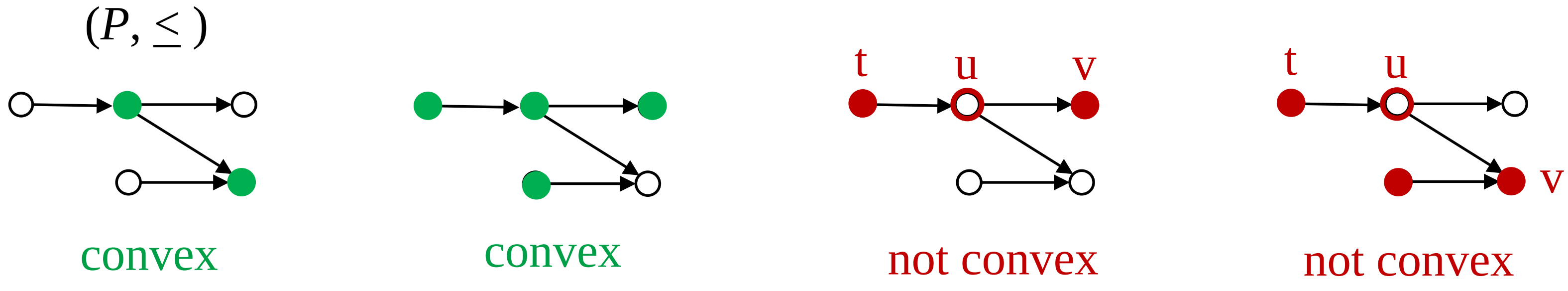
- (3) implies (using the Axiom of Choice and transfinite induction) that a point in the convex hull of any set $S \subseteq X$ is in a convex hull of some *finite* subset of S (the closure operator of $\mathcal{F}$ is *domain finite*)

3a. Other Notions of Convexity : Poset Convexity

Joint work with Laurence Wolsey (CORE)

A subset $C \subseteq P$ of a poset (partially ordered set) $(P, \leq)$ is **convex** (aka, an interval) iff

$$[t \in C \text{ and } t \leq \boldsymbol{u} \leq v \in C\,] \Rightarrow \boldsymbol{u} \in C$$



3a.Convex Subsets in a Poset (2)

A subset $C \subseteq P$ of a poset (partially ordered set) $(P, \leq)$ is **convex** (aka, an interval) iff

$$[t \in C \text{ and } t \leq \boldsymbol{u} \leq v \in C\,] \Rightarrow \boldsymbol{u} \in C$$

Examples:

- Subset of tasks assigned to a contractor in Project Planning (*subcontractor work package*)
- Subset of tasks assigned to a station in Assembly Line Balancing
  - [Frédéric Meunier & Mustapha El Lemdani, 2012]
- Set of blocks (or jobs) processed in a given year (period) in Open Pit Mine (or Project) Scheduling
- One-dimensional special case:
  - Consecutive periods in unit commitment [e.g., Jon Lee et al., 2004]
  - Contiguous drawpoints in an underground mine tunnel [Anita Parkinson 2012]

3a. Convex Subsets in a Poset : Four Questions

A. Polyhedral Description:

- Given a finite poset $P = (V, \leq)$, determine an explicit, finite system of linear inequalities describing the **convex hull** $C_P \subseteq \Re^V$ of the characteristic vectors of all (poset) convex subsets in P

B. Separation Problem:

- Given poset P and a vector $\mathbf{x^0} \in \Re^V$, decide whether $x^0 \in C_P$ and, if not, produce a linear inequality that is satisfied by the characteristic vectors of all convex subsets in P and is violated by x^0

C. Optimization Problem:

- Given poset P and a weight vector $\mathbf{w} \in \Re^V$, find a convex subset $S^* \subseteq V$ with maximum total weight $w(S^*) = \Sigma_{u \in S^*} w(u)$

D. Extended Formulation:

- Given poset P, determine a **compact** (i.e., polynomial-size) extended formulation of that convex hull C_P

3a. Solving the Optimization Problem

Closures in a poset

- A subset $T \subseteq V$ in a poset $P = (V, \leq)$ is a **closure** (aka, terminal subset, upper ideal, filter) iff

$$[t \in T \text{ and } t \leq u] \Rightarrow u \in T$$

- A characterization of poset convex subsets:

Lemma: A subset S in a poset P is convex iff

$S = T \setminus T'$ for some closures T and T' in P

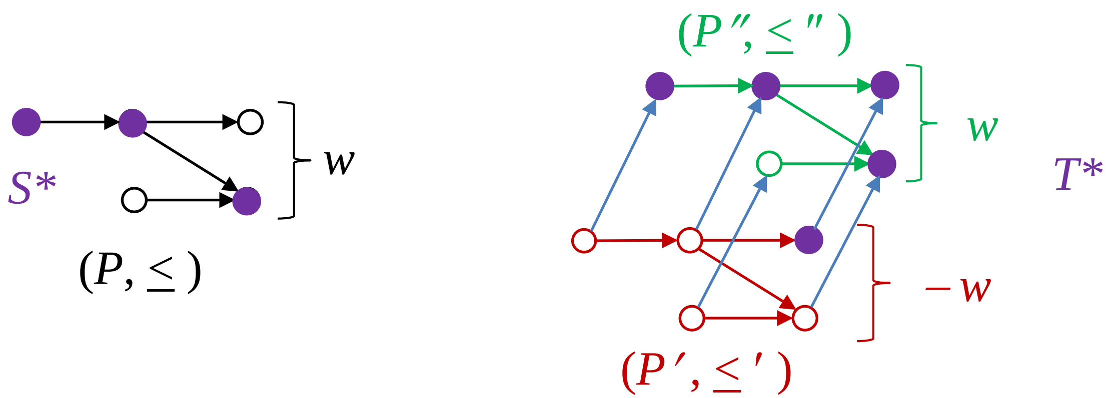
Maximum Weight Closure problem:

- Given poset P and a weight vector $w \in \Re^V$, find a closure $T^* \subseteq V$ with maximum total weight $w(T^*)$
- Solved in strongly polytime as a minimum s-t-cut problem in a related network
  - Rhys (1970); also Balinski (1970) and Picard (1976)

3a. Solving the Optimization Problem (2)

Solving the Maximum Weight Poset *Convex* Subset problem:

- Define poset $(P' \cup P'', \preceq_\cup)$ where
  - $(P', \preceq')$ and $(P'', \preceq'')$ are two copies of $(P, \preceq)$
  - $\preceq_\cup$ is induced by $\preceq'$, $\preceq''$, and $v' \preceq_\cup v''$ for all $v \in P$



- Let weights $w(v') = -w(v)$ and $w(v'') = w(v)$ for all $v \in P$

Proposition: T^* is a maximum-weight terminal subset in $(P' \cup P'', \preceq_\cup)$ iff $S^* := \{v \in P : v' \notin T \text{ and } v'' \in T\}$ is a maximum weight convex subset in $(P, \preceq)$

3a. Extended Formulation

- Poset closures have a well-known compact ideal formulation [Rhys, 1970]:

$$\{y \in \Re^V : 0 \le y \le 1, \text{ and } y_u \le y_v \text{ for all } u \to v\}$$

- The reduction between the optimization problems implies:

<u>Theorem:</u> Given a poset P, let V' and V'' be two copies of V. Then

$$E_P = \{(x, y', y'') \in \Re^{V \times V' \times V''} : \; x = y'' - y'$$
$$0 \le y' \le y'' \le 1$$
$$y'_u \le y'_v \text{ and } y''_u \le y''_v \text{ for all } u \to v\}$$

is a compact ideal extended formulation of poset convex subsets of P

3a. Proof Overview

Optimization Problem
solved as a mincut problem

↓

Linear-sized, ideal
Extended Formulation

3a. Alternating Inequalities and their Separation

- A sequence $c = (c_1, \ldots, c_{l(c)})$ of elements of poset P is a **chain** (a totally ordered subset) iff $c_1 < c_2 < \ldots < c_{l(c)}$
- Its **alternating vector a^c**$\in \Re^V$ has components

$$a_u^c = \begin{cases} +1 & \text{if } u = c_i \text{ for some odd } i \\ -1 & \text{if } u = c_i \text{ for some even } i \\ 0 & \text{otherwise} \end{cases}$$

- Chain c is **odd** if its length $l(c)$ is odd
- **Odd(P)** = set of all odd chains in P

Lemma (**Validity** of poset alternating inequalities):

The characteristic vector x of every convex subset in a poset P satisfies the **alternating inequalities**

$$a^c x \leq 1 \quad \text{for all } c \in \text{Odd}(P)$$

- i.e., $x_{c_1} - x_{c_2} + x_{c_3} - \ldots + x_{c_{l(c)}} \leq 1$

3a. Separation Problem for Alternating Inequalities

- Given poset P and a vector $x^0 \in \Re^V$, decide whether x^0 satisfies all alternating inequalities and, if not, produce a violated inequality, i.e., a chain $c \in \text{Odd}(P)$ such that $a^c x^0 > 1$

Dynamic Programming:

- $\text{Pred}(v) = \{u \in V : u < v\}$ set of all (strict) predecessor of $v \in V$
- **Pred*(v)** = $\text{Pred}(v) \setminus \cup_{u \in \text{Pred}(v)} \text{Pred}(u)$ that of its immediate predecessors
- **Odd(v)** [resp., **Even(v)**] the set of all odd [resp., even] chains in P that end at or before v
  - the empty chain $\emptyset \in \text{Even}(v)$ for all v
- **DP value functionals** F and $G : V \to \Re$

$$F(v) = \max\{a^c x^0 : c \in \text{Odd}(v)\}$$
$$G(v) = \max\{a^c x^0 : c \in \text{Even}(v)\}$$

3a. Separation for Alternating Inequalities (2)

- DP value functionals F and $G \in \Re^V$

$$F(v) = \max\{a^c x^0 : c \in \mathrm{Odd}(v)\}$$
$$G(v) = \max\{a^c x^0 : c \in \mathrm{Even}(v)\}$$

- **DP recursions:**

$$F(v) = \begin{cases} x^0_v & \text{if } \mathrm{Pred}^*(v) = \emptyset \\ \max_{u \in \mathrm{Pred}^*(v)} \{\max\{F(u), G(u) + x^0_v\}\} & \text{o/w} \end{cases}$$

$$G(v) = \begin{cases} 0 & \text{if } \mathrm{Pred}^*(v) = \emptyset \\ \max_{u \in \mathrm{Pred}^*(v)} \{\max\{G(u), F(u) - x^0_v\}\} & \text{o/w} \end{cases}$$

Lemma: Using these DP recursions solves the separation problem for the alternating inequalities in linear time

3a. Proof Overview (2)

Optimization Problem solved as a mincut problem

↓

Linear-sized, ideal **Extended Formulation**

Validity of the **Alternating Inequalities**

↓

Linear-time DP algorithm for their **Separation Problem**

3a. DP Functionals: Properties

- **DP recursions:**

$$F(v) = \begin{cases} x^0_v & \text{if } \mathrm{Pred}^*(v) = \varnothing \\ \max_{u \in \mathrm{Pred}^*(v)} \{\max\{F(u), G(u) + x^0_v\}\} & \text{o/w} \end{cases}$$

$$G(v) = \begin{cases} 0 & \text{if } \mathrm{Pred}^*(v) = \varnothing \\ \max_{u \in \mathrm{Pred}^*(v)} \{\max\{G(u), F(u) - x^0_v\}\} & \text{o/w} \end{cases}$$

(1) F and G are nondecreasing w.r.t. poset order $\leq$

- i.e., $u \leq v$ implies $F(u) \leq F(v)$ and $G(u) \leq G(v)$

(2) $F(v) - G(v) = x^0_v$

- If $\mathrm{Pred}^*(v) \neq \varnothing$ then

$$G(v) + x^0_v = \max_{u \in \mathrm{Pred}^*(v)} \{\max\{G(u) + x^0_v , F(u)\}\} = F(v)$$

3a. DP Functionals and the Extended Formulation

Lemma (Properties of the DP functionals):

(1) F and G are nondecreasing w.r.t. poset order $\leq$

(2) $x^0 = F - G$

Recall the Extended Formulation:

$$E_P = \{(x, y', y'') \in \Re^{V \times V' \times V''} : \; x = y'' - y'$$
$$0 \leq y' \leq y'' \leq 1$$
$$y'_u \leq y'_v \text{ and } y''_u \leq y''_v \text{ for all } u \to v \; \}$$

Proposition: If $\mathbf{x^0 \geq 0}$ and **all** $\mathbf{F(v) \leq 1}$ then $(x^0, y', y'') \in E_p$ where $y'_v = G(v)$ and $y''_v = F(v)$ for all $v \in V$

Corollary: $x^0 \in C_P$ iff $x^0 \geq 0$ and satisfies all the alternating inequalities.

- This solves the Polyhedral Description question!

3a. In Summary

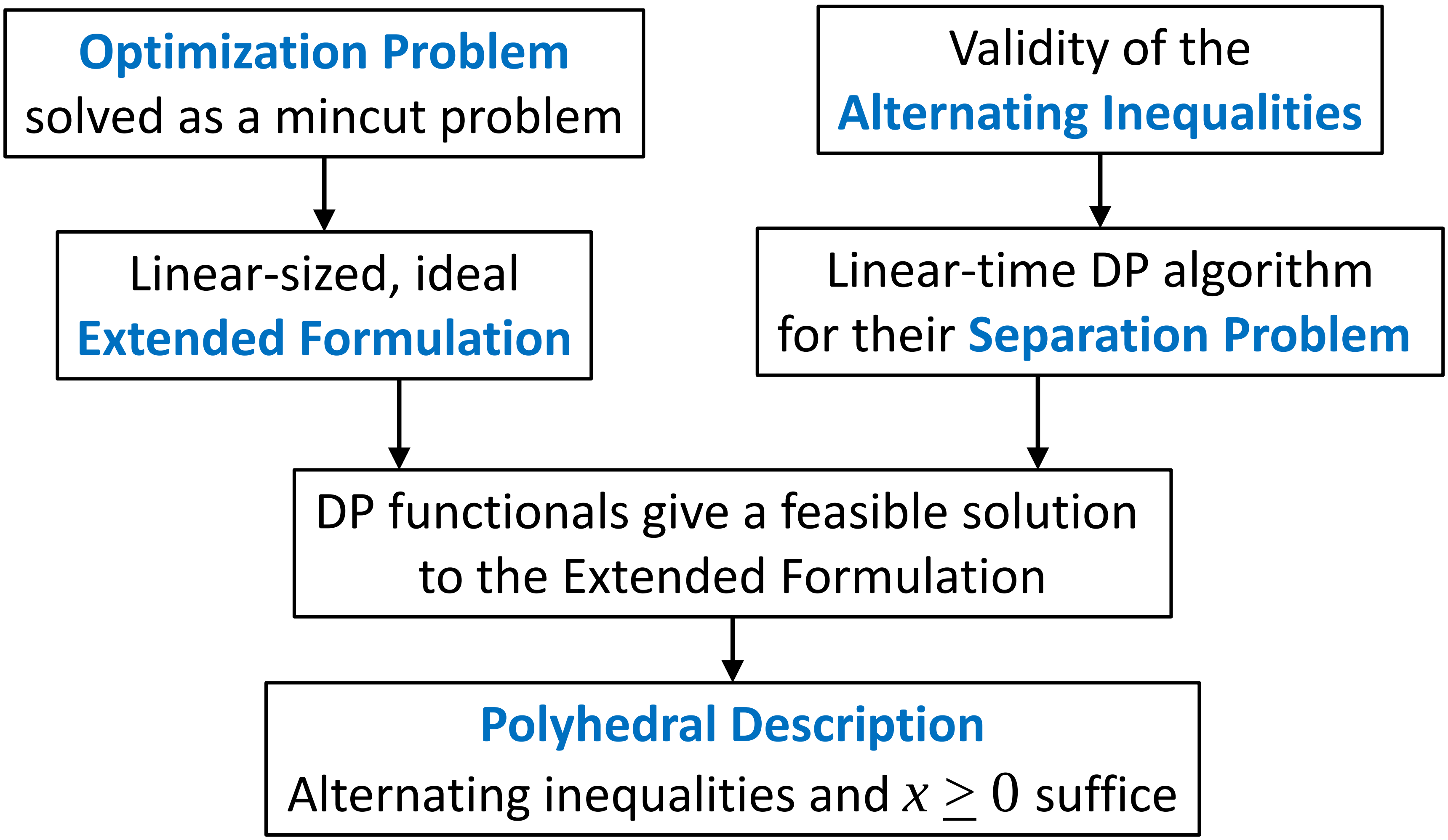
Theorem: Let P be a given finite poset.

(i) The optimization problem for convex subsets of P is solvable in strongly polynomial time as an s-t-cut problem

(ii) The alternating inequalities plus the non-negativity constraints $x \geq 0$ form the minimal linear inequality system defining the convex hull of the characteristic vectors of all convex subsets in P.

(iii) The separation problem for this convex hull is solvable in linear time.

3a. In Summary (2): Proof Method



3b. Other Notions of Convexity : Geodesic Convexity and Related Notions

A subset $C \subseteq X$ of a metric space is (geodesic) **convex** iff C contains all shortest s-t paths (in X) for all $s,t \in C$

- Geodesic convex subsets form a convex structure
  - a generalization of standard convexity in $\mathbf{R}^d$

Examples:

- In $\mathbf{R}^d$ (or $\mathbf{Z}^d$) with the L_1 metric, this gives the box convexity
  - convex sets are boxes (rectangles)
  - the optimization problem is easy
- Geodesic convexity in graphs, with (nonnegative) edge lengths…

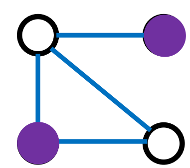
3b. Geodesic Convexity (2)

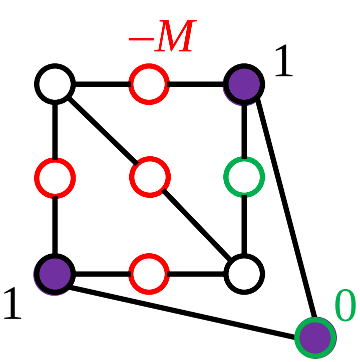
Theorem: The Maximum Weight Geodesic Convex Subset problem in a graph with *unit edge lengths* cannot be approximated in polytime to any factor $n^{1-\varepsilon}$ with $\varepsilon > 0$, unless P = NP

Proof: similar to previous proof, also using the Maximum Independent Set (MIS) problem: given instance $G = (V, E)$ of MIS, define a unit-edge-length graph $G' = (V', E')$ as follows:

- split each edge uv of the complete graph $K(V)$ with a node $p(u,v)$
  - so u-$p(u,v)$-v is the unique shortest u-v path in G'
- Let all edge lengths = 1, and node weights $w_u = 1$

$$w_{p(u,v)} = \begin{cases} -M < -|V| & \text{if } uv \in E \\ 0 & \text{otherwise} \end{cases}$$


graph G

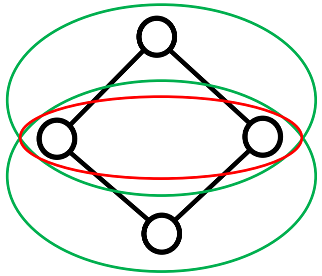


graph G'

3b. Geodesic Convexity and Related Notions (3)

How about subsets $C \subseteq X$ that contain <u>some</u> shortest s-t paths (in X) for every pair $s,t \in C$?

- Such **weakly geodesic** subsets do <u>not</u> form a convex structure
  - not stable under intersection



Examples:

- The police officer assigned to a quadrant should be able to remain in his/her quadrant while going as quickly as possible from any quadrant point to any other quadrant point
- In (connected) graphs with *zero edge lengths* this defines connected subsets
  - The Optimization problem for connected subsets is NP-hard to approximate within any constant factor
    [E. Alvarez-Miranda, I. Ljubic, P. Mutzel (2013)]

Theorem: The Maximum Weight Weakly Geodesic Convex Subset problem in a graph with *unit edge lengths* cannot be approximated in polytime to any factor $n^{1-\varepsilon}$ with $\varepsilon > 0$, unless P = NP

- The same proof applies

Thank you for
your attention

Questions?
Comments?